On convergence of multiscale methods

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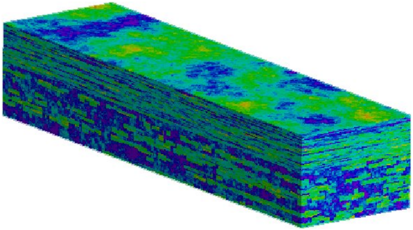
Multiscale Problems

Applications such as composite materials or

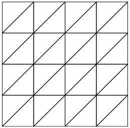


▷ flow in a porous medium

require numerical solution of partial differential equations with rough data (module of elasticity, conductivity, or permeability).

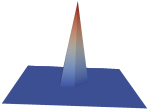
Major challenge: Features on multiple non-separated scales.

Finite Elements (FE) – Methodology

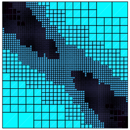


The numerical solution of PDEs by FEM consists of

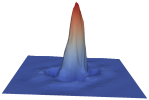
- construction of an “appropriate” FE mesh
- choosing (local) basis functions (of variable degree of approximation)



An optimal construction should be adapted to the local behavior of the exact solution and, hence, should take into account



- local singularities of the solution
  (e.g. singularities at re-entrant corners)
- effects of singular perturbations in the solutions
  (e.g. boundary layers)
- scales and amplitudes of rough coefficients



Outline

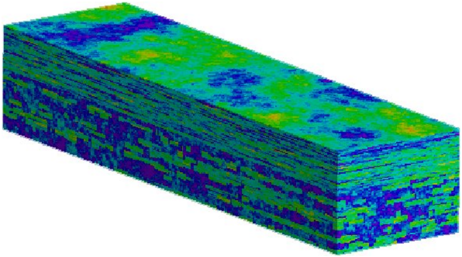
1. **Setting and Motivation**
2. Multiscale Method and Convergence
3. Full Discretization and Numerical Experiments
4. Applications
5. Conclusion

Model Multiscale Problem

Poisson's equation

$$-\operatorname{div} A\nabla u = f \quad \text{in } \Omega \qquad u = 0 \quad \text{on } \partial\Omega$$

with data $f \in L^2(\Omega)$ and $0 < \alpha \leq A \in L^\infty(\Omega)$



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Poisson's equation (variational form): $u \in V := H^1_0(\Omega)$ s.t.

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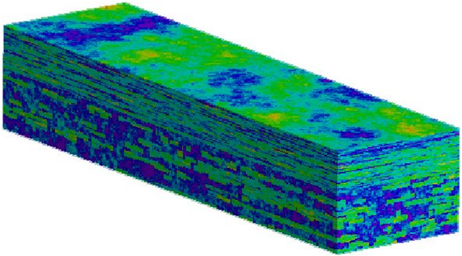

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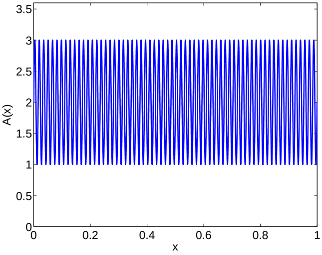
Example (periodic coefficient): $A(x) = 2 + \sin(2\pi x/\varepsilon)$, $\varepsilon = 2^{-6}$, $f = 1$



oscillatory coefficient

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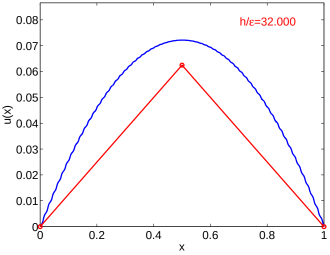
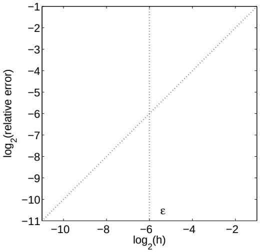


solution and P1-FEM-approximation



$\log_2(H^1(\Omega) - \text{error})$ vs. $\log_2(h)$

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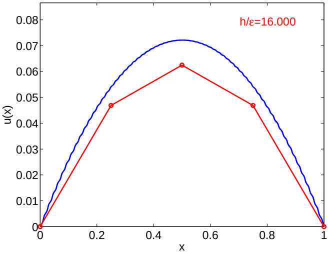

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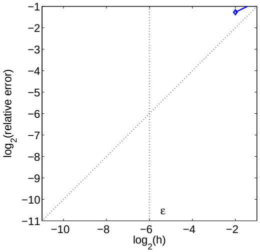

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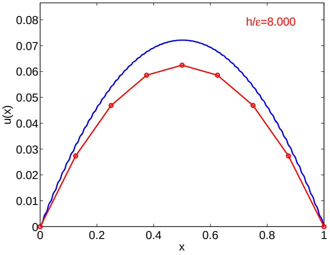

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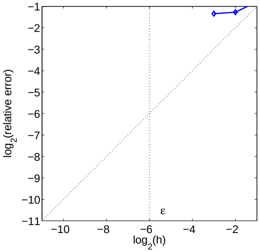

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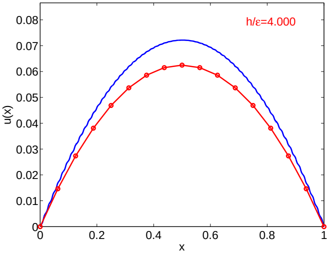

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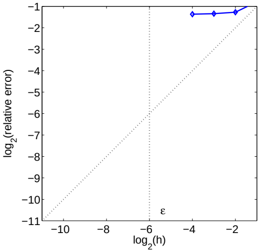

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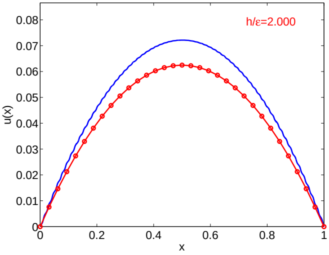

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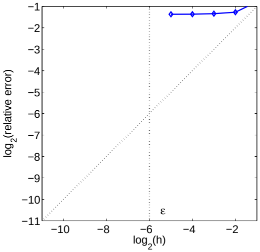

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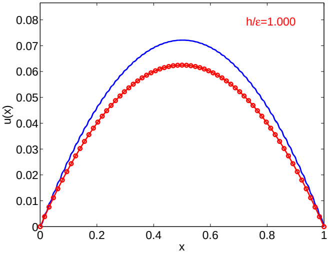

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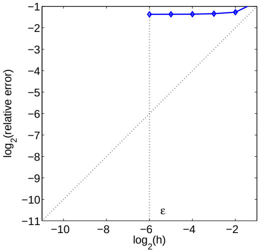

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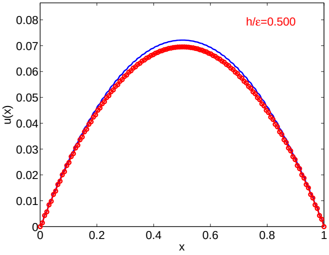

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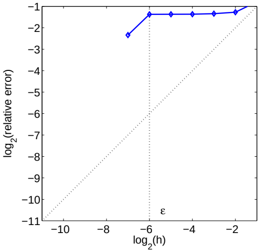

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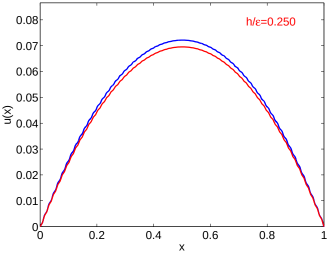

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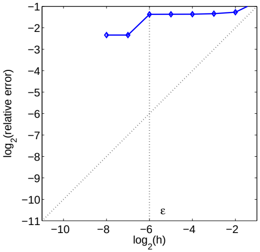

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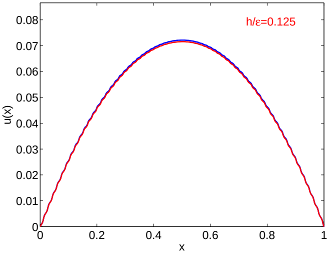

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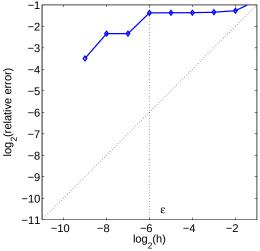

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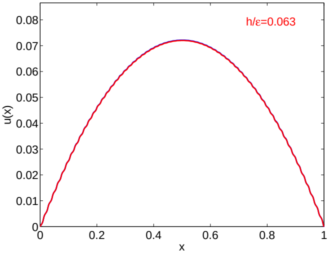

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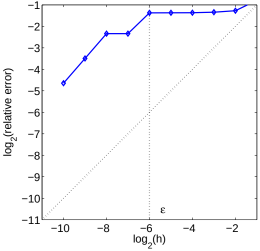

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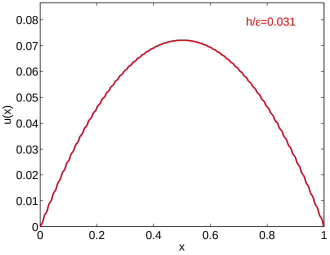

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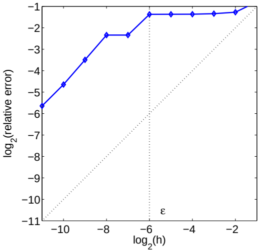

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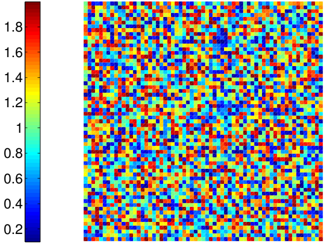
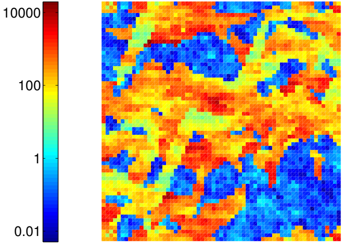
Examples (rough coefficients)



random material (academic)



porous medium (SPE10 benchmark)

Objectives

Without any assumptions on scales ...

- Construction of an upscaled variational problem based on a generalized FEM
  (coarse mesh $\mathcal{T}$ of size H & modified nodal basis functions)
- Computation of basis functions involves solution of PDE only on local patches of coarse elements with diameter $\approx \log(1/H)$
- Error estimate

$$|||u - u_H^{\text{ms}}||| := \|A^{1/2}\nabla(u - u_H^{\text{ms}})\| \leq C(f)H$$

with $C(f)$ independent of scales of A

A. Målqvist and D. Peterseim.
Localization of Elliptic Multiscale Problems.
ArXiv e-prints, Oct. 2011.

Some Known Methods

- Upscaling techniques: Durlofsky et al. 98, Nielsen et al. 98
- Variational multiscale method: Hughes et al. 95, Arbogast 04, Larson-Målqvist 05, Nolen et al. 08, Nordbotten 09
- Multiscale FEM: Hou-Wu 96, Efendiev-Ginting 04, Aarnes-Lie 06
- Residual free bubbles: Brezzi et al. 98
- Multiscale finite volume method: Jenny et al. 03
- Heterogeneous multiscale method: Engquist-E 03, E-Ming-Zhang 04
- Equation free: Kevrekidis et al. 05
- Metric based upscaling: Owhadi et al. 06
- ...

Common idea

Local approximations (in parallel) on a fine scale are used to modify a coarse scale space or equation

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Remark

Error analysis rely on strong assumptions such as scale separation and periodicity

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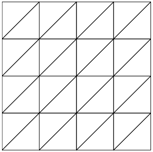
Multiscale Decomposition

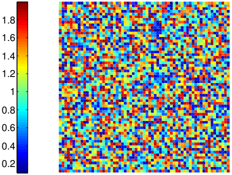
- (coarse) FE mesh $\mathcal{T}$ with parameter H
- P1-FE space $V_H := \{v \in V \mid \forall T \in \mathcal{T}, v|_T \in P_1(T)\}$
- $\mathfrak{I}_{\mathcal{T}} : V \to V_H$ quasi-interpolation operator



Decomposition

$$V = V_H \oplus V^{\mathrm{f}} \quad \text{with } V^{\mathrm{f}} := \text{kernel}\, \mathfrak{I}_{\mathcal{T}} = \{v \in V \mid \mathfrak{I}_{\mathcal{T}} v = 0\}$$

Example:



rough coefficient

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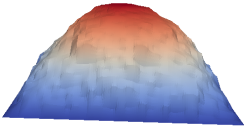
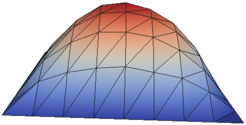


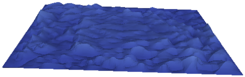
$u \in V$

$=$



$u_H \in V_H$

$+$



$u^{\mathrm{f}} \in V^{\mathrm{f}}$

Orthogonalization

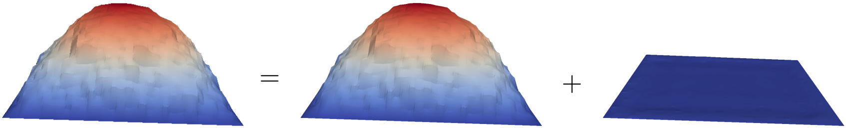
- For each $v \in V_H$ define finescale projection $\mathfrak{F}v \in V^{\mathrm{f}}$ by

$$a(\mathfrak{F}v, w) = a(v, w) \quad \text{for all } w \in V^{\mathrm{f}}$$

Orthogonal Decomposition

$$V = V_H^{\mathrm{ms}} \oplus V^{\mathrm{f}} \quad \text{with } V_H^{\mathrm{ms}} := (V_H - \mathfrak{F}V_H)$$

Example:



$u \in V \quad = \quad u_H^{\mathrm{ms}} \in V_H^{\mathrm{ms}} \quad + \quad u^{\mathrm{f}} \in V^{\mathrm{f}}$

Error Analysis (Perfect decomposition)

Lemma

$$|||u - u_H^{\mathrm{ms}}||| \leq C_{\mathrm{ol}} C_{\mathfrak{I}_\mathcal{T}} \alpha^{-1} ||Hf||_{L^2(\Omega)}$$

Sketch of proof:

- recall $||v - \mathfrak{I}_\mathcal{T} v||_{L^2(T)} \leq C_{\mathfrak{I}_\mathcal{T}} H ||\nabla v||_{L^2(\omega_T)}$ with $\omega_T := \cup\{K \in \mathcal{T} \mid T \cap K \neq \emptyset\}$ [Carstensen/Verfürth '99]
- orthogonal decomposition yields $u^{\mathrm{f}} := u - u_H^{\mathrm{ms}} \in V^{\mathrm{f}}$
- $\mathfrak{I}_\mathcal{T} u^{\mathrm{f}} = 0$, interpolation error estimate, and finite overlap of the patches ω_T conclude the proof

$$|||u^{\mathrm{f}}|||^2 = a(\underbrace{u^{\mathrm{f}} + u_H^{\mathrm{ms}}}_{=u}, u^{\mathrm{f}}) = F(u^{\mathrm{f}}) = F(u^{\mathrm{f}} - \mathfrak{I}_\mathcal{T} u^{\mathrm{f}})$$

$$\leq \sum_{T \in \mathcal{T}} ||f||_{L^2(T)} ||u^{\mathrm{f}} - \mathfrak{I}_\mathcal{T} u^{\mathrm{f}}||_{L^2(T)} \leq C_{\mathrm{ol}} C_{\mathfrak{I}_\mathcal{T}} \alpha^{-1} ||Hf||_{L^2(\Omega)} |||u^{\mathrm{f}}||| \qquad \square$$

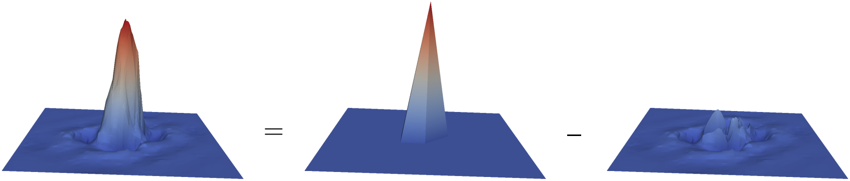
Modified Nodal Basis

- $\mathcal{N}$ denotes set of interior vertices of $\mathcal{T}$
- $\lambda_x \in V_H$ denotes classical nodal basis function ($x \in \mathcal{N}$)
- $\phi_x = \mathfrak{F}\lambda_x \in V^{\mathrm{f}}$ denotes finescale correction of λ_x ($x \in \mathcal{N}$)

Ideal multiscale FE space

$$V_H^{\mathrm{ms}} = \operatorname{span}\{\lambda_x - \phi_x \mid x \in \mathcal{N}\}$$

Example



$\lambda_x - \phi_x \in V_H^{\mathrm{ms}}$ = $\lambda_x \in V_H$ − $\phi_x \in V^{\mathrm{f}}$

Modified Nodal Basis

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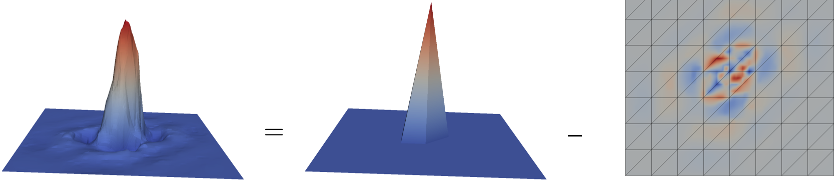
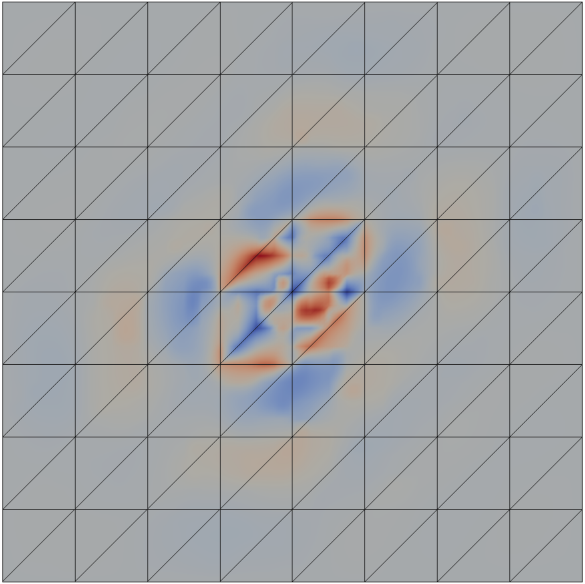


$\lambda_x - \phi_x \in V_H^{\mathrm{ms}}$ $=$ $\lambda_x \in V_H$ $-$ $\phi_x \in V^{\mathrm{f}}$

Modified Nodal Basis



Regularity of Nodal Basis

Lemma (H^2-Regularity)

Given a convex domain Ω and a diffusion coefficient $A \in C^1(\Omega)$ it holds $\lambda_x - \phi_x \in H^2(\Omega) \cap H^1_0(\Omega)$.

Sketch of proof:

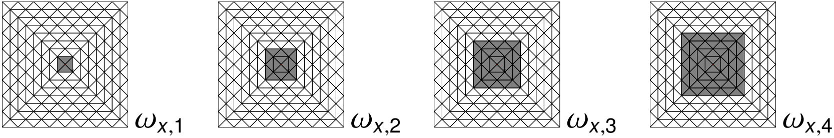
- $\phi_x \in H^1_0(\Omega)$ since $a(\phi_x, v) = a(\lambda_x, v)$ for all $v \in V^f \subset H^1_0(\Omega)$.
- consider the auxiliary problem, find $z \in H^1_0(\Omega)$ such that,

$$a(z, v) = a(\lambda_x - \phi_x, \mathfrak{I}_{\mathcal{T}} v) := l(v), \quad \text{for all } v \in H^1_0(\Omega). \tag{1}$$

  Since $l \in L^2(\Omega)$ we conclude $z \in H^2(\Omega) \cap H^1_0(\Omega)$.
- Since z is unique solution and $z = \lambda_x - \phi_x$ fulfill (1) the Lemma follows.

Localization

- Define nodal patches of k-th order $\omega_{x,k}$ about $x \in \mathcal{N}$



- Localized corrections $\phi_{x,k} \in V^{\mathrm{f}}(\omega_{x,k}) := \{v \in V^{\mathrm{f}} \mid v|_{\Omega \setminus \omega_{x,k}} = 0\}$ solve

$$a(\phi_{x,k}, w) = a(\lambda_x, w) \quad \text{for all } w \in V^{\mathrm{f}}(\omega_{x,k})$$

Localized multiscale FE spaces

$$V_{H,k}^{\mathrm{ms}} = \operatorname{span}\{\lambda_x - \phi_{x,k} \mid x \in \mathcal{N}\}$$

The Multiscale Method

Multiscale approximation seeks $u^{\text{ms}}_{H,k} \in V^{\text{ms}}_{H,k}$ such that

$$a(u^{\text{ms}}_{H,k}, v) = F(v) \quad \text{for all } v \in V^{\text{ms}}_{H,k}$$

Remarks:

- $\dim V^{\text{ms}}_{H,k} = |\mathcal{N}| = \dim V_H$
- basis functions of the multiscale method have local support and are totally independent
- overlap of the supports is proportional to the parameter k
- error analysis suggests $k \approx \log \frac{1}{H}$
- method can take advantage of periodicity

Error Analysis

Lemma (Truncation error)

There exist $C_1 < \infty$ and $\gamma < 1$ independent of x, k, H such that

$$|||\phi_x - \phi_{x,k}||| \le C_1 \gamma^k |||\phi_x|||.$$

Sketch of proof:

- By introducing a cut of function $\zeta_{x,\ell k} = 0$ in $\omega_{x,\ell(k-1)}$ and $\zeta_{x,\ell k} = 1$ in $\Omega \setminus \omega_{x,\ell k}$ we conclude: $|||\phi_x - \phi_{x,\ell k}||| \lesssim |||\phi_x - (1 - \zeta_{x,\ell k})\phi_x||| \lesssim \|\zeta_{x,\ell k}\|_{L^\infty(\Omega)} |||\phi_x|||_{\Omega\setminus\omega_{x,\ell(k-1)}} + \|\nabla\zeta_{x,\ell k}\|_{L^\infty(\Omega)} \|\phi_x - \mathfrak{I}_{\mathcal{T}}\phi_x\|_{L^2(\Omega\setminus\omega_{x,\ell(k-1)})} \lesssim |||\phi_x|||_{\Omega\setminus\omega_{x,\ell(k-1)}}$.
- $|||\phi_x|||^2_{\Omega\setminus\omega_{x,\ell(k-1)}} \le (A\zeta^2_{x,\ell(k-1)}\nabla\phi_x, \nabla\phi_x) = (A\nabla\phi_x, \nabla(\zeta^2_{x,\ell(k-1)}\phi_x)) - 2(A\zeta_{x,\ell(k-1)}\phi_x\nabla\zeta_{x,\ell(k-1)}, \nabla\phi_x) \lesssim \ell^{-1}\|H^{-1}\|_{L^\infty(\Omega)}\|\phi_x\|_{L^2(\Omega\setminus\omega_{x,\ell(k-2)})}|||\phi_x|||_{\Omega\setminus\omega_{x,\ell(k-2)}} \lesssim \ell^{-1}|||\phi_x|||^2_{\Omega\setminus\omega_{x,\ell(k-2)}}$.
- Repeat $|||\phi_x|||^2_{\Omega\setminus\omega_{x,\ell(k-1)}} \lesssim (C_2/\ell)^{k-1}|||\phi_x|||^2 := \gamma^{k-1}|||\phi_x|||^2$.

Error Analysis

Theorem (Main result)

$$|||u - u^{\text{ms}}_{H,k}||| \leq C_2 \left(k^d \|H^{-1}\|_{L^\infty(\Omega)} \gamma^k \|f\|_{L^2(\Omega)} + \|Hf\|_{L^2(\Omega)} \right)$$

holds with a constant C_2 that does not depend on H, k, f, or u.

Sketch of proof:

- Let $\tilde{u}^{\text{ms}}_{H,k} = \sum_{x \in \mathcal{N}} u^{\text{ms}}_H(x)(\lambda_x - \phi_{x,k})$ and note $|||u - u^{\text{ms}}_{H,k}|||^2 \leq |||u - \tilde{u}^{\text{ms}}_{H,k}|||^2$ since $u^{\text{ms}}_{H,k}$ is a projection.
- We split the error $u - \tilde{u}^{\text{ms}}_{H,k} = (u - u^{\text{ms}}_H) + (u^{\text{ms}}_H - \tilde{u}^{\text{ms}}_{H,k})$ and note $|||u - u^{\text{ms}}_H||| \lesssim \|Hf\|_{L^2(\Omega)}$ using previous Lemma.
- Finally $|||u^{\text{ms}}_H - \tilde{u}^{\text{ms}}_{H,k}|||^2 \leq \sum_{x \in \mathcal{N}} u^{\text{ms}}_H(x)^2 |||\phi_x - \phi_{x,k}|||^2 \lesssim \sum_{x \in \mathcal{N}} u^{\text{ms}}_H(x)^2 \gamma^{2k} |||\phi_x|||^2 \lesssim k^{2d} \|H^{-1}\|^2_{L^\infty(\Omega)} \gamma^{2k} \|f\|^2_{L^2(\Omega)}$.

Error Analysis

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$$|||u - u^{\mathrm{ms}}_{H,k}||| \leq C_2 \left(k^d \|H^{-1}\|_{L^\infty(\Omega)} \gamma^k \|f\|_{L^2(\Omega)} + \|Hf\|_{L^2(\Omega)} \right)$$

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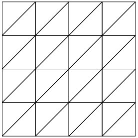
Theorem holds without any assumptions on scales or regularity!

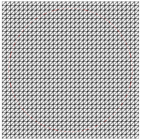
Outline

Full Discretization

- Finescale mesh



$\mathcal{T}$ mesh refinement $\rightsquigarrow$



$\mathcal{T}_h$ with $h \leq H$

- Reference FE space

$$V_h := \{v \in V \mid \forall T \in \mathcal{T}(\Omega), v|_T \in P_p(T)\}$$

- Reference FE solution $u_h \in V_h$ solves

$$a(u_h, v) = F(v) \quad \text{for all } v \in V_h$$

- Fully discrete corrections $\phi^h_{x,k} \in V^f_h(\omega_{x,k}) := V^f(\omega_{x,k}) \cap V_h$ satisfy

$$a(\phi^h_{x,k}, w) = a(\lambda_x, w) \;\; \text{for all } w \in V^f_h(\omega_{x,k})$$

Full Discretization

Fully discrete multiscale FE spaces

$$V_{H,k}^{\mathrm{ms},h} = \mathrm{span}\{\lambda_x - \phi_{x,k}^h \mid x \in \mathcal{N}\}$$

Fully discrete multiscale approximation $u_{H,k}^{\mathrm{ms},h} \in V_{H,k}^{\mathrm{ms},h}$ satisfies

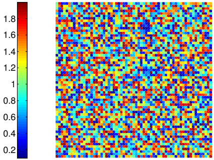
$$a(u_{H,k}^{\mathrm{ms},h}, v) = F(v) \quad \text{for all } v \in V_{H,k}^{\mathrm{ms},h}$$

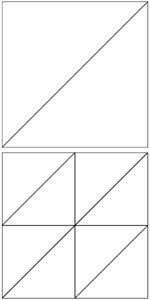
Theorem (Error estimate)

$$|||u - u_{H,k}^{\mathrm{ms},h}||| \leq C_3 \left(|||u - u_h||| + k^d \|H^{-1}\|_{L^\infty(\Omega)} \gamma^k \|f\|_{L^2(\Omega)} + \|Hf\|_{L^2(\Omega)} \right)$$

holds with a constant C_3 that does not depend on H, h, k, f, or u.

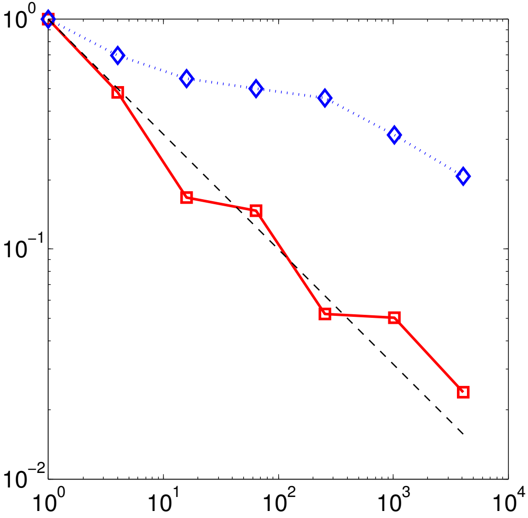
Numerical Experiment I





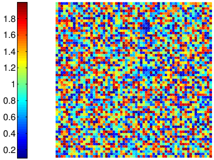
$H = 2^{-1}, 2^{-2}, \dots, 2^{-7}$

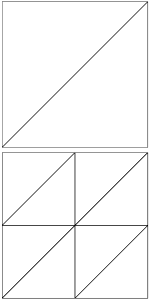
$h = 2^{-9}$, $k = \log(1/H)$



$|||u_h - u^{\mathrm{ms},h}_{H,k}|||$ vs. #dof

Numerical Experiment I





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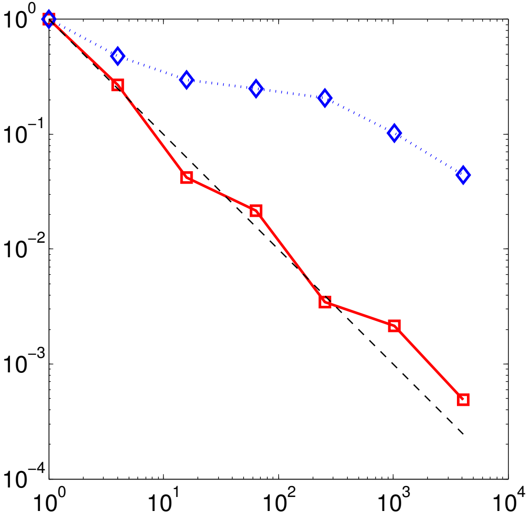
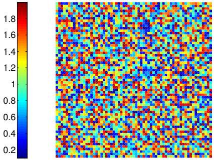
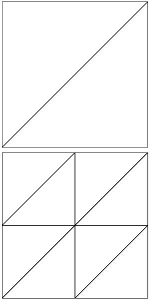


$\|u_h - u_{H,k}^{\mathrm{ms},h}\|$ vs. #dof

Numerical Experiment I





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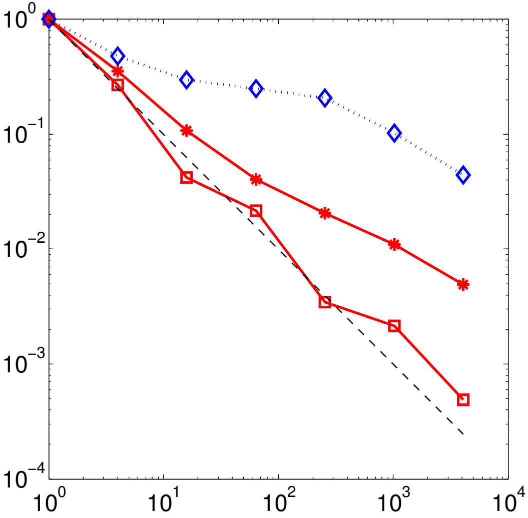
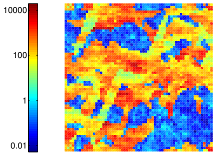
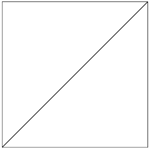
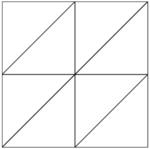


$\|u_h - \mathfrak{I}_{\mathcal{T}} u_{H,k}^{\text{ms},h}\|$ vs. #dof

Numerical Experiment II







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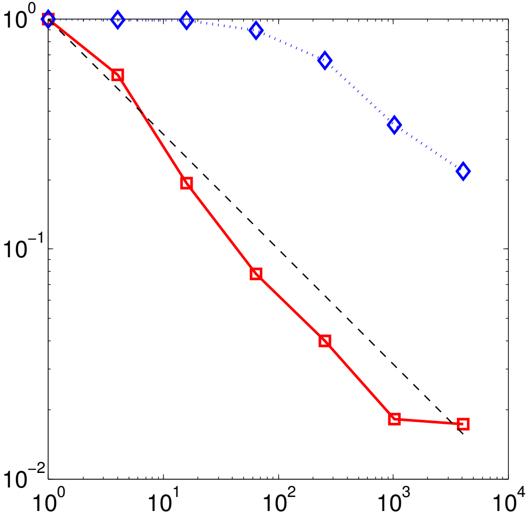
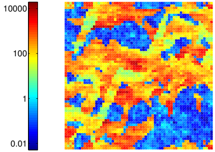
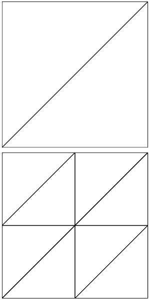


$|||u_h - u^{\mathrm{ms},h}_{H,k}|||$ vs. #dof

Numerical Experiment II





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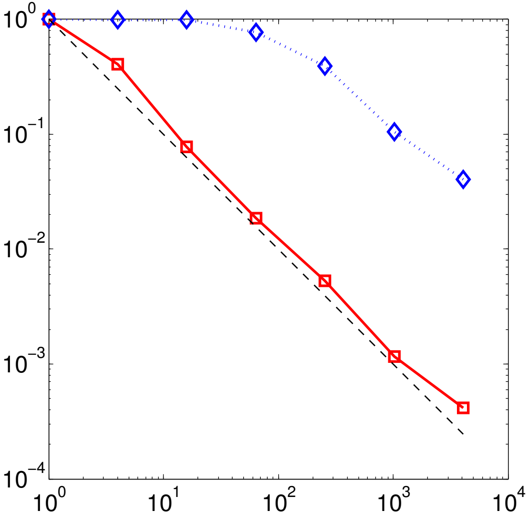
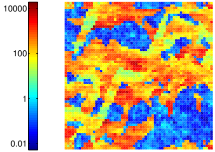
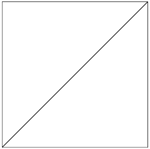
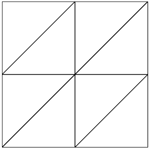


$\|u_h - u^{ms,h}_{H,k}\|$ vs. #dof

Numerical Experiment II







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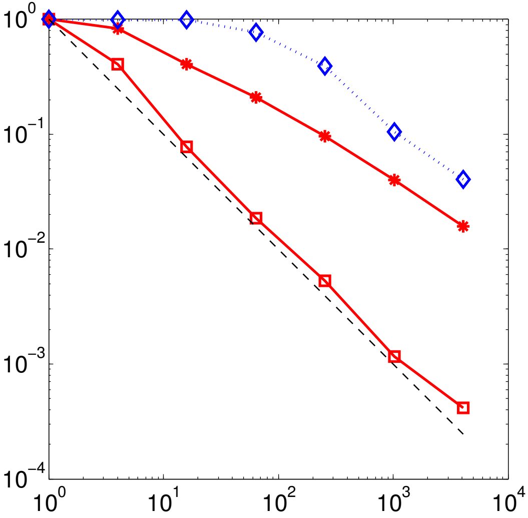

$\|u_h - \mathfrak{I}_{\mathcal{T}} u^{ms,h}_{H,k}\|$ vs. #dof

Outline

Applications

Oil reservoir simulation



Find pressure p and water concentration s such that:

$$-\nabla \cdot k\mu(s)\nabla p = q, \quad \dot{s} - \nabla \cdot [f(s)\mu(s)k\nabla p] = g, \quad \text{in } \Omega,$$

where k is permeability, $\mu(s)$ the total mobility, f fractional flow, and g, q sink and source terms.

Applications

$$-\nabla \cdot [k\mu(s)\nabla p] = q, \quad \dot{s} - \nabla \cdot [f(s)\mu(s)k\nabla p] = g, \quad \text{in } \Omega.$$

Advantages with Multiscale Approach

- An elliptic problem need to be solved in every time step with similar diffusion tensor since $A = k\mu(s)$, where k is independent of time and $\mu(s)$ only changes at the water front.
- In a Monte Carlo framework, where uncertainty in k is considered, only moments $a(\lambda_x - \phi_x, \lambda_y - \phi_y)$ need to be computed locally and passed up to the coarse scale, in order to solve the course scale equation.
- Individual local problems can be solved with different patch size and different resolution $\phi^h_{x,k}$ which makes adaptivity available.

Applications

$$-\nabla \cdot [k\mu(s)\nabla p] = q, \quad \dot{s} - \nabla \cdot [f(s)\mu(s)k\nabla p] = g, \quad \text{in } \Omega.$$

Remaining Challenges

- Multiscale approach for the (time dependent) transport equation
- Treatment of non-linearities.
- Efficient implementation to be able to attack real engineering problems (3D).

Outline

1. Setting and Motivation
2. Multiscale Method and Convergence
3. Full discretization and Numerical Experiments
4. Applications
5. **Conclusion**

Conclusion and Outlook

Conclusion

- A new variational multiscale FEM yields scale-independent textbook convergence and, hence, establishes reliable computational approximation of multiscale problems.
- Numerical experiments confirms the theoretical results. Furthermore numerical results are not sensitive to high contrast.

Outlook

- Design and error analysis of reliable multiscale methods for parabolic and hyperbolic problems.
- Treatment of high contrast also in the analysis and uncertainty.