Numerical simulation of beam network models

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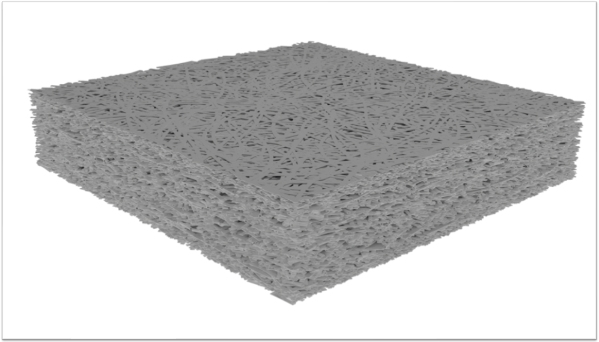
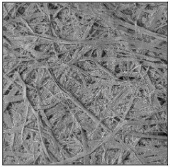
Motivation: Simulation of paperboard

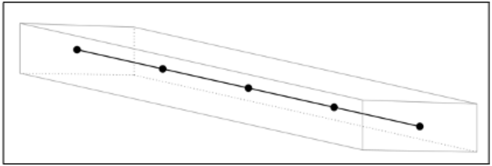


- Collaboration with Fraunhofer Chalmers Centre (FCC)[1] and packaging company Tetra Pak
- Simulation of mechanical properties (tensile/bending strength)

[1] Görtz, Numerical homogenization of network models and micro-mechanical simulation of paperboard, PhD thesis, 2024

Motivation: Simulation of paperboard



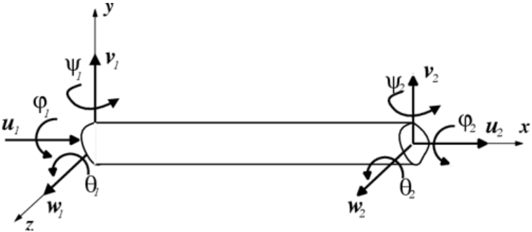


- Wood fibres modelled by hollow flattened slender cylinders
- Simplification: Timoshenko beams with rigid joints
- The displacement solves a linear system of equations $Au = F$
- A is SPD, sparse but large and ill-conditioned
- Direct methods are used (FCC)

Main goal: derive and analyze an efficient iterative method

Outline

The Timoshenko[2] beam model



- 1D model of the elastic deformation of a 3D beam
- Assumption: the cross sections remains plain after deformation
- Takes shear deformation into account
- Six degrees of freedom (centreline displacement and cross-section rotation)

[2] Timoshenko, On the correction for shear of the differential equation for transverse vibrations of prismatic bars, London Edinburgh Philos. Mag. and J. Sci., 1921

Governing equation[3] (single beam)

$$
\begin{aligned}
-C_{\boldsymbol{n}}(\partial_x \boldsymbol{u}_{\mathfrak{e}} + \boldsymbol{i}_{\mathfrak{e}} \times \boldsymbol{r}_{\mathfrak{e}}) &= \boldsymbol{n}_{\mathfrak{e}} & \qquad -C_{\boldsymbol{m}}\partial_x \boldsymbol{r}_{\mathfrak{e}} &= \boldsymbol{m}_{\mathfrak{e}} \\
\partial_x \boldsymbol{n}_{\mathfrak{e}} &= \boldsymbol{f}_{\mathfrak{e}} & \qquad \partial_x \boldsymbol{m}_{\mathfrak{e}} + \boldsymbol{i}_{\mathfrak{e}} \times \boldsymbol{n}_{\mathfrak{e}} &= \boldsymbol{g}_{\mathfrak{e}}
\end{aligned}
$$

- Unit vector in direction of $\mathfrak{e}$, $\boldsymbol{i}_{\mathfrak{e}} : \mathfrak{e} \to \mathbb{R}^3$
- Centre line displacement, $\boldsymbol{u}_{\mathfrak{e}} : \mathfrak{e} \to \mathbb{R}^3$
- Cross-section rotation, $\boldsymbol{r}_{\mathfrak{e}} : \mathfrak{e} \to \mathbb{R}^3$
- Stress from normal and shear forces: $\boldsymbol{n}_{\mathfrak{e}} : \mathfrak{e} \to \mathbb{R}^3$
- Moment from torsion and bending, $\boldsymbol{m}_{\mathfrak{e}} : \mathfrak{e} \to \mathbb{R}^3$
- Material parameter, $C_{\boldsymbol{n}}$, $C_{\boldsymbol{m}}$ symmetric $\mathbb{R}^3 \times \mathbb{R}^3$ depending on Young's modulus, Shear modulus, and cross-section.
- Distributed force $\boldsymbol{f}_{\mathfrak{e}} : \mathfrak{e} \to \mathbb{R}^3$ and moment $\boldsymbol{g}_{\mathfrak{e}} : \mathfrak{e} \to \mathbb{R}^3$

[3] Carrera et. al., Beam Structures, Wiley 2011

Continuity and balance conditions[4]

The network is represented by a graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$.

1. *Continuity of solution:* $\boldsymbol{u}_{\mathfrak{e}}(\mathfrak{n}) = \boldsymbol{u}_{\mathfrak{n}}$ and $\boldsymbol{r}_{\mathfrak{e}}(\mathfrak{n}) = \boldsymbol{r}_{\mathfrak{n}}$
2. *Dirichlet boundary nodes:* $\boldsymbol{u}_{\mathfrak{n}} = \boldsymbol{u}^D_{\mathfrak{n}}$ and $\boldsymbol{r}_{\mathfrak{n}} = \boldsymbol{r}^D_{\mathfrak{n}}$, $\mathfrak{n} \in \mathcal{N}_D$
3. *Balance equations:* Let $[\![\cdot]\!]_{\mathfrak{n}}$ be a summation at $\mathfrak{n}$ and $\nu_{\mathfrak{e}} = \pm 1$:

$$[\![\boldsymbol{n}_{\mathfrak{e}} \nu_{\mathfrak{e}}]\!]_{\mathfrak{n}} = \boldsymbol{f}_{\mathfrak{n}} \qquad [\![\boldsymbol{m}_{\mathfrak{e}} \nu_{\mathfrak{e}}]\!]_{\mathfrak{n}} = \boldsymbol{g}_{\mathfrak{n}}$$

[4] Lagnese et. at. Modeling, analysis and control of dynamic elastic multi-link structures, Birkhäuser Boston, 1994

Outline

HDG[5,6] discretization

Aim: reduce dofs in global solve to dofs at joints

- *Primal variables:* $\bar{\boldsymbol{u}}_{\mathfrak{e}}, \bar{\boldsymbol{r}}_{\mathfrak{e}} \in V_{h,p}^{\mathfrak{e}} := (\mathbb{P}_{h,p}(\mathfrak{e}))^3$ for all edges $\mathfrak{e} \in \mathcal{E}$
- *Dual variables:* $\bar{\boldsymbol{n}}_{\mathfrak{e}}, \bar{\boldsymbol{m}}_{\mathfrak{e}} \in V_{h,p}^{\mathfrak{e}}$ for all edges $\mathfrak{e} \in \mathcal{E}$
- *Hybrid variables:* $\bar{\boldsymbol{u}}_{\mathfrak{n}}, \bar{\boldsymbol{r}}_{\mathfrak{n}} \in \mathbb{R}^3$ for all $\mathfrak{n} \in \mathcal{N} \setminus \mathcal{N}_D$

[5] Cockburn et. al. Unified hybridization of discontinuous Galerkin, mixed, and continuous Galerkin methods for second order elliptic problems, SINUM, (2009)

[6] Rupp et. al. PDEs on hypergraphs and networks of surfaces, M2AN, (2022)

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For all $\bar{\boldsymbol{p}}, \bar{\boldsymbol{q}}, \bar{\boldsymbol{v}}, \bar{\boldsymbol{w}} \in V^{\mathfrak{e}}_{h,p}$:

$$\begin{aligned}
-(C_{\boldsymbol{n}}^{-1}\bar{\boldsymbol{n}}_{\mathfrak{e}}, \bar{\boldsymbol{p}})_{\mathfrak{e}} \quad + (\bar{\boldsymbol{u}}_{\mathfrak{e}}, \partial_x \bar{\boldsymbol{p}})_{\mathfrak{e}} - (\boldsymbol{i}_{\mathfrak{e}} \times \bar{\boldsymbol{r}}_{\mathfrak{e}}, \boldsymbol{p})_{\mathfrak{e}} &= \langle \bar{\boldsymbol{u}}_{\mathfrak{n}}, \bar{\boldsymbol{p}}\nu_{\mathfrak{e}}\rangle_{\mathfrak{e}} \\
-(C_{\boldsymbol{m}}^{-1}\bar{\boldsymbol{m}}_{\mathfrak{e}}, \bar{\boldsymbol{q}})_{\mathfrak{e}} \quad + (\bar{\boldsymbol{r}}_{\mathfrak{e}}, \partial_x \bar{\boldsymbol{q}})_{\mathfrak{e}} &= \langle \bar{\boldsymbol{r}}_{\mathfrak{n}}, \bar{\boldsymbol{q}}\nu_{\mathfrak{e}}\rangle_{\mathfrak{e}} \\
(\partial_x \bar{\boldsymbol{n}}_{\mathfrak{e}}, \bar{\boldsymbol{v}})_{\mathfrak{e}} &= (\boldsymbol{f}_{\mathfrak{e}}, \bar{\boldsymbol{v}})_{\mathfrak{e}} \\
(\boldsymbol{i}_{\mathfrak{e}} \times \bar{\boldsymbol{n}}_{\mathfrak{e}}, \bar{\boldsymbol{w}})_{\mathfrak{e}} + (\partial_x \bar{\boldsymbol{m}}_{\mathfrak{e}}, \bar{\boldsymbol{w}})_{\mathfrak{e}} &= (\boldsymbol{g}_{\mathfrak{e}}, \bar{\boldsymbol{w}})_{\mathfrak{e}}
\end{aligned}$$

[5] Cockburn et. al. Unified hybridization of discontinuous Galerkin, mixed, and continuous Galerkin methods for second order elliptic problems, SINUM, (2009)

[6] Rupp et. al. PDEs on hypergraphs and networks of surfaces, M2AN, (2022)

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For all $\bar{\boldsymbol{p}}, \bar{\boldsymbol{q}}, \bar{\boldsymbol{v}}, \bar{\boldsymbol{w}} \in V_{h,p}^\mathfrak{e}$: (penalty parameter $\tau_\mathfrak{e} > 0$)

$$
\begin{aligned}
&-(C_{\boldsymbol n}^{-1}\bar{\boldsymbol n}_\mathfrak{e}, \bar{\boldsymbol p})_\mathfrak{e} + (\bar{\boldsymbol u}_\mathfrak{e}, \partial_x \bar{\boldsymbol p})_\mathfrak{e} - (\boldsymbol i_\mathfrak{e} \times \bar{\boldsymbol r}_\mathfrak{e}, \boldsymbol p)_\mathfrak{e} = \langle \bar{\boldsymbol u}_\mathfrak{n}, \bar{\boldsymbol p}\nu_\mathfrak{e}\rangle_\mathfrak{e} \\
&-(C_{\boldsymbol m}^{-1}\bar{\boldsymbol m}_\mathfrak{e}, \bar{\boldsymbol q})_\mathfrak{e} + (\bar{\boldsymbol r}_\mathfrak{e}, \partial_x \bar{\boldsymbol q})_\mathfrak{e} = \langle \bar{\boldsymbol r}_\mathfrak{n}, \bar{\boldsymbol q}\nu_\mathfrak{e}\rangle_\mathfrak{e} \\
&(\partial_x \bar{\boldsymbol n}_\mathfrak{e}, \bar{\boldsymbol v})_\mathfrak{e} + \tau_\mathfrak{e}\langle \bar{\boldsymbol u}_\mathfrak{e}, \bar{\boldsymbol v}\rangle_\mathfrak{e} = (\boldsymbol f_\mathfrak{e}, \bar{\boldsymbol v})_\mathfrak{e} + \tau_\mathfrak{e}\langle \bar{\boldsymbol u}_\mathfrak{n}, \bar{\boldsymbol v}\rangle_\mathfrak{e} \\
&(\boldsymbol i_\mathfrak{e} \times \bar{\boldsymbol n}_\mathfrak{e}, \bar{\boldsymbol w})_\mathfrak{e} + (\partial_x \bar{\boldsymbol m}_\mathfrak{e}, \bar{\boldsymbol w})_\mathfrak{e} + \tau_\mathfrak{e}\langle \bar{\boldsymbol r}_\mathfrak{e}, \bar{\boldsymbol w}\rangle_\mathfrak{e} = (\boldsymbol g_\mathfrak{e}, \bar{\boldsymbol w})_\mathfrak{e} + \tau_\mathfrak{e}\langle \bar{\boldsymbol r}_\mathfrak{n}, \bar{\boldsymbol w}\rangle_\mathfrak{e}
\end{aligned}
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[5] Cockburn et. al. Unified hybridization of discontinuous Galerkin, mixed, and continuous Galerkin methods for second order elliptic problems, SINUM, (2009)

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The discrete balance equations reads

$$[\![\bar{\boldsymbol{n}}_{\mathfrak{e}}\nu_{\mathfrak{e}} + \tau_{\mathfrak{e}}(\bar{\boldsymbol{u}}_{\mathfrak{e}} - \bar{\boldsymbol{u}}_{\mathfrak{n}})]\!]_n = \boldsymbol{f}_{\mathfrak{n}}, \qquad [\![\bar{\boldsymbol{m}}_{\mathfrak{e}}\nu_{\mathfrak{e}} + \tau_{\mathfrak{e}}(\bar{\boldsymbol{r}}_{\mathfrak{e}} - \bar{\boldsymbol{r}}_{\mathfrak{n}})]\!]_n = \boldsymbol{g}_{\mathfrak{n}}$$

[5] Cockburn et. al. Unified hybridization of discontinuous Galerkin, mixed, and continuous Galerkin methods for second order elliptic problems, SINUM, (2009)

[6] Rupp et. al. PDEs on hypergraphs and networks of surfaces, M2AN, (2022)

HDG discretization

Global system:

$$A\bar{z}_h = F,$$

where $\bar{z}_h = (\bar{\boldsymbol{u}}_{\mathfrak{n}}, \bar{\boldsymbol{r}}_{\mathfrak{n}})$ with 6 dofs per joint.

- Independent local solves on edges are needed to form A and F
- F contains applied forces, moments and boundary data
- A is sparse, SPD but ill-conditioned
- Spectral equivalence for the continuous formulation:

$$\alpha v^\top L v \leq v^\top A v \leq \beta v^\top L v \quad \forall v$$

  where $v^\top L v = \frac{1}{2}\sum_{x\sim y} \frac{|v(x)-v(y)|^2}{|x-y|}$ (weighted graph laplacian)
- For future reference: $v^\top M v = \frac{1}{2}\sum_{x\sim y}(|v(x)|^2 + |v(y)|^2)|x-y|$ (weighted mass-type matrix)

A priori error bound[7,8]

Theorem (Convergence of HDG method)

If $\tau_\mathfrak{e} \sim h_\mathfrak{e}^s$ *for some* $s \in \{-1, 0, 1\}$ *and* $\boldsymbol{u}_\mathfrak{e}, \boldsymbol{r}_\mathfrak{e}, \boldsymbol{n}_\mathfrak{e}, \boldsymbol{m}_\mathfrak{e} \in H^{p+1}(\mathfrak{e})$ *for all* $\mathfrak{e} \in \mathcal{E}$, *then it holds*

$$\left[\sum_{\mathfrak{e}\in\mathcal{E}} \left[\|\boldsymbol{u}_\mathfrak{e} - \bar{\boldsymbol{u}}_\mathfrak{e}\|_\mathfrak{e}^2 + \|\boldsymbol{r}_\mathfrak{e} - \bar{\boldsymbol{r}}_\mathfrak{e}\|_\mathfrak{e}^2\right]\right]^{1/2} \lesssim h^{p+1-s^+},$$

$$\left[\sum_{\mathfrak{e}\in\mathcal{E}} \left[\|\boldsymbol{n}_\mathfrak{e} - \bar{\boldsymbol{n}}_\mathfrak{e}\|_\mathfrak{e}^2 + \|\boldsymbol{m}_\mathfrak{e} - \bar{\boldsymbol{m}}_\mathfrak{e}\|_\mathfrak{e}^2\right]\right]^{1/2} \lesssim h^{p+1-|s|},$$

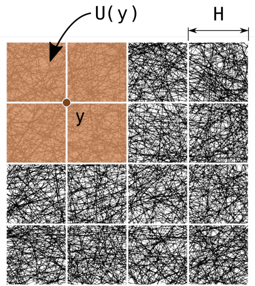
where $s^+ := \max(s, 0)$.

[7] Celiker, Cockburn, Shi, Hybridizable DG methods for Timoshenko beams, JSC (2010)

[8] Rupp, Hauck, M., Arbitrary order approximations at constant cost for Timoshenko beam network models, arXiv:2407.14388

Outline

Geometric coarsening[9]



- $\mathcal{T}_H$ is a mesh of boxes
- $\hat{V}_H$ is Q1-FEM with basis $\{\varphi_y\}_y$
- $V_H \subset \hat{V}_H$ satisfy the boundary conditions
- Clément type interpolation operator

$$\mathcal{I}_H v = \sum_{\text{free DoFs } y} \bar{v}_{U(y)} \varphi_y \in V_H$$

Lemma (Stability and approximability of $\mathcal{I}_H$)

For all $v \in V$ and for $H \geq R_0 > 0$,

$$H^{-1}|v - \mathcal{I}_H v|_M + |\mathcal{I}_H v|_L \leq C|v|_L,$$

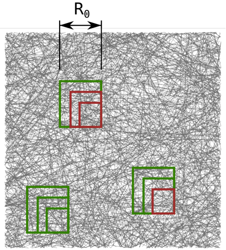
where $C = C_d \mu \sqrt{\sigma}$. ($V$ and V_H will have 3 components here)

[9] Görtz, Hellman, M., Iterative solution of spatial network models by subspace decomposition, Math. Comp. (2024)

Network homogeneity

The network must resemble a homogeneous material on coarse scales $H \geq R_0$.

1. *Homogeneity:* Let $B_H(x)$ be a box at x of side length H, with $H \geq R_0$. We assume limited density variation

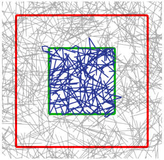


$$1 \leq \frac{\max_x |1|^2_{M,B_H(x)}}{\min_x |1|^2_{M,B_H(x)}} \leq \sigma(R_0)$$

Limited density variation on scales larger than R_0.

Network connectivity

2. *Connectivity:* For all $H \geq R_0$ and $x \in \Omega$ there is a connected subgraph $\mathcal{G}'$ that contains



- all edges with one endpoint in $B_H(x)$
- only edges with endpoints contained in $B_{H+R_0}(x)$

Network connectivity

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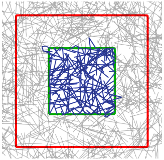


- all edges with one endpoint in $B_H(x)$
- only edges with endpoints contained in $B_{H+R_0}(x)$

Consider $L'\phi = \lambda M'\phi$, $\lambda_1 = 0$, $\lambda_2 > 0$ (Algebraic connectivity[10]):

$$|v - \bar{v}|_{M,B_H} \leq |v - \bar{v}|_{M'} \leq \lambda_2^{-1/2}|v - \bar{v}|_{L'} \leq \lambda_2^{-1/2}|v|_{L,B_{H+R_0}}$$

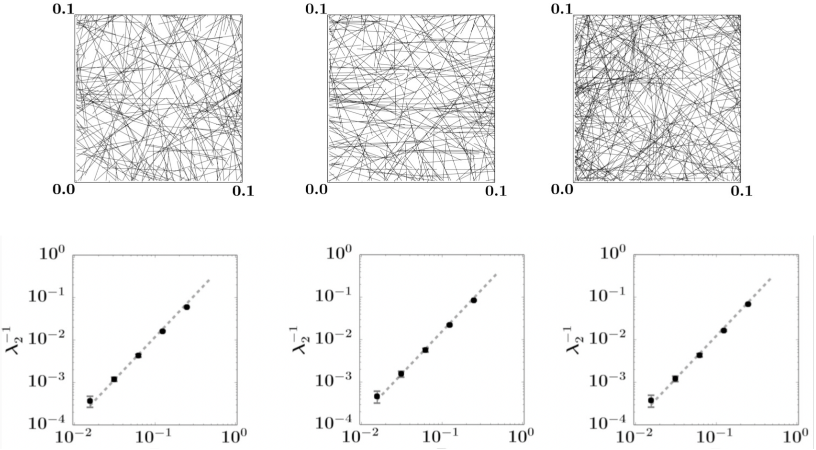
If $\mathcal{G}'$ fulfills an iso-perimetric inequality $\lambda_2 \sim H^{-2}$ and therefore

$$\lambda_2^{-1/2} = \mu(R_0)H$$

[10] Chung, Spectral graph theory, AMS, 1997

Example: Connectivity $\lambda_2^{-1/2} \approx \mu H$

Finite length fibers $r = 0.05$ and $|1|_M^2 = 1000$, $\Omega = [0,1]^2$



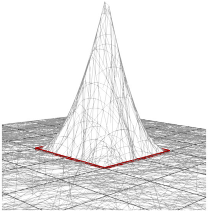
H varies from 2^{-2} to 2^{-6}. Here $R_0 \sim 2^{-6}$.

Subspace decomposition preconditioner[11]

Let $V = V_0 + V_1 + \cdots + V_m$ with

$$V_0 := V_H$$

$$V_i := \{\boldsymbol{v} \in V : \ \mathrm{supp}(\boldsymbol{v}) \subset U_i\}$$



Define $P_i\colon \ V \times V \to V_i \times V_i$ such that

$$(AP_i v, w) = (Av, w)$$

for all w and form $P := P_0 + P_1 + \cdots + P_m$.

- $BAz = BF$, with preconditioner $P = BA$
- Preconditioned conjugate gradient method.
- Semi-iterative: direct method on decoupled problems

[11] Kornhuber & Yserentant, Numerical homogenization of elliptic multiscale problemes by subspace decomposition, MMS, 2016

Convergence analysis

Lemma (Properties of the decomposition)

If the interpolation bound holds and $A \sim L$ with constants α and β, then for $H \geq 2R_0$ at least one decomposition $v = \sum_{j=0}^{m} v_j$ satisfies:

$$\sum_{j=0}^{m} |v_j|_A^2 \leq C_1 |v|_A^2, \quad C_1 = C_d \beta \alpha^{-1} \sigma \mu^2$$

and every decomposition satisfies $|v|_A^2 \leq C_2 \sum_{j=0}^{m} |v_j|_A^2$ with $C_2 = C_d$.

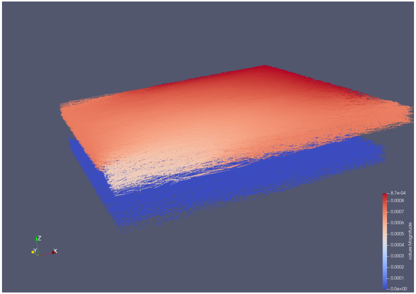
Theorem (Convergence of PCG)

With $\kappa = C_1 C_2$, $H > 2R_0$, it holds

$$|z - z^{(\ell)}|_A \leq 2 \left(\frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1} \right)^{\ell} |z - z^{(0)}|_A .$$

Outline

Example: Elastic deformation of paper



- 4 mm x 4 mm paper
- 615K edges and 424K nodes
- We study stretching of the paper caused by Dirichlet boundary conditions (upper right)
- HDG discretization with $p = 5$ and $\tau = 1$
- Preconditioner with $8 \times 8 \times 1$ element in coarse space

Example: Elastic deformation of paper

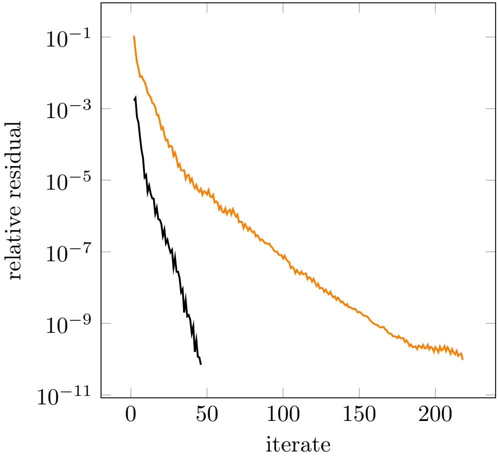
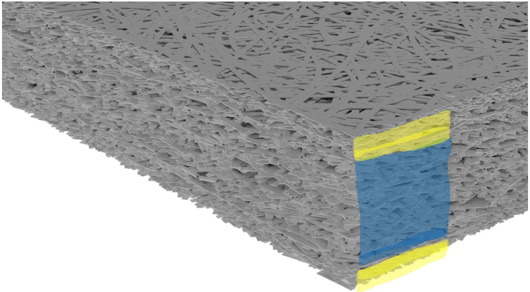


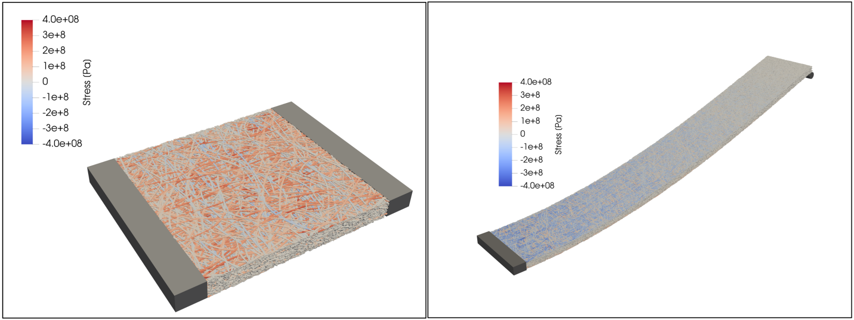
Figure: Convergence of PCG: constant material parameters (black) and realistic (orange).

Engineering application (FCC/Stora Enso)



- Three-ply paperboard
- Grammage: $400 g/m^2$
- Measure: (tensile) 4mm $\times$ 4mm (bending) 50mm $\times$ 4mm
- Dofs: (tensile) 16M (bending) 200M

Engineering application (FCC/Stora Enso)



- Solver converges in 60 iterations (practical purposes)
- Validated on various commercial paperboards
- Results consistent with experimental data[12]

[12] Görtz et. al., Iterative method for large-scale Timoshenko beam models assessed on commercial-grade paperboard, Computational Mechanics (2025)

Outline

1. The Timoshenko beam model
2. Hybridized formulation
3. Iteration by subspace decomposition
4. Numerical examples
5. **Multiscale approach**
6. Conclusion and future work

Multiscale approach

Given a symmetric positive bilinear form defined by

$$a(v, w) = (Av, w)$$

and an interpolant $\mathcal{I}_H : \mathcal{V} \to \mathcal{V}_H$ we can formulate an LOD method. Let $\mathcal{W} = \ker(\mathcal{I}_H)$ and

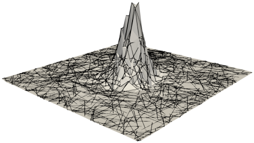
$$\mathcal{V}_H^{\text{ms}} = \{v \in \mathcal{V} : a(v, w) = 0 \text{ for all } w \in \mathcal{W}\}.$$

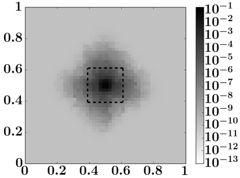
We seek $z_H^{\text{ms}} \in \mathcal{V}_H^{\text{ms}}$ such that

$$a(z_H^{\text{ms}}, v) = F(v), \quad \text{for all } v \in \mathcal{V}_H^{\text{ms}}.$$

- k-layer patches are used to localize computations to form $\mathcal{V}_H^{\text{ms},k}$
- Elementwise localization of rhs by $A = \sum_{T \in \mathcal{T}_H} A_T$

Convergence





Decay is proven using the optimality of the preconditioner in the W space[13][14] or by classical cut-off argument[15].

Theorem (Convergence of LOD)

With $\kappa = C_1 C_2$, $H > 2R_0$, it holds

$$|z - z_H^{ms,(k)}|_A \leq C(H + e^{-ck})|z|_A.$$

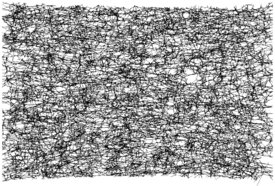
[13] Kornhuber et. al. An analysis of a class of variational multiscale methods based on subspace decomposition, Math. Comp. (2018)

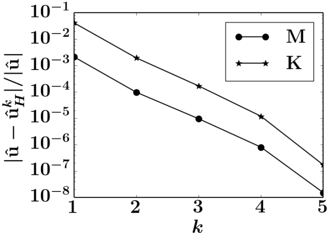
[14] Edelvik et. al., Numerical homogenization of spatial network models, CMAME (2024)

[15] Hauck et. al., An algebraic multiscale method for spatial network models, preprint

Fibre network application

Euler-Bernoulli beam model[16], pure boundary displacement. Unit square, 20k randomly places sticks of length 0.05, 320k dofs.





- $\hat{u}$ is the reference solution: $K\hat{u} = 0$ with Dirichlet bc $\hat{u}(\Gamma) = g$
- $\hat{u}_H^k$ is the LOD approximation with k-layer patches, $H = 2^{-5}$
- K is the system matrix
- M is a diagonal mass matrix

[16] Edelvik et. al., Numerical homogenization of spatial network models, CMAME (2024)

Outline

Conclusions and future works

Robust iterative approach to solve spatial network models with applications in the paper industry

- Görtz, Hellman, M., *Iterative solution of spatial network models by subspace decomposition,* Math. Comp. 93 (2024)
- Rupp, Hauck, M., *Arbitrary order approximations at constant cost for Timoshenko beam network models,* arXiv:2407.14388
- Görtz, Kettil, M., Fredlund, and Edelvik, *Iterative method for large-scale Timoshenko beam models assessed on commercial-grade paperboard,* Computational Mechanics 75 (2025)
- Edelvik, Görtz, Hellman, Kettil and M., *Numerical homogenization of spatial network models,* Comput. Methods Appl. Mech. Engrg. 418 (2024)

Conclusions and future works

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Future work: δ-overlap in DD, algebraic coarsening, algebraic LOD applied to the HDG setting, elastic wave propagation, and large (local) deformations