Numerical upscaling of perturbed diffusion problems

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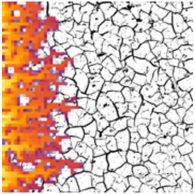
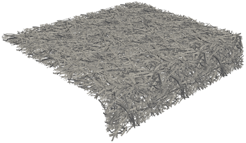
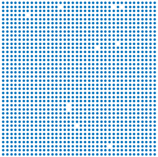
Heterogeneous materials

We consider heterogeneous materials with perturbations

- ▷ time dependency
- ▷ local deformation
- ▷ defects

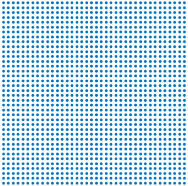
Propagating front in a porous material, large deformation in network model, and random defects in a composite.

Two issues: changing and rapidly varying data.

Outline

Elliptic model problem

The Poisson equation with a reference $\hat{A}$ (left) and perturbed A (right) diffusion that fulfills $0 < \alpha \leq \hat{A}, A \leq \beta$.



On weak form we have: find $\hat{u} \in V := H_0^1(\Omega)$ and $u \in V$ such that

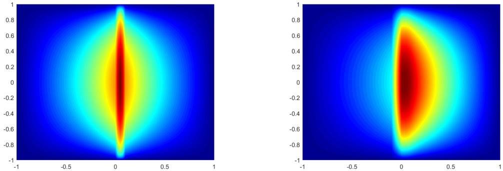
$$\hat{a}(\hat{u}, v) := \int_\Omega (\hat{A}\nabla\hat{u}) \cdot \nabla v \, dx = \int_\Omega f \cdot v \, dx \quad \text{for all } v \in V,$$

$$a(u, v) := \int_\Omega (A\nabla u) \cdot \nabla v \, dx = \int_\Omega f \cdot v \, dx \quad \text{for all } v \in V.$$

Modelling error

$$\|A^{1/2}\nabla(u-\hat{u})\|^2_{L^2(\Omega)} = (A\nabla(u-\hat{u}), \nabla(u-\hat{u}))$$
$$= ((\hat{A}-A)\nabla\hat{u}, \nabla(u-\hat{u})) + (f-f, u-\hat{u})$$
$$\leq \alpha^{-1}\|\hat{A}-A\|_{L^\infty(\Omega)}\|\hat{A}^{1/2}\nabla\hat{u}\|_{L^2(\Omega)}\|A^{1/2}\nabla(u-\hat{u})\|_{L^2(\Omega)}.$$

We conclude $\|A^{1/2}\nabla(u-\hat{u})\|_{L^2(\Omega)} \leq C\alpha^{-3/2}\|f\|_{L^2(\Omega)}\|A-\hat{A}\|_{L^\infty(\Omega)}$.

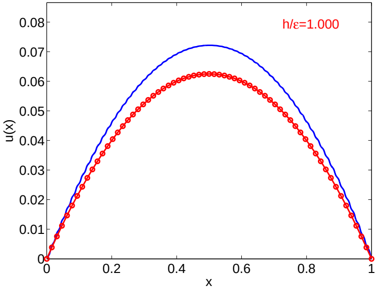


Here $\hat{A} = 1 - 0.9\chi_{\{0<x<0.1\}}$, $A = 1 - 0.9\chi_{\{-0.1<x<0\}}$, and $f = \chi_{\{0<x<0.1\}}$.

The full problem has to be resolved for each perturbation.

Rapid data variation

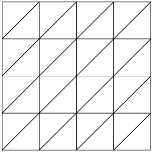
We let $A(x) = 2 + \sin(2\pi x/\epsilon)$, $\epsilon = 2^{-6}$, and $f = 1$.



- FEM on mesh $h \geq \epsilon$ leads to averaging of the diffusion and wrong effective behavior.
- A priori error bound:
  $\|A^{1/2}\nabla(u - u_h)\|_{L^2(\Omega)} \leq Ch\|u\|_{H^2(\Omega)} \approx Ch\epsilon^{-1}$.

Outline

Orthogonal decomposition



- (coarse) FE mesh $\mathcal{T}_H$ with parameter $H > h$
- P1-FE space $V_H := \{v \in V \mid \forall T \in \mathcal{T}_H, v|_T \in P_1(T)\}$
- $\mathfrak{I}_{\mathcal{T}} : V \to V_H$ some interpolation operator

Decomposition

$$V = V_H \oplus V^{\mathrm{f}} \quad \text{with } V^{\mathrm{f}} := \text{kernel}\, \mathfrak{I}_{\mathcal{T}} = \{v \in V \mid \mathfrak{I}_{\mathcal{T}} v = 0\}$$

- For each $v \in V_H$ define finescale projection $Qv \in V^{\mathrm{f}}$ by

$$a(Qv, w) = a(v, w) \quad \text{for all } w \in V^{\mathrm{f}}$$

***a*-Orthogonal Decomposition**

$$V = V_H^{\mathrm{ms}} \oplus V^{\mathrm{f}} \quad \text{with } V_H^{\mathrm{ms}} := (V_H - QV_H)$$

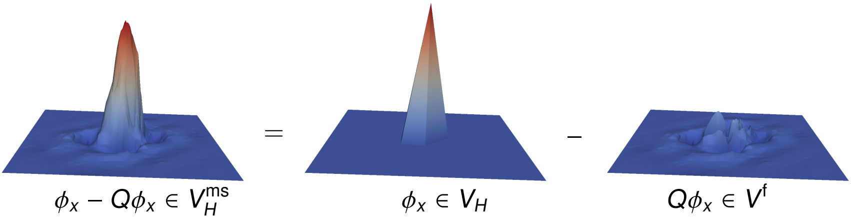
Modified nodal basis

- $\phi_x \in V_H$ denotes classical nodal basis function ($x \in \mathcal{N}$)
- $Q\phi_x \in V^{\mathrm{f}}$ denotes the finescale correction of ϕ_x ($x \in \mathcal{N}$)

Multiscale FE space

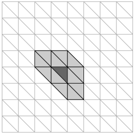
$$V_H^{\mathrm{ms}} = \operatorname{span}\{\phi_x - Q\phi_x \mid x \in \mathcal{N}\}$$

Example

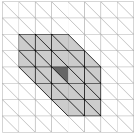


Localization

- Define nodal patches of ℓ-th order $\omega_{T,\ell}$ about $T \in \mathcal{T}_H$



$\omega_{T,1}$



$\omega_{T,2}$

- Correctors $Q_\ell^T \phi_x \in V^{\mathrm{f}}(\omega_{T,\ell}) := \{v \in V^{\mathrm{f}} \mid v|_{\Omega \setminus \omega_{T,\ell}} = 0\}$ solve

$$a(Q_\ell^T \phi_x, w) = \int_T A \nabla \phi_x \cdot \nabla w \, dx \quad \text{for all } w \in V^{\mathrm{f}}(\omega_{T,\ell})$$

Localized multiscale FE spaces

$$V_{H,\ell}^{\mathrm{ms}} = \mathrm{span}\{\phi_x - \sum_{T \in \mathcal{T}_H} Q_\ell^T \phi_x \mid x \in \mathcal{N}\}$$

Fully discrete method (PG-LOD)

We discretize the fine scales and let

$$V_{H,\ell}^{\text{ms},h} = \text{span}\{\phi_x - \sum_{T\in\mathcal{T}} Q_{\ell,h}^T \phi_x \mid x \in \mathcal{N}\}$$

Fully discrete multiscale approximation $u_{H,\ell}^{\text{ms},h} \in V_{H,\ell}^{\text{ms},h}$

$$a(u_{H,\ell}^{\text{ms},h}, v) = (f, v) \quad \text{for all } v \in V_H$$

Theorem (A priori error bound)

$$|||u_h - u_{H,\ell}^{\text{ms},h}||| \leq C'(H + e^{-c\ell}) = CH,$$

with C independent of A' and $\ell \approx |\log(H)|$.

Elfverson, Ginting, and Henning, On multiscale methods in PG formulation, Numer. Math., 2015.

Comments on the choice of method

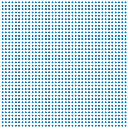
1. The multiscale representation can be reused (evolution, iteration, samples).
2. Local changes in data leads to local recomputations.
3. LOD allows a priori error bounds for non-periodic data.
4. The PG formulation reduces communication.

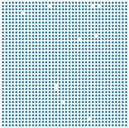
- GFEM 1983-, Babuška-Osborn, Melenk, Lipton, ...
- VMS 1995-, Hughes et.al., Larson-M., Hughes-Sangalli, ...
- MsFEM 1996-, Hou & Wu, Efendiev et.al., ...
- HMM 2003-, Engquist & E, ...
- LOD 2013-, M. & Peterseim, Henning et.al., ...
- Baysian NH, Gamblets, 2010-, Owhadi et.al.

There are many other related methods not listed here.

Outline

Local recomputation of basis function





We assume we have computed $\hat{Q}_\ell^T \phi_x \in V^{\mathrm{f}}(\omega_{T,\ell})$ associated with $\hat{A}$

$$\hat{a}(\hat{Q}_\ell^T \phi_x, w) = \int_T \hat{A} \nabla \phi_x \cdot \nabla w \, dx \quad \text{for all } w \in V^{\mathrm{f}}(\omega_{T,\ell})$$

We only recompute for $T \in \mathcal{T}_H \setminus \tilde{\mathcal{T}}_H$ and let

$$\tilde{Q}_\ell^T = \begin{cases} \hat{Q}_\ell^T, & T \in \tilde{\mathcal{T}}_H \\ Q_\ell^T, & T \in \mathcal{T}_H \setminus \tilde{\mathcal{T}}_H \end{cases}$$

We let $\tilde{V}_{H,\ell}^{\mathrm{ms}} = \mathrm{span}(\{\phi_x - \sum_{T \in \mathcal{T}_H} \tilde{Q}_\ell^T \phi_x\})$.

Local recomputation of basis function

We let $v, w \in V_H$ and define

$$\tilde{a}(v,w) = \sum_{T\in\mathcal{T}_H} (\tilde{A}_T\nabla v, \nabla w)_T + (\tilde{A}_T\nabla\tilde{Q}_\ell^T v, \nabla w)_{\omega_\ell(T)}$$

where

$$\tilde{A}_T = \begin{cases} \hat{A}, & T\in\tilde{\mathcal{T}}_H \\ A, & T\in\mathcal{T}_H\setminus\tilde{\mathcal{T}}_H \end{cases}$$

We let $\tilde{u} := (1-\sum_T \tilde{Q}_\ell^T)\tilde{u}_H$, where $\tilde{u}\in V_H$ solves

$$\tilde{a}(\tilde{u}_H, w) = (f,w), \quad \forall w\in V_H$$

- We note that $\hat{A}|_{\omega_\ell(T)}$ on T is arbitrary in this formulation. It can e.g. refer to different time steps for different T.

A priori error analysis

Theorem (A priori error bound)

If $\ell \approx |\log(H)|$ *and* $\max_{T \in \tilde{\mathcal{T}}} e_T < TOL$*, where*

$$e_T := \max_{v|_T, v \in V_H} \frac{\|(\tilde{A} - A)A^{-1/2}\nabla(\chi_T - \tilde{Q}_\ell^T)v\|_{L^2(\omega_{T,\ell})}}{\|A^{1/2}\nabla v\|_{L^2(T)}}$$

then $|||u_h - \tilde{u}||| \leq CH + C_\ell TOL$*.*

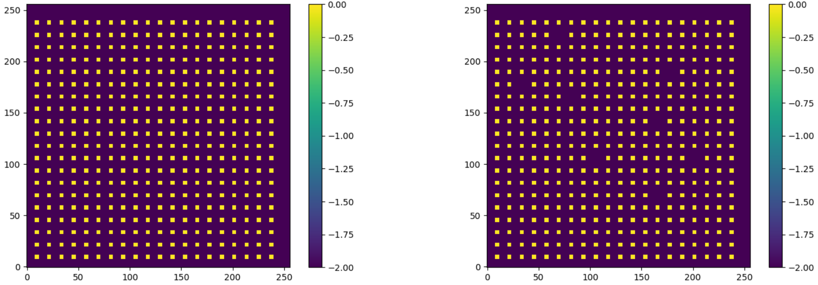
- e_T is computable and independent of Q_ℓ^T.
- e_T is large if both $A - \tilde{A}$ and $\tilde{Q}_\ell^T v$ are large.
- We only recompute for elements T where $e_T >$ TOL.

Hellman and M., Numerical homogenization of elliptic PDEs with similar coefficients, MMS, 2019.

Outline

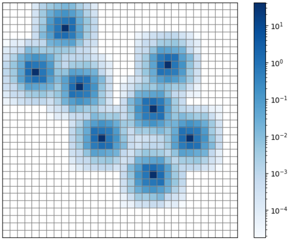
Numerical examples (defects)

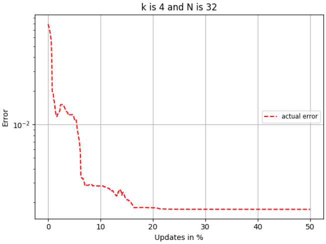
We consider an example with 2% defects with $f = \chi_{[1/8,7/8]\times[1/8,7/8]}$ and the diffusion taking values 1 (dots) and 0.01.



We let $H = 2^{-5}$, $h = 2^{-9}$, $\ell = 4$, and increase the amount of updates.

Numerical examples (defects)





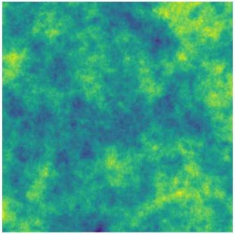
Error indicator (left) and error as a function of updates (right).

Right hand side correction eliminates H-dependency: find $u_f^T \in V^{\mathrm{f}}$ such that

$$a(u_f^T, v) = (f, v)_T$$

for all $v \in V^{\mathrm{f}}$. The full solution is $u^{\mathrm{full}} = u^{\mathrm{ms}} + \sum_{T \in \mathcal{T}_H} u_f^T$.

Numerical examples (two phase flow)



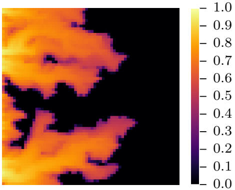
$$-\nabla \cdot \lambda(s) K \nabla u = 0 \qquad \dot{s} - \nabla \cdot \lambda_w(s) \nabla u = 0$$

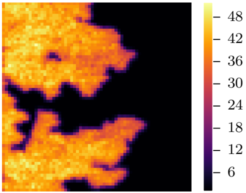
- K piecewise constant, log-normal, correlation length 0.05.
- The total mobility $\lambda(s) = s^3 + (1-s)^3$ and $\lambda_w(s) = s^3$.
- Pressure u has Dirichlet 1 (left) and 0 (right).
- Initial condition $s(0) = 0$.
- $s = 1$ on the inflow for the saturation.

Numerical examples (two phase flow)

Let $0 < t_1 < t_2 < \dots < t_n < \dots t_N$ and use a dG0 upwind forward Euler type scheme[1] for the saturation with the pressure and saturation solved sequentially i.e. diffusion $A_n(x) = \lambda(s_{n-1})K(x)$.

- We let $h = 2^{-9}$, $H = 2^{-6}$, $N = 2000$, and $\tau = 1/N$.
- We allow $\{\tilde{A}_T\}_{T \in \mathcal{T}_H}$ to be evaluated at different times.



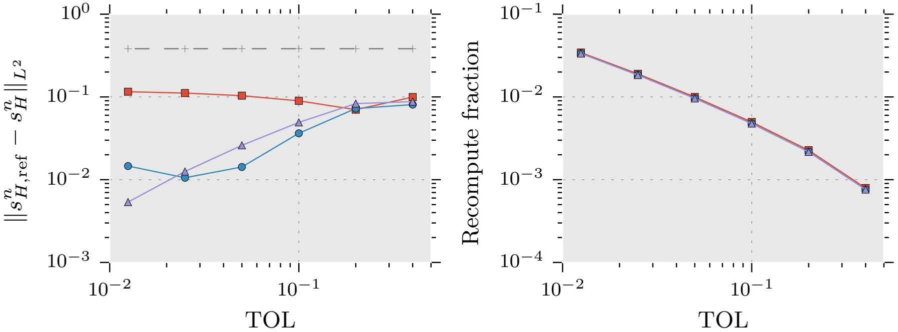


With $\ell = 2$, $\mathrm{TOL} = 0.05$ we get below 2% L^2-error in s_H^{2000} and on average 20.3 recomputations per coarse element i.e. 1%.

[1] Odsæter et. al. Postprocessing of non-conservative flux ..., CMAME 2017

Numerical examples (two phase flow)

We let $\ell = 1, 2, 3$ and vary TOL in the example.



Dashed is standard Galerkin on coarse mesh. Solution improves with $\ell = 1$ (squares) $\ell = 2$ (circles) $\ell = 3$ (diamonds). More recomputations with decreasing TOL (right).

Only coarse scale saturations are computed to minimize storage.

Outline

Comments and future work

- We derive error indicators driving local recomputation of multiscale basis functions.
- We prove a priori error bounds for the proposed method.
- We show that multiscale methods can be useful also for problems with locally varying diffusion.
- Future work: application to random diffusion problems. Sample diffusion locally and recompute basis when needed.

Thank you for your attention!