Adaptive Variational Multiscale Methods

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Outline and Papers

Outline

- Model problem, elliptic linear pde
- Variational multiscale method, symmetric version
- Derivation of proposed method, examples
- A posteriori error estimation
- Adaptivity
- Application to oil reservoir problem
- Convection dominated problem
- Future work

Papers

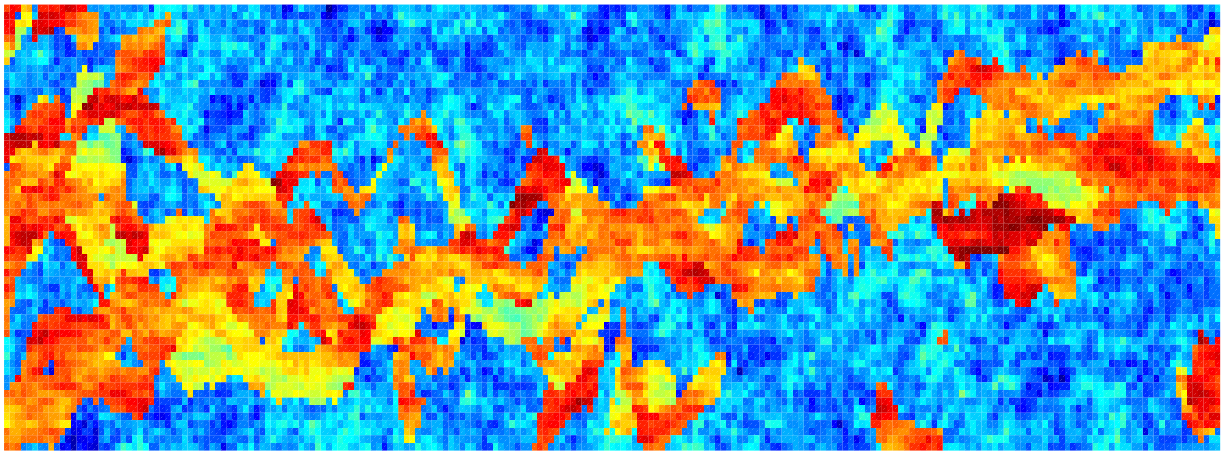
- M.G. Larson and A. Målqvist, *Adaptive Variational Multiscale Methods Based on A Posteriori Error Estimation: Energy Norm Estimates for Elliptic Problems*, CMAME 2007
- M.G. Larson and A. Målqvist, *A Mixed Adaptive Variational Multiscale Method with Applications in Oil Reservoir Simulation* M3AS 2009 (accepted)

Model Problem

Poisson equation: Find u such that

$$-\nabla \cdot a\nabla u = f \quad \text{in } \Omega,$$
$$a\partial_n u = 0 \quad \text{on } \partial\Omega,$$

where $a(x) \geq a_0 > 0$ bounded, $f \in L^2(\Omega)$ with $\int_\Omega f\,dx = 0$, and Ω polygonal domain.



Model problem

Weak form (standard): Find $u \in \mathcal{V}$ such that

$$a(u,v) = (a\nabla u, \nabla v) = (f,v) = l(v) \quad \text{for all } v \in \mathcal{V},$$

where $(v,w) = \int_\Omega vw\,dx$.

We can also formulate the problem on mixed form: let $-\nabla \cdot u_2 = f$ and $u_2 = a\nabla u_1$ to get,

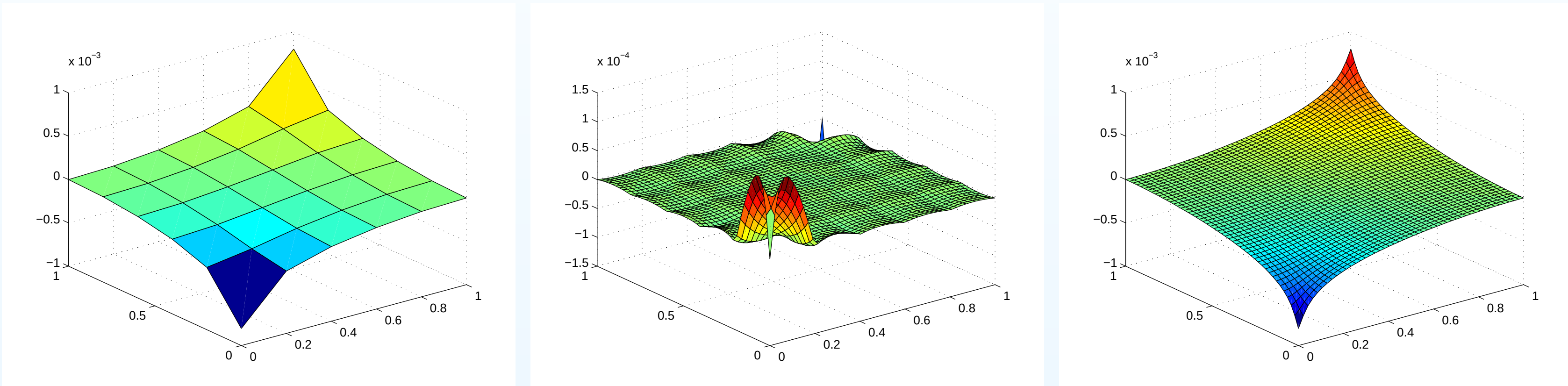
Weak form (mixed): Find $\{u_1, u_2\} \in \mathcal{V}$ such that

$$a(u,v) = (\frac{1}{a}u_2, v_2) + (u_1, \nabla \cdot v_2) - (\nabla \cdot u_2, v_1) = (f, v_1) = l(v),$$

for all $\{v_1, v_2\} \in \mathcal{V}$.

Variational Multiscale Method: Hughes et. al. $95, 98$

We split the space $\boxed{\mathcal{V}_c \oplus \mathcal{V}_f = \mathcal{V} \quad \big(= H^1(\Omega)/\mathbf{R} \big)}$



- $\mathcal{V}_c$ is a finite dimensional approximation of $\mathcal{V}$. (finite element space)
- $\mathcal{V}_f$ can be chosen in different ways
  - Hierarchical basis
  - $L^2(\Omega)$-orthogonal to $\mathcal{V}_c$
  - Wavelet modified hierarchical basis

Symmetric Variational Multiscale Method

Starting from the model problem: find $u \in \mathcal{V}$ such that

$$a(u, v) = l(v) \quad \text{for all } v \in \mathcal{V}$$

and setting

$$u = u_c + u_f \quad v = v_c + v_f$$

we get: find $u_c + u_f \in \mathcal{V}_c \oplus \mathcal{V}_f$ such that

$$\boxed{a(u_c + u_f, v_c + v_f) = l(v_c + v_f) \quad \text{for all } v_c + v_f \in \mathcal{V}_c \oplus \mathcal{V}_f}$$

Note that $u_f \in \mathcal{V}_f$ satisfies the equation

$$a(u_f, v_f) = l(v_f) - a(u_c, v_f) \quad \text{for all } v_f \in \mathcal{V}_f$$

Fine Scale Equations

Given the fine scale equation

$$a(u_f, v_f) = l(v_f) - a(u_c, v_f) \quad \text{for all } v_f \in \mathcal{V}_f$$

we let $u_f = u_{f,l} + u_{f,c} \in \mathcal{V}_f$ with

$$a(u_{f,l}, v_f) = l(v_f) \quad \text{for all } v_f \in \mathcal{V}_f$$
$$a(u_{f,c}, v_f) = -a(u_c, v_f) \quad \text{for all } v_f \in \mathcal{V}_f$$

Let $\mathcal{T} : \mathcal{V}_c \to \mathcal{V}_f$ denote the solution operator $u_{f,c} = \mathcal{T} u_c$. We get

$$u = u_c + \mathcal{T} u_c + u_{f,l}$$
$$a(u_c + \mathcal{T} u_c + u_{f,l}, v_c + v_f) = l(v_c + v_f)$$

for all $v_c \in \mathcal{V}_c$ and $v_f \in \mathcal{V}_f$.

Coarse Scale Equations

Since $u_{f,l}$ is directly determined we get the following problem for u_c: find $u_c \in \mathcal{V}_c$ such that

$$a(u_c + \mathcal{T} u_c, v_c + \mathcal{T} v_c) = l(v_c + \mathcal{T} v_c) - a(u_{f,l}, v_c + \mathcal{T} v_c)$$

for all $v_c \in \mathcal{V}_c$.

- Here we chose $v_f = \mathcal{T} v_c$ to get a symmetric formulation
- Note that $a((I + \mathcal{T})v_c, v_f) = 0$ and $l(v_f) - a(u_{f,l}, v_f) = 0$ i.e. $I + \mathcal{T}$ decouples the problem. Any choice of $v_f \in \mathcal{V}_f$ is ok.
- In standard VMS $v_f = 0$ in this step and thus when approximating the local effects using numerical or analytical tools the resulting method usually gives non-symmetric matrix.

Approximation of Fine Scale Solutions

- Let $\tilde{\mathcal{T}}$ be a computable approximation of $\mathcal{T}$
- Let $U_{f,l}$ be a computable approximation of $u_{f,l}$

We get the method: find $U_c \in \mathcal{V}_c$ such that

$$\boxed{a(U_c + \tilde{\mathcal{T}} U_c, v_c + \tilde{\mathcal{T}} v_c) = l(v_c + \tilde{\mathcal{T}} v_c) - a(U_{f,l}, v_c + \tilde{\mathcal{T}} v_c)}$$

for all $v_c \in \mathcal{V}_c$. On matrix form this leads to,

$$\boxed{KU_c = b}$$

Given U_c, $U_{f,l}$, and $\tilde{\mathcal{T}}$, U_f can be computed.

Compare with MsFEM (Hou et. al. 97) where basis functions are modified using local computations.

Construction of $\tilde{\mathcal{T}}$

Recall that $u_c = \sum_i u_{c,i} N_{c,i}$ with $\{N_{c,i}\}$ a basis in $\mathcal{V}_c$ and let

$$a(\mathcal{T} N_{c,i}, v_f) = -a(N_{c,i}, v_f) \quad \text{for all } v_f \in \mathcal{V}_f$$

By linearity

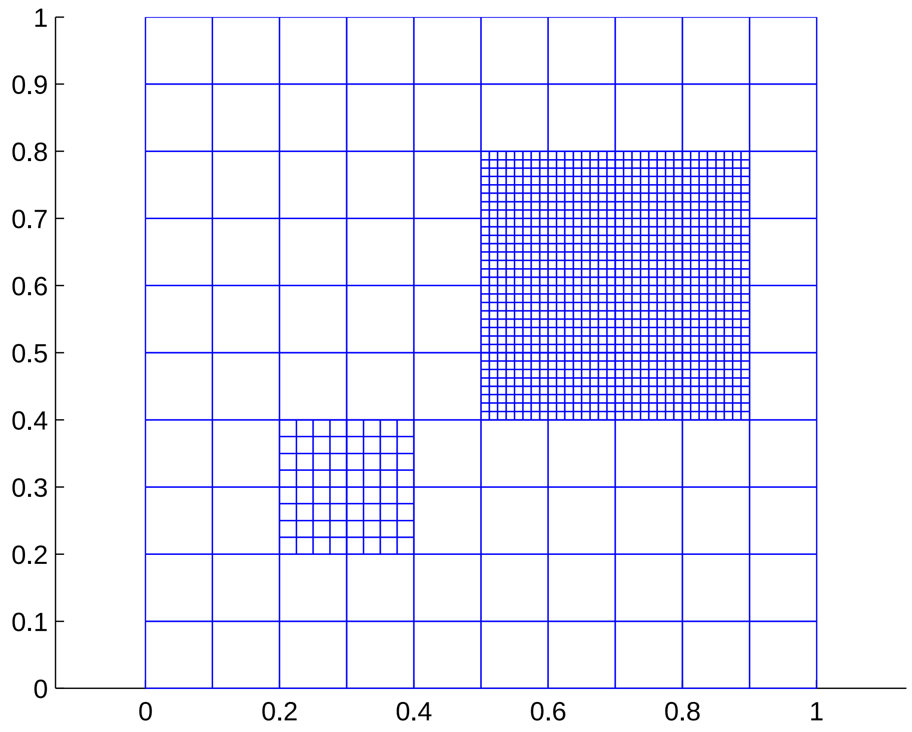
$$\mathcal{T} u_c = \sum_i u_{c,i} \mathcal{T} N_{c,i}$$

and thus we are led to computing $\mathcal{T} N_{c,i}$ for each coarse basis function $N_{c,i}$.

We define $\tilde{\mathcal{T}}$ by solving these problems approximately by

- Restricting to a localized patch problem $\text{supp}(N_{c,i}) \subset \omega_i$
- Discretizing using a fine subgrid on ω_i

Refinement and Layers



We let H be coarse scale mesh size and h be fine scale mesh size. Further we let L denote the number of layers of coarse elements in the patch. Typically homogeneous Dirichlet boundary condition are used.

Construction of $U_{f,l}$

Recall that $u_{f,l} \in \mathcal{V}_f$ solves

$$a(u_{f,l}, v_f) = l(v_f) \quad \text{for all } v_f \in \mathcal{V}_f$$

Using a partition of unity φ_i we can split the right hand side as follows $l(v_f) = \sum_i l(\varphi_i v_f)$ to get,

$$u_{f,l} = \sum_i u_{f,l,i}$$

$$a(u_{f,l,i}, v_f) = l(\varphi_i v_f)$$

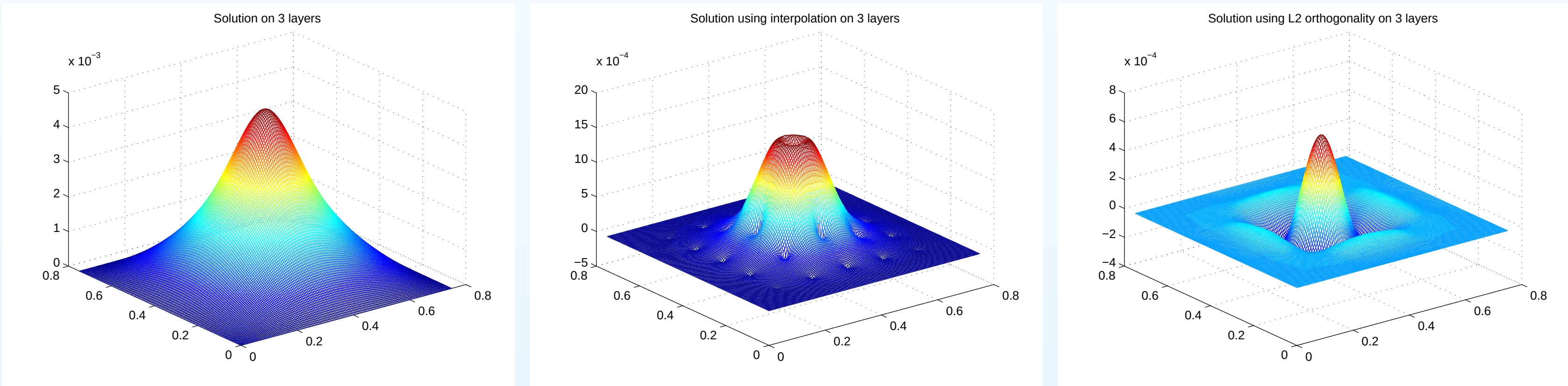
Again we find an approximation by restricting to patches and discretizing the subgrid.

Simple Observation About Decay in $\mathcal{V}_f$

Consider,

$$-\Delta u = \varphi_i \quad \text{in } \Omega, \qquad u = 0 \text{ on } \partial\Omega,$$

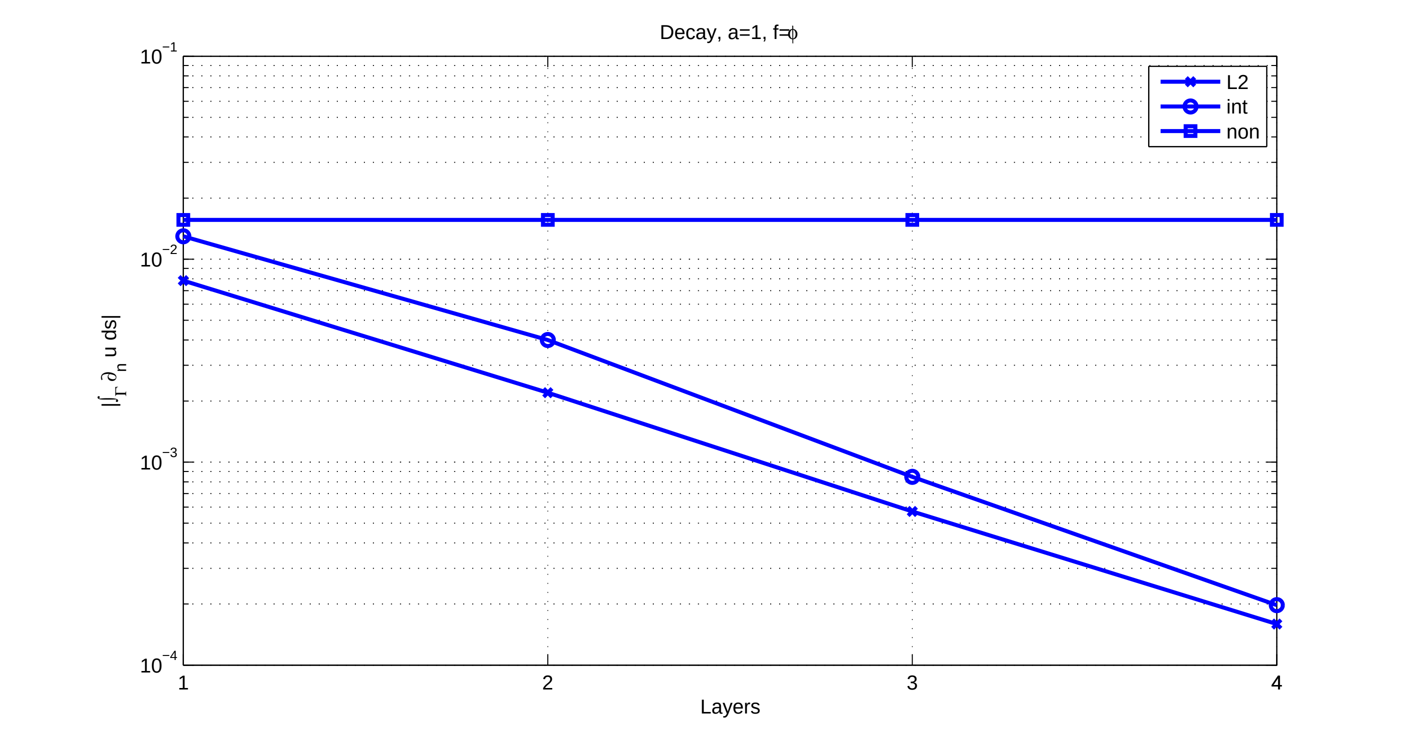
where φ_i has local support in center of Ω. The weak form reads: find $u \in \mathcal{W}$ s.t., $(\nabla u, \nabla v) = (\varphi_i, v)$ for all $v \in \mathcal{W}$.



To the left $\mathcal{W} = \mathcal{V}_c \oplus \mathcal{V}_f$, middle $\mathcal{W} = \mathcal{V}_f$ using hierarchical split, and right $\mathcal{W} = \mathcal{V}_f$ using L^2-orthogonal split.

Simple Observation About Decay in $\mathcal{V}_f$

Decay of flux integrated over the boundary.



We see exponential decay with respect distance measured in nof coarse elements. This effect gives rapid convergence as the patch size increases.

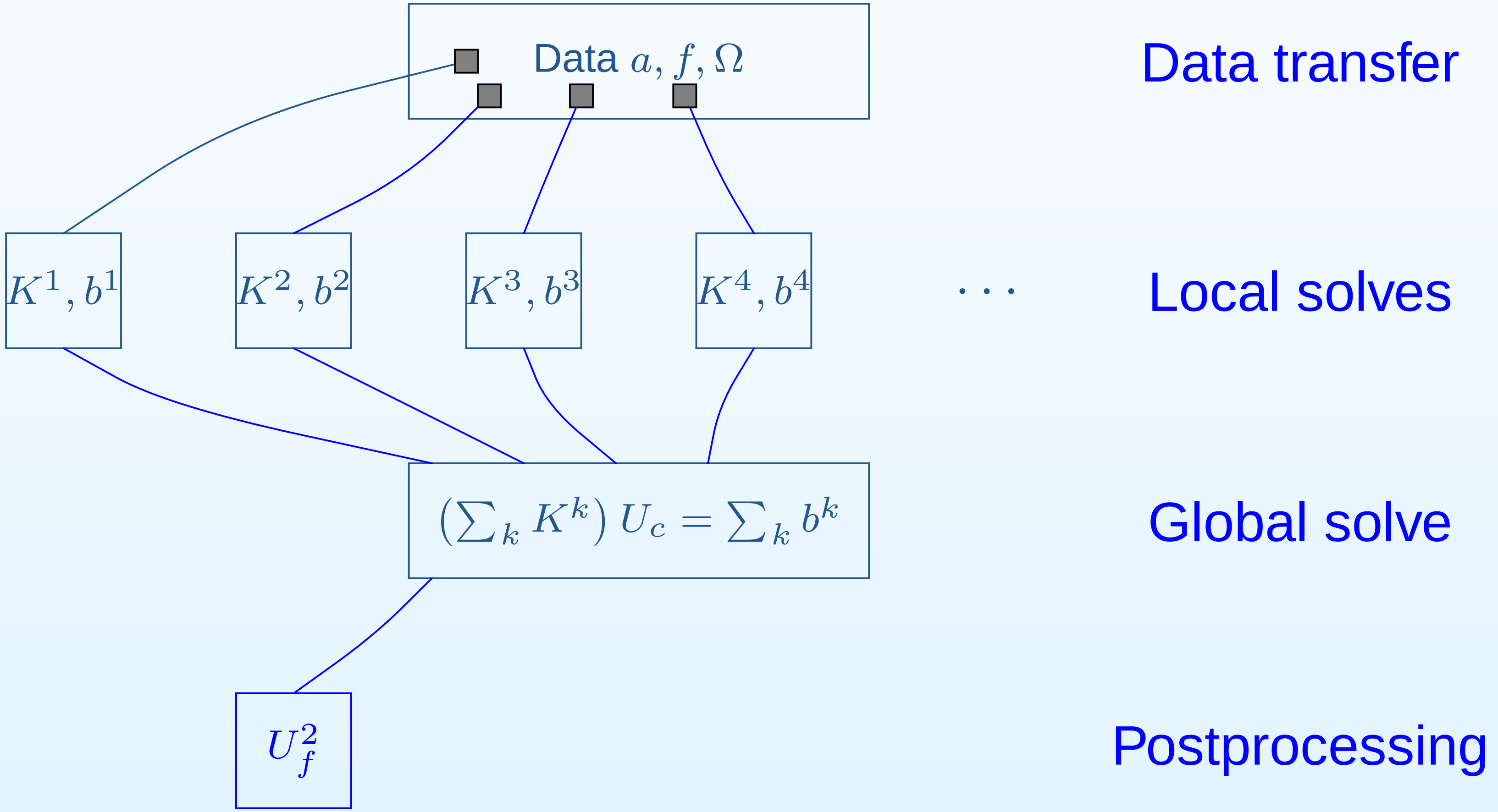
Summary of the Method so Far

- Fine scale equations are divided into a u_c dependent part and a u_c independent part
- The equations are decoupled
- We note rapid decay which allows us to restrict local solutions to patches
- We use local problems to modify coarse scale equation
- Fine scale features can be reconstructed given the coarse scale solution

We will show a posteriori error estimates and adaptive strategies later in the talk.

Parallel Structure

One local problem for each coarse dof, minimal communication.

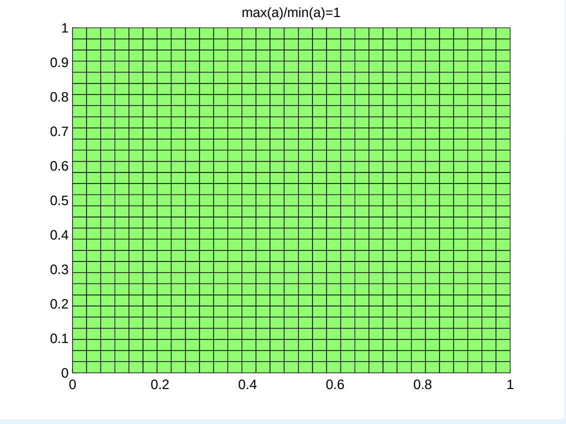


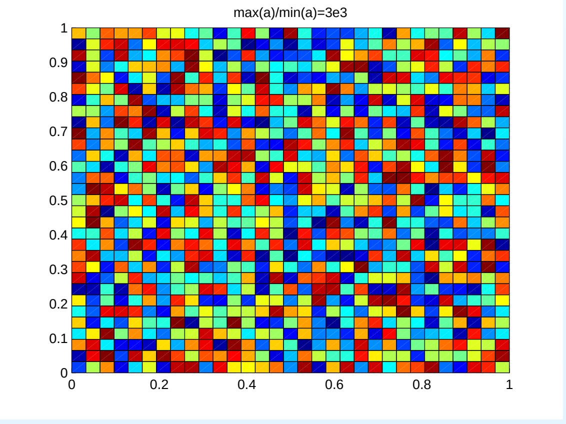
Application to a Mixed Problem

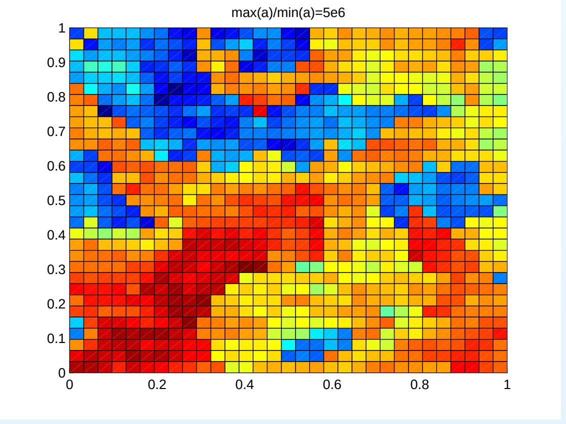
Poisson equation on mixed form:

$$\begin{cases} \frac{1}{a}\boldsymbol{\sigma} - \nabla u = 0 & \text{in } \Omega \\ -\nabla \cdot \boldsymbol{\sigma} = f & \text{in } \Omega \\ n \cdot \boldsymbol{\sigma} = 0 & \text{on } \Gamma \end{cases}$$

where the permeability a is constant, random, or taken from the SPE data set (upperness in log-scale),







Splitting Based on RT-elements

We use lowest order RT basis functions together with piecewise constants.

- Let π_c is the RT-interpolant onto $\mathcal{V}_c$ and P_c be the L^2-projection onto W_c
- We define $\boldsymbol{\sigma} = \pi_c \boldsymbol{\sigma} + (I - \pi_c)\boldsymbol{\sigma}$ and thus $\boldsymbol{\sigma}_f = (I - \pi_c)\boldsymbol{\sigma} \in \mathcal{V}_f \quad \boldsymbol{\sigma}_c = \pi_c \boldsymbol{\sigma} \in \mathcal{V}_c$.
- Further we define $u = P_c u_c + (1 - P_c)u = u_c + u_f \in \mathcal{W}_c \oplus \mathcal{W}_f$.
- Thus we are using an L^2-orthogonal splitting in the scalar variable.

Hierarchical split for lagrangian elements leads to nodal exactness in the coarse solution while here we get exactness of average values on coarse elements.

Some Terms Disappear

Find $\boldsymbol{\sigma}_c \in \mathcal{V}_c$, $\boldsymbol{\sigma}_f \in \mathcal{V}_f$, $u_c \in \mathcal{W}_c$, and $u_f \in \mathcal{W}_f$ such that,

$$\begin{cases} (\frac{1}{a}(\boldsymbol{\sigma}_c + \boldsymbol{\sigma}_f), \boldsymbol{v}_c + \boldsymbol{v}_f) + (u_c + u_f, \nabla \cdot (\boldsymbol{v}_c + \boldsymbol{v}_f)) = 0 \\ \qquad -(\nabla \cdot (\boldsymbol{\sigma}_c + \boldsymbol{\sigma}_f), w_c + w_f) = (f, w_c + w_f) \\ (\frac{1}{a}\boldsymbol{\sigma}_f, \boldsymbol{v}_f) + (u_f, \nabla \cdot \boldsymbol{v}_f) = -(\frac{1}{a}\boldsymbol{\sigma}_c, \boldsymbol{v}_f) - (u_c, \nabla \cdot \boldsymbol{v}_f) \\ \qquad -(\nabla \cdot \boldsymbol{\sigma}_f, w_f) = (f, w_f) + (\nabla \cdot \boldsymbol{\sigma}_c, w_f) \end{cases}$$

for all $\boldsymbol{v}_c \in \mathcal{V}_c$, $\boldsymbol{v}_f \in \mathcal{V}_f$, $w_c \in \mathcal{W}_c$, and $w_f \in \mathcal{W}_f$.

Since for coarse elements K

$$(w_f, \nabla \cdot \boldsymbol{v}_c) = \sum_K \nabla \cdot \boldsymbol{v}_c \int_K w_f \, dx = 0,$$

$$(w_c, \nabla \cdot \boldsymbol{v}_f) = \sum_K w_c \int_K \nabla \cdot \boldsymbol{v}_f \, dx = \sum_K w_c \int_{\partial K} \boldsymbol{n} \cdot \boldsymbol{v}_f \, ds = 0.$$

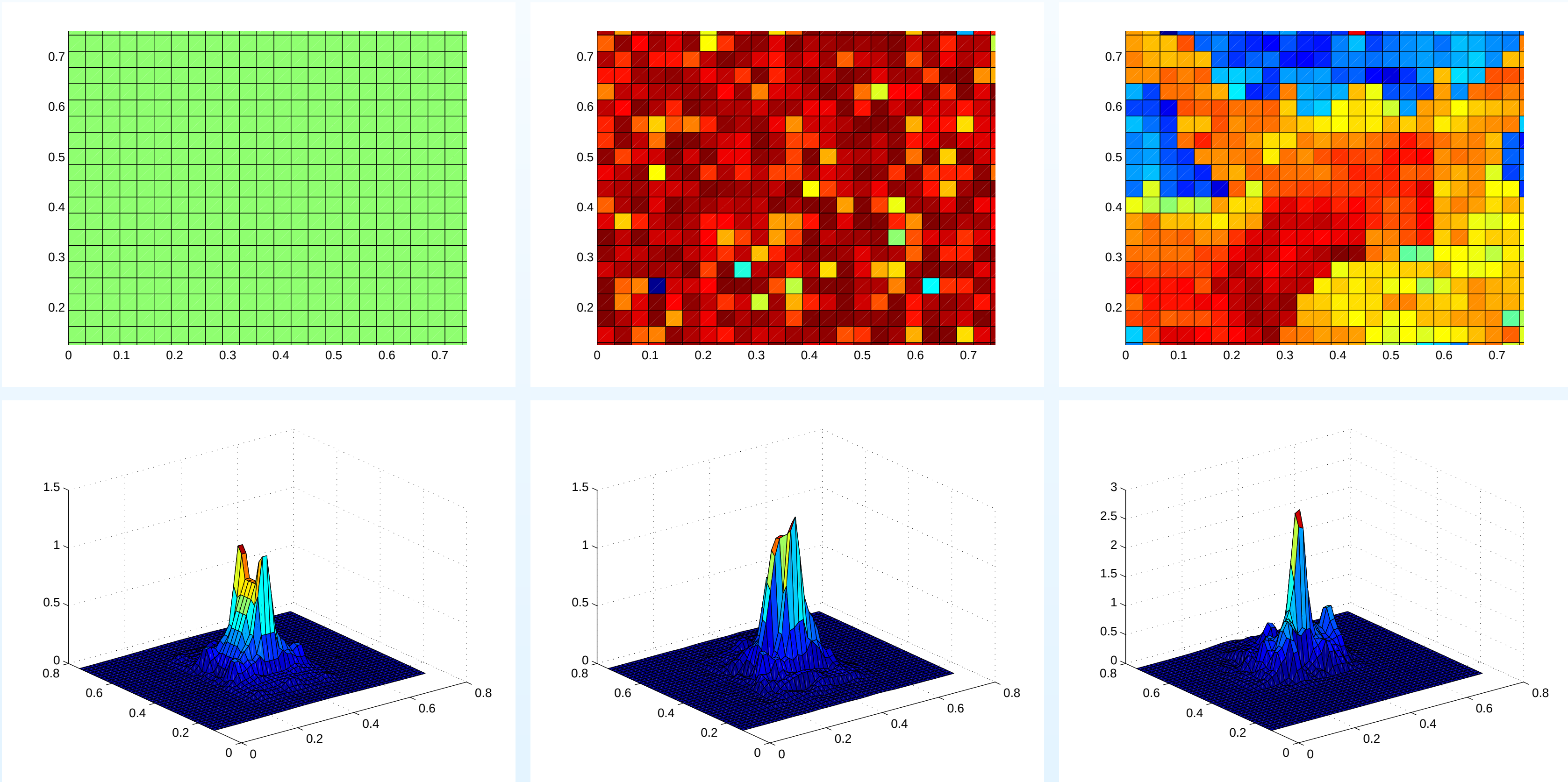
Approximate Fine Scales

$$\begin{cases} (\frac{1}{a}\boldsymbol{\sigma}_f, \boldsymbol{v}_f) + (u_f, \nabla \cdot \boldsymbol{v}_f) = -(\frac{1}{a}\boldsymbol{\sigma}_c, \boldsymbol{v}_f) \\ -(\nabla \cdot \boldsymbol{\sigma}_f, w_f) = (f, w_f) \end{cases}$$

- We apply the abstract framework
- Divide the fine scale problem into contributions from the coarse scale part $\boldsymbol{\sigma}_c$ and right hand side f
- Let $\boldsymbol{\sigma}_c = \sum_i \sigma_{c,i}\phi_i$ where ϕ_i are the Raviart-Thomas basis functions. Solve the local problem driven by the basis functions (one problem for each basis function)
- Localize by restricting the problem to a patch and using homogeneous Neumann conditions
- Discretize using a suitable subgrid

Example of Local Solutions $\boldsymbol{\xi}_i = \mathcal{T}\boldsymbol{\phi}_i$

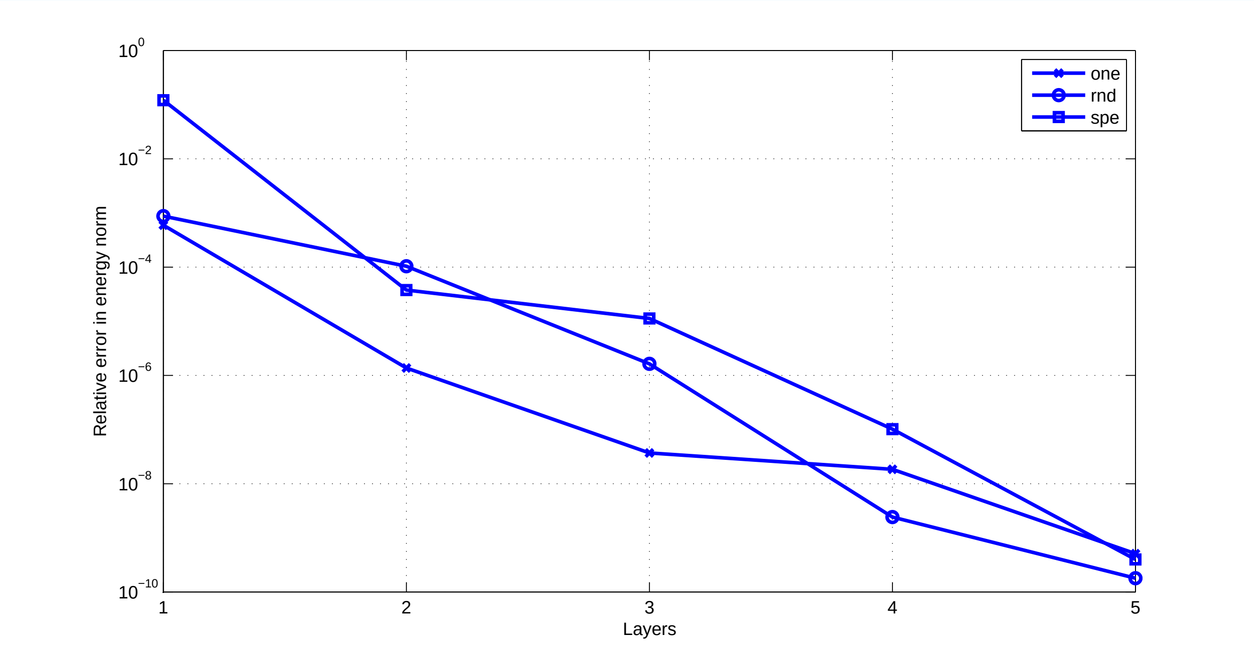
$$\begin{cases} (\frac{1}{a}\boldsymbol{\xi}_i, \boldsymbol{v}_f) + (\beta_i, \nabla \cdot \boldsymbol{v}_f) = -(\frac{1}{a}\boldsymbol{\phi}_i, \boldsymbol{v}_f) \\ -(\nabla \cdot \boldsymbol{\xi}_i, w_f) = 0. \end{cases}$$



We use 3 layer patches and plot absolute value of the flux $|\boldsymbol{\xi}_i|$.

Example of Convergence

- Reference mesh has 32×32 elements
- The coarse mesh has 8×8 elements.
- We let $f = 1$ lower left corner and $f = -1$ in upper right, otherwise $f = 0$.



Error compared to reference solution.

Adaptive VMS

The Adaptive Variational Multiscale Method (AVMS) builds on the following ingredients:

- Error estimation framework
- Adaptive strategy for tuning of critical discretization parameters

The method is designed so that:

$$\boxed{\text{error} \rightarrow 0 \text{ when } h \rightarrow 0 \text{ and } L \rightarrow \infty}$$

- A priori error estimates in progress.
- To circumvent difficulties with choosing discretization parameters h and L we use an adaptive algorithm based on a posteriori error estimates

A Posteriori Error Estimate (standard version, Dirichlet)

The following energy norm bound holds

$$\|u - U\|_a^2 = \|\sqrt{a}\nabla(u - U)\|^2 \leq C_a \sum_i \left(R_{\omega_i}^2 + R_{\partial\omega_i}^2\right)$$

where

$$R_{\omega_i}^2 = \|h(f\phi_i + \nabla \cdot (a\nabla(U_{c,i}\phi_i + U_{f,i})))\|_{\omega_i}^2 + \sum_{K \in \omega_i} \|h^{1/2}[a\partial_n U_{f,i}]\|_{\partial K \setminus \partial\omega_i}^2$$

$$R_{\partial\omega_i}^2 = \|h^{1/2} a\partial_n U_{f,i}\|_{\partial\omega_i \setminus \Gamma}^2,$$

where $U_{f,i} = U_{c,i}\tilde{\mathcal{T}}\phi_i + U_{f,l,i}$.

Similar linear functional estimates have also been derived using a dual problem.

A Posteriori Error Estimate (mixed version, Neumann)

The following energy norm bound holds

$$\|\boldsymbol{\sigma} - \boldsymbol{\Sigma}\|_a^2 = \|\frac{1}{\sqrt{a}}(\boldsymbol{\sigma} - \boldsymbol{\Sigma})\|^2 \leq C_a \sum_i \left(R_{\omega_i}^2 + R_{\partial\omega_i}^2\right)$$

where

$$R_{\omega_i}^2 = \|\frac{1}{a}(\Sigma_c^i \boldsymbol{\phi}_i + \boldsymbol{\Sigma}_{f,i}) - \nabla U_{f,i}^*\|_{\omega_i}^2 + \|\frac{h}{a}(f\psi_i + \nabla \cdot (\Sigma_c^i \boldsymbol{\phi}_i + \boldsymbol{\Sigma}_{f,i}))\|_{\omega_i}^2$$
$$+ \sum_{K \in \omega_i} \|h^{-1/2}[U_{f,i}^*]\|_{\partial K}^2$$

$$R_{\partial\omega_i}^2 = \|h^{-1/2} U_{f,i}^*\|_{\partial\omega_i \setminus \Gamma}^2$$

U^* is a post processed version (Lovadina and Stenberg 06) of U, $C_a \sim \|\sqrt{a}\|_{L^\infty(\omega_i)}$.

Adaptive Strategy

We have the error bound

$$\|\boldsymbol{\sigma} - \boldsymbol{\Sigma}\|_a^2 \leq C_a \sum_i \left(R_{\omega_i}^2 + R_{\partial\omega_i}^2\right)$$

1. Let $h = H/2$ and $L = 1$ for all i.
2. Compute the solution $\{U, \boldsymbol{\Sigma}\}$.
3. Calculate residuals for each coarse RT basis functions.
4. Mark large entries.
5. For marked entries $R_{\omega_i}^2$ let $h := h/2$.
6. For marked entries $R_{\partial\omega_i}^2$ let $L := L + 1$.
7. Return to 1 or stop if estimators are small enough.

Application in Oil Reservoir Simulation

We seek the water saturation s (oil is $1-s$) that solves the system of a pressure and a transport equation,

$$\boxed{\begin{aligned} \frac{1}{a\lambda(s)}\boldsymbol{\sigma} - \nabla u &= 0 \quad \text{in } \Omega, \\ -\nabla \cdot \boldsymbol{\sigma} &= q \quad \text{in } \Omega, \\ \boldsymbol{n} \cdot \boldsymbol{\sigma} &= 0 \quad \text{on } \Gamma, \end{aligned}}$$

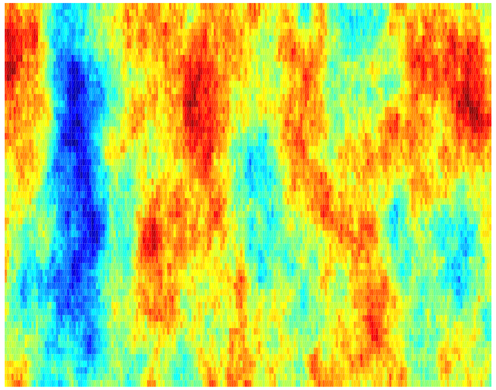
$$\dot{s} + \boldsymbol{\sigma} \cdot \nabla f(s) = 0,$$

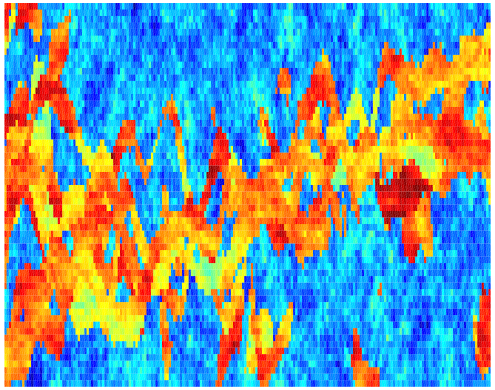
$f(s)$ referred to as fractional flow function, $\lambda(s)$ is total mobility, and q is a source term.

This is a simple model of two phase flow. Note the two way coupling, $\lambda(s)$ is one except at the water front.

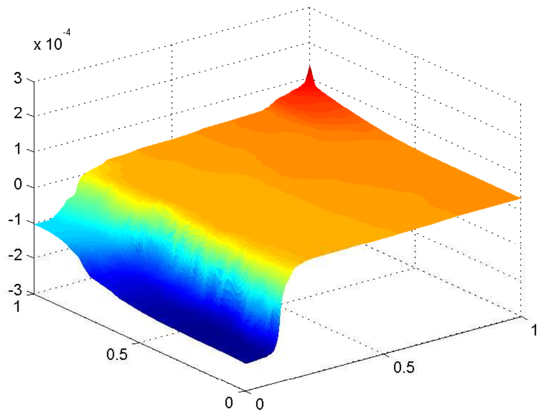
Application in Oil Reservoir Simulation

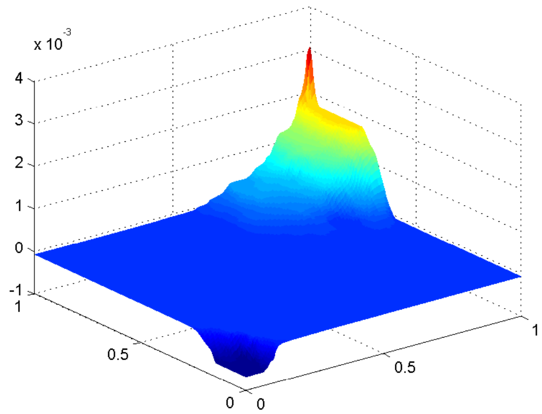
Layer 1 and 50 in the SPE comparative sol. proj. (log scale).





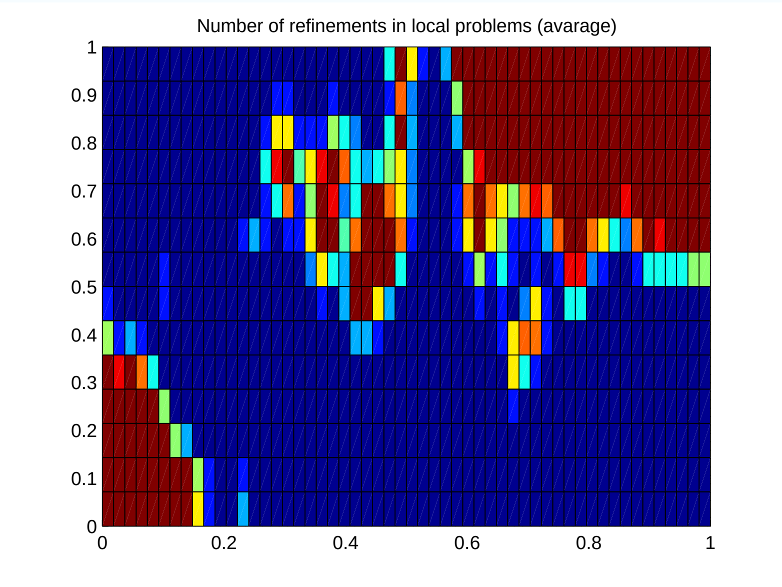
Plot of the sol. (pressure), $q = 1$ upper right $q = -1$ lower left.

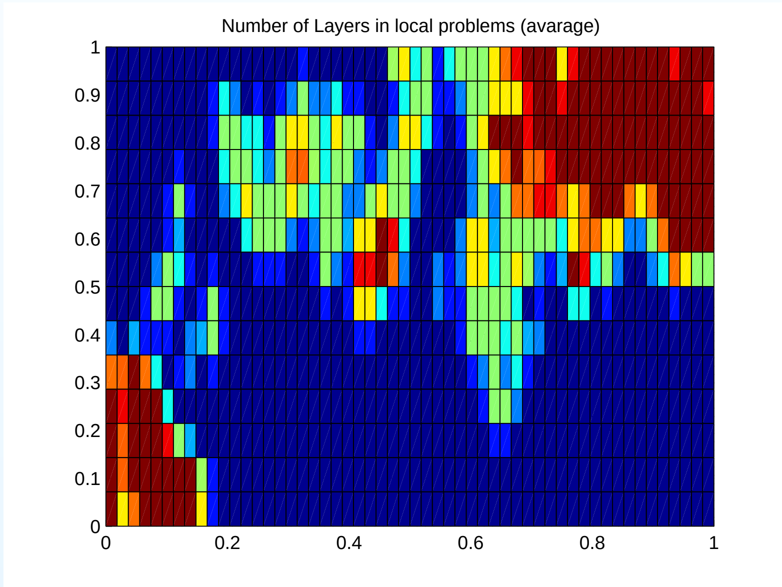




Refinements and layers SPE50

We use 55×15 coarse elements and a reference mesh with 440×120 elements.

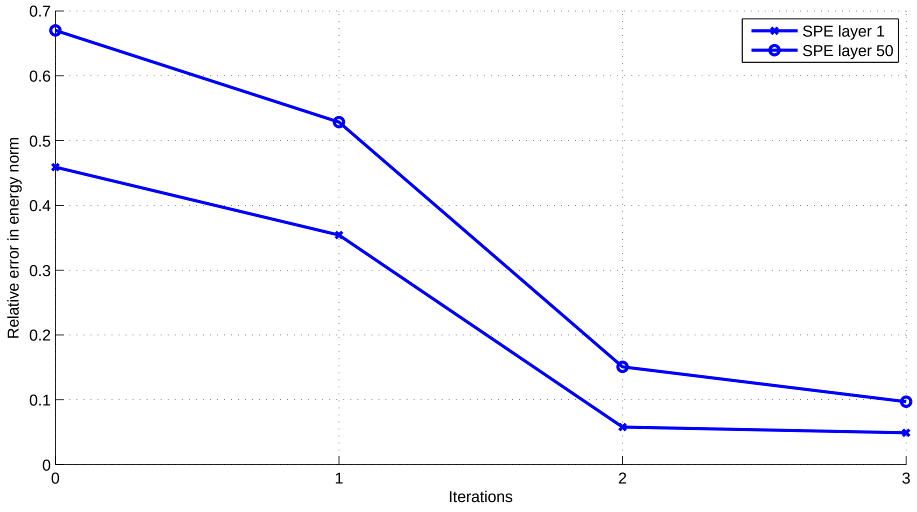




We start the adaptive algorithm with one refinement and one layer in all local problems. After three iterations in the algorithm marking 30%.

Convergence of Adaptive Algorithm

We compare error in energy norm with reference solution.

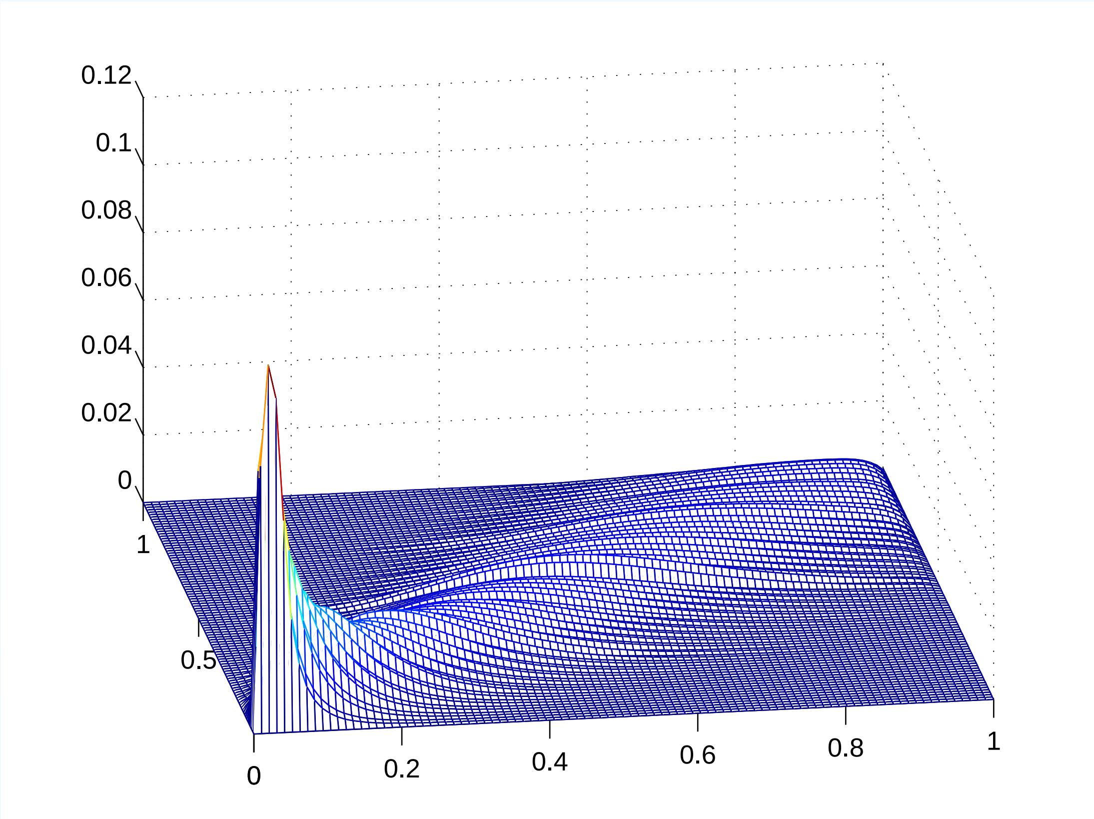


- Critical areas are found
- A majority of the patches uses one layer and one refinement.
- As the water front travels only local problems at the front need to be recomputed.

Convection Dominated Problem

$$-\epsilon \triangle u + \nabla \cdot (bu) = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

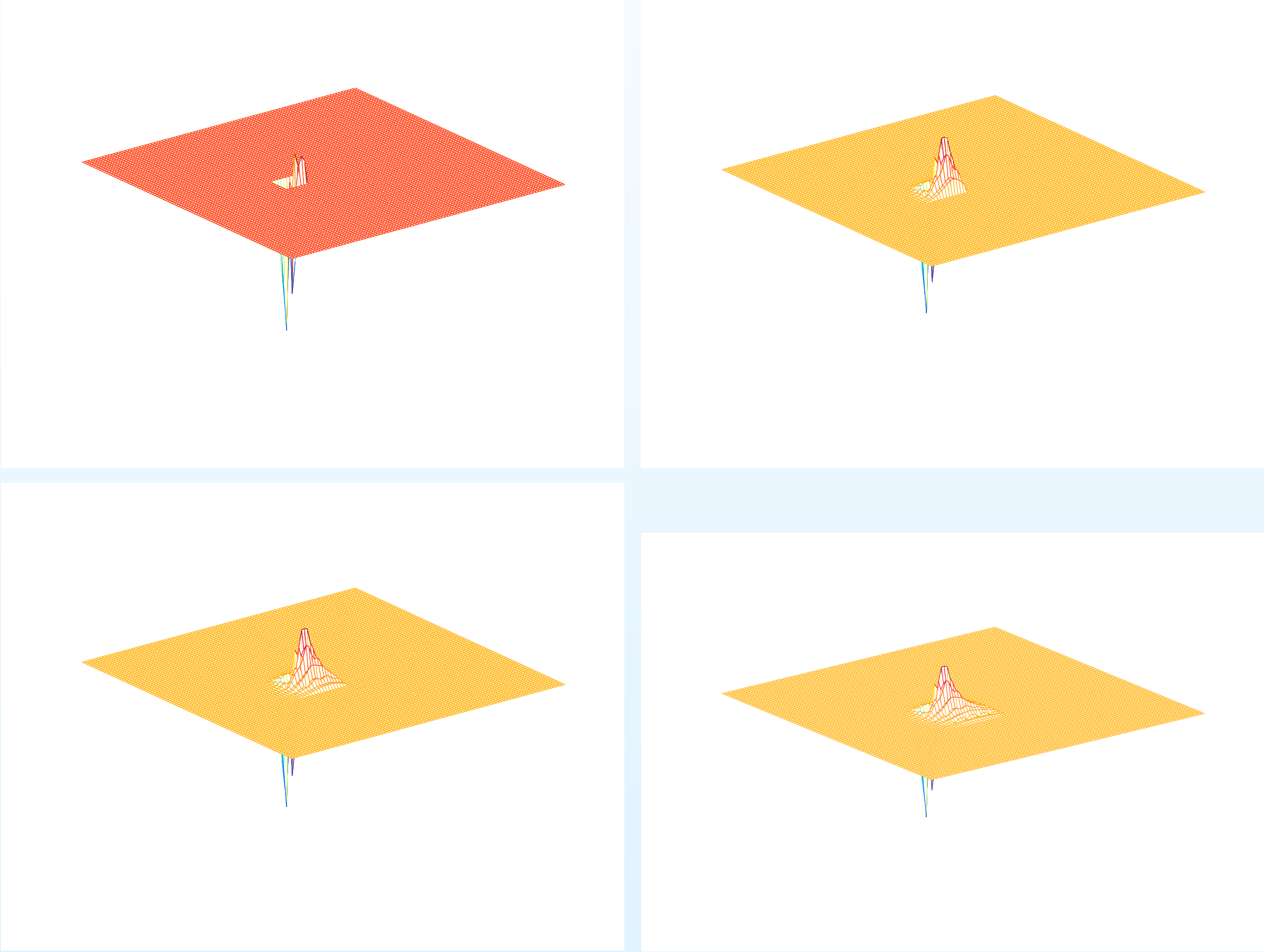
where $\epsilon = 0.01$, $f = 1$ lower left corner, $b = [b_x, b_x]$, b_x oscillates between 0.01 and 1.



Solutions to Local Problems $U_{f,i}$

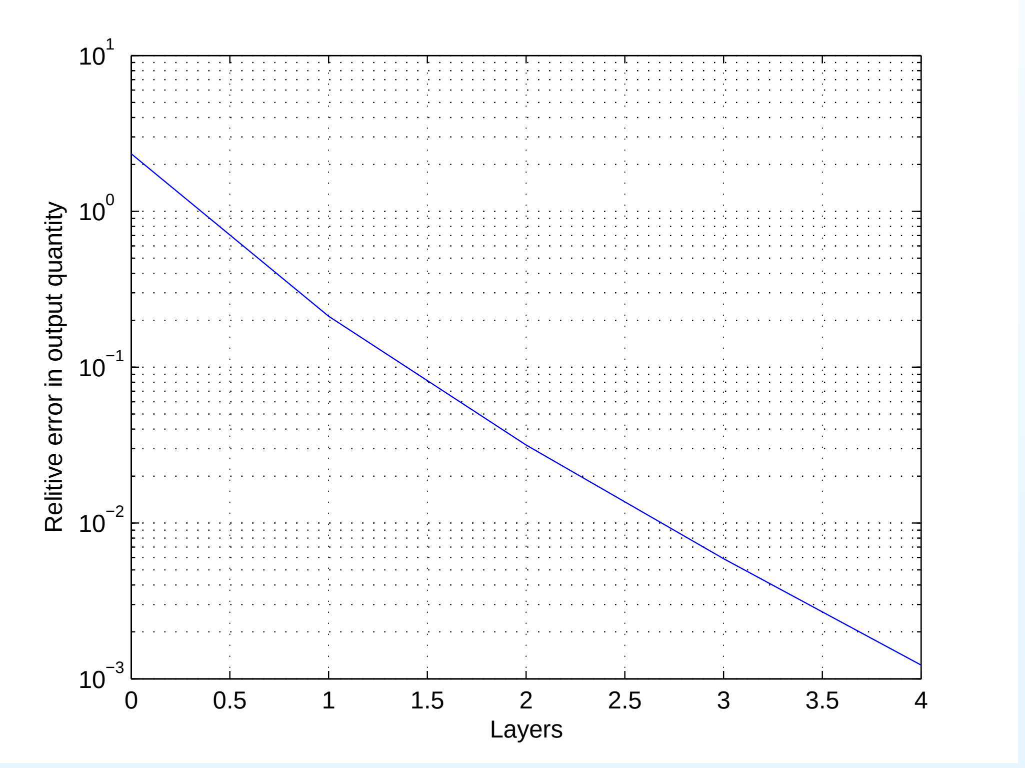
$$(\epsilon\nabla U_{f,i}, \nabla v)+(\nabla\cdot(bU_{f,i}), v) = (f\phi_i, v)-(\epsilon\nabla(U_{c,i}\phi_i), \nabla v)-(\nabla\cdot(bU_{c,i}\phi_i), v)$$

for all $v \in \mathcal{V}_f|_{\omega_i}$. We use hierarchical split.



Error in Multiscale Solution

Let $H = 1/24$, $h = H/4$ and study relative error $(U - U_{\text{ref}}, 1)/(U_{\text{ref}}, 1)$ compared to reference solution.



We observe a very clear exponential decay. Note that the error using standard Galerkin on the coarse mesh is very high.

Summary

The AVMS provides:

- Systematic technique for construction of a computable approximation of the fine scale part of the solution using decoupled localized subgrid problems.
- A posteriori error estimation framework (also for goal functionals)
- Adaptive algorithms for automatic tuning of critical discretization parameters
- Its applicable to a range of equations (only linear at this point)

The decay in $\mathcal{V}_f$ together with the adaptive strategy makes the method efficient.

Future Work

- Scale up numerics, parallel code, 3D. (PhD student)
- A priori error analysis, capture decay.
- More then two scales.
- Use Discontinuous Galerkin with L^2 orthogonal split between the scales.
- Multiscale approach to the coupled transport-pressure equation. (Time dependent problems Nordbotten 09)
- Tests on more realistic data, compare with other methods.