Domain decomposition methods

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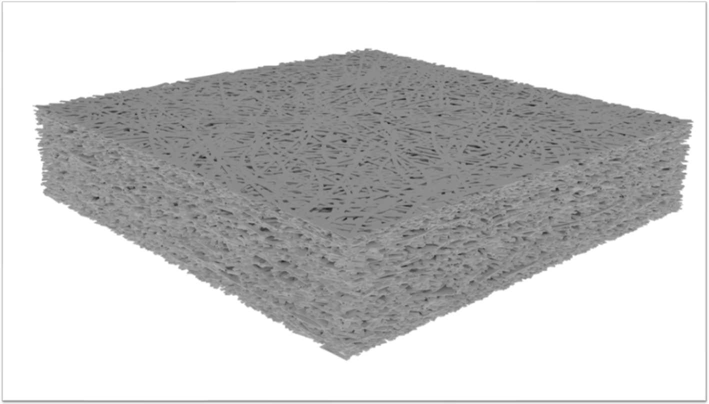
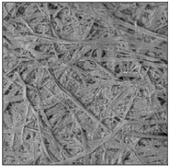
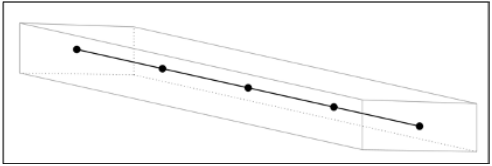
Motivating example: Simulation of paperboard



- Mechanical properties (tensile/bending strength)
- Collaboration with Fraunhofer Chalmers Centre (FCC), paper making industry (Stora Enso) and packaging (Tetra Pak)

Motivating example: Simulation of paperboard



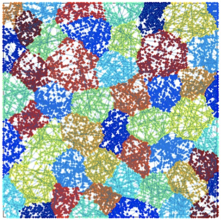


- Paper fibres are hollow flattened slender cylinders
- Model: network of connected beams
- The displacement solves a linear system $Au = f$
- A is SPD, sparse but large (200M dofs) and ill-conditioned
- Direct solvers overload memory, iterative solvers (AMG, PCG) fail to converge

How to develop an efficient robust solver for this problem?

Outline

Domain decomposition (DD)[1]



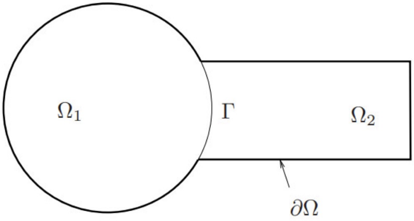
- $Au = f$ is solved iteratively over the subdomains $\Omega = \cup_{i=1}^{n} \Omega_i$

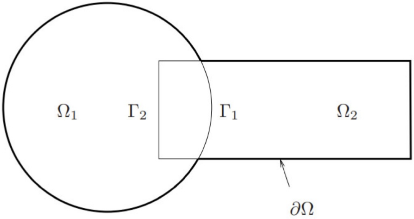
$$A_i u_i = f_i, \quad \text{in } \tilde{\Omega}_i$$

- Non-overlapping ($\tilde{\Omega}_i = \Omega_i$) or overlapping ($\tilde{\Omega}_i \supset \Omega_i$) domains? How to pass information between domains?
- Decompose into subdomain before or after discretization?
- DD as a solver or as a preconditioner?

[1] Alternating Schwarz 1870, numerical solution of PDEs 1950-

Nonoverlapping vs. overlapping

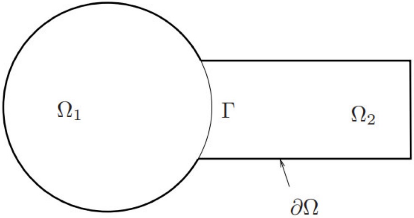


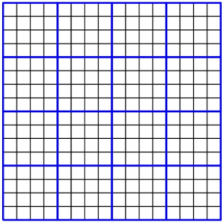


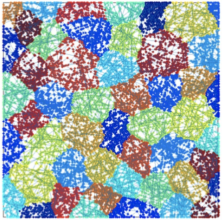
- *Nonoverlapping:* less overhead, transmission condition on the interface (DN, NN, RR) with a parameter s
- *Overlapping:* freedom in geometry of subdomains, interface conditions, overlap parameter δ

Choice of parameters (s, δ) affects performance.

Continuous, discretized, or algebraic level



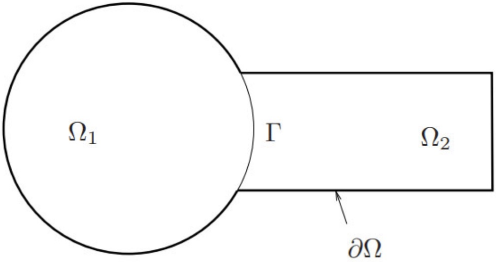




$-\Delta u = f$ $\qquad A_h u_h = f_h$ $\qquad Ax = b$

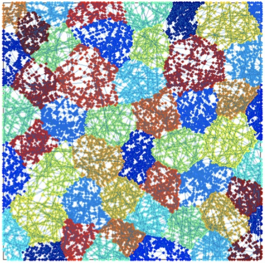
- A DD algorithm is optimal if the rate of convergence is independent the size of the system.
- A DD algorithm is scalable if the convergence does not deteriorate when the number of subdomains grows.

DD as a solver or as a preconditioner



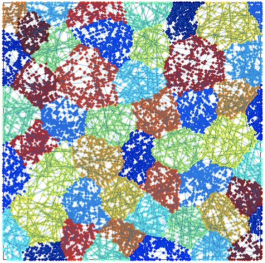
- Solve $Au = f$ by iteration with transmission condition
- Rewrite as an equation on Γ: $Su_\Gamma = g_\Gamma$
- DD can be formulated as a preconditioned Richardson iteration: $u_\Gamma^{n+1} = u_\Gamma^n + B(g_\Gamma - Su_\Gamma^n)$
- Convergence depends on the condition number $\kappa(BS)$
- Optimal but typically not scalable

DD in PCG



- Solve $Au = f$ with the conjugate gradient method
- Convergence rate depends on $\kappa(A) \gg 1$
- Instead $BAu = Bf$ is considered with $B \sim A^{-1}$ and $\kappa(BA) \sim 1$
- CG: given some v we need to compute the action BAv
- The action of B can again be an iteration of a DD algorithm

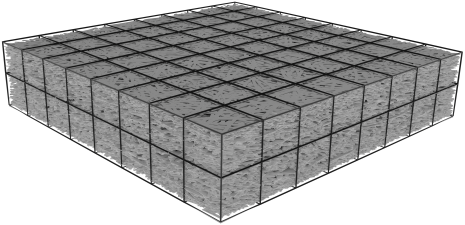
Additive Schwarz preconditioner



- Solve one coarse grid problem $A_H z_0 = Av$ in Ω (for scalability)
- Solve overlapping local problems $A_i z_i = Av$, in $\tilde{\Omega}_i,\ i = 1, \dots, n$
- Let $B(Av) = z_0 + z_1 + \cdots + z_n$ (additive)
- $B(f)$ can be constructed in a similar way
- Optimal and scalable $\kappa(BA) \sim 1$

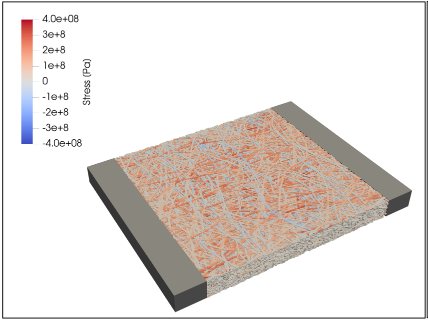
Outline

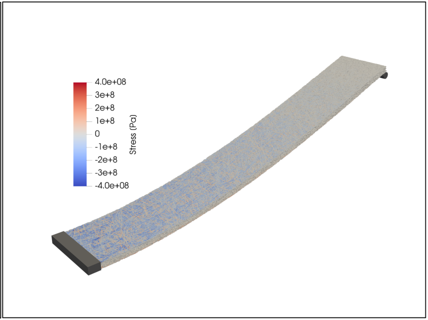
Example: Simulation of paperboard



- Preconditioned conjugate gradient method
- Additive Schwarz preconditioner: overlapping subdomains ($\delta \sim H$) with coarse grid correction
- Subproblems $A_i u_i = f_i$ solved using a direct solver
- Optimal convergence under assumptions on the underlying graph
- Rapid convergence (50-100) iterations, very robust

Example: Simulation of paperboard





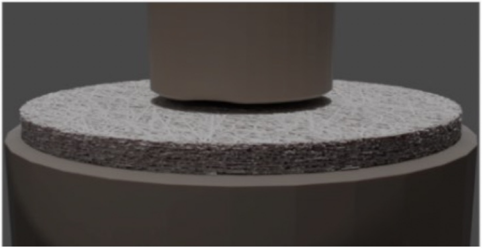
- Simulation of tensile strength and bending strength
- Numerical simulation agrees with experiments[2]

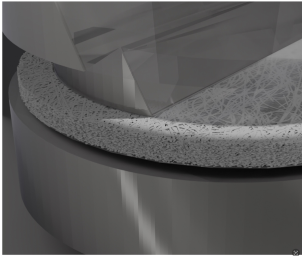
DD was crucial for the efficient solution of these problems

[2] M. Görtz, PhD thesis, 2024

Example: Conclusion and future challenges

- DD is still relevant
- Well understood for linear problems

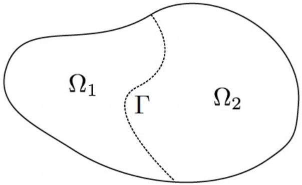




- Applications: large scale deformation, compression, folding, crack propagation, ...
- Nonlinear models
- DD for evolution problems: elastic wave propagation

Outline

Emil's contributions



- Theoretical foundation for non-overlapping DD for non-linear elliptic problems
- Theoretical foundation for non-overlapping DD for non-linear parabolic problems
- Investigation of the main techniques (DN, NN, RR) and their performance, including parameter studies