# *Adaptive Variational Multiscale Methods with Applications in Oil Reservoir Simulation*

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# Outline and Papers

## Outline

- The Variational Multiscale Method (VMS)
- Symmetric VMS
- Error estimates and adaptivity
- Mixed formulation
- Numerical examples
- Conclusions

## Papers

- M.G. Larson and A. Målqvist, *Adaptive Variational Multiscale Methods Based on A Posteriori Error Estimation: Energy Norm Estimates for Elliptic Problems*, CMAME 2007
- M.G. Larson and A. Målqvist, *A Mixed Adaptive Variational Multiscale Method with Applications in Oil Reservoir Simulation* M3AS 2009

## Model Problem

The framework covers a range of equations.

**Poisson Equation.** Find $u : \Omega \to \mathbf{R}$ such that

$$-\nabla \cdot a\nabla u = f \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

where $a(x) \geq a_0 > 0$ bounded with multiscale features, $f \in L^2$, and $\Omega$ is a domain in $\mathbf{R}^d$, $d = 1, 2, 3$

**Weak Form.** Find $u \in H_0^1(\Omega)$ such that

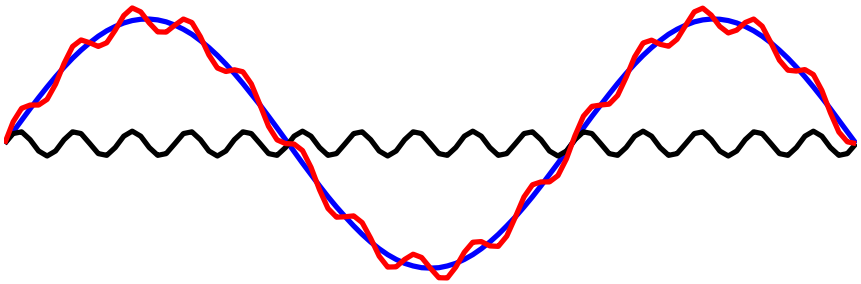
$$a(u, v) = (a\nabla u, \nabla v) = (f, v) \quad \text{for all } v \in H_0^1(\Omega),$$

where $(v, w) = \int_\Omega v \cdot w \, dx$.

# Variational Multiscale Method

We introduce two spaces $\mathcal{V}_c$ and $\mathcal{V}_f$ such that

$$\boxed{\mathcal{V}_c \oplus \mathcal{V}_f = H_0^1(\Omega)}$$



- $\mathcal{V}_c$ is a finite dimensional approximation of $H_0^1(\Omega)$. (finite element space)
- $\mathcal{V}_f$ can be chosen in different ways
  - Hierarchical basis
  - $L^2(\Omega)$-orthogonal to $\mathcal{V}_c$
  - Wavelet modified hierarchical basis

## Construction of Symmetric VMS

Starting from the model problem: find $u \in \mathcal{V}$ such that

$$a(u, v) = l(v) \quad \text{for all } v \in \mathcal{V}$$

and setting

$$u = u_c + u_f \quad v = v_c + v_f$$

we get: find $u_c + u_f \in \mathcal{V}_c \oplus \mathcal{V}_f$ such that

$$a(u_c + u_f, v_c + v_f) = l(v_c + v_f) \quad \text{for all } v_c + v_f \in \mathcal{V}_c \oplus \mathcal{V}_f$$

Note that $u_f \in \mathcal{V}_f$ satisfies the equation

$$a(u_f, v_f) = l(v_f) - a(u_c, v_f) \quad \text{for all } v_f \in \mathcal{V}_f$$

# Fine Scale Equations

Given the fine scale equation

$$a(u_f, v_f) = l(v_f) - a(u_c, v_f) \quad \text{for all } v_f \in \mathcal{V}_f$$

we let

$$u_f = u_{f,l} + u_{f,c}$$

with

- $u_{f,l} \in \mathcal{V}_f$ the contribution from the right hand side

$$a(u_{f,l}, v_f) = l(v_f) \quad \text{for all } v_f \in \mathcal{V}_f$$

- $u_{f,c} \in \mathcal{V}_f$ the contribution from the coarse scale part

$$a(u_{f,c}, v_f) = -a(u_c, v_f) \quad \text{for all } v_f \in \mathcal{V}_f$$

# Fine Scale Effects Coarse Scale

Let $\mathcal{T} : \mathcal{V}_c \to \mathcal{V}_f$ denote the solution operator to

$$a(u_{f,c}, v_f) = -a(u_c, v_f) \quad \text{for all } v_f \in \mathcal{V}_f$$

i.e.

$$u_{f,c} = \mathcal{T} u_c$$

Using the resulting decomposition

$$u = u_c + \mathcal{T} u_c + u_{f,l}$$

we get

$$a(u_c + \mathcal{T} u_c + u_{f,l}, v_c + v_f) = (f, v_c + v_f)$$

for all $v_c \in \mathcal{V}_c$ and $v_f \in \mathcal{V}_f$.

## Symmetric VMS

Since $u_{f,l}$ is directly determined we get the following problem for $u_c$: find $u_c \in \mathcal{V}_c$ such that

$$a(u_c + \mathcal{T} u_c, v_c + \mathcal{T} v_c) = (f, v_c + \mathcal{T} v_c) - a(u_{f,l}, v_c + \mathcal{T} v_c)$$

for all $v_c \in \mathcal{V}_c$.

- Here we chose $v_f = \mathcal{T} v_c$ to get a symmetric formulation
- Note that in the standard VMS procedure one basically sets $v_f = 0$ in this step and thus the resulting problem is not necessarily symmetric
- However, when the exact operator $\mathcal{T}$ is used, the methods are equivalent.
- Note also that $a((I + \mathcal{T})v_c, v_f) = 0$ i.e., the exact $\mathcal{T}$ decouples the problem.

## Symmetric VMS

Let

- $\tilde{\mathcal{T}}$ be a computable approximation of $\mathcal{T}$
- $U_{f,l}$ be a computable approximation of $u_{f,l}$

We get the method: find $U_c \in \mathcal{V}_c$ such that

$$a(U_c + \tilde{\mathcal{T}} U_c, v_c + \tilde{\mathcal{T}} v_c) = (f, v_c + \tilde{\mathcal{T}} v_c) - a(U_{f,l}, v_c + \tilde{\mathcal{T}} v_c)$$

for all $v_c \in \mathcal{V}_c$.

On matrix form this leads to,

$$K_{\text{mod}} U_c = b_{\text{mod}}$$

Given $U_c$, $U_{f,l}$, and $\tilde{\mathcal{T}}$, $U_f$ can be computed.

## Construction of $\tilde{\mathcal{T}}$

Recall that $u_c = \sum_i u_{c,i} N_{c,i}$ with $\{N_{c,i}\}$ a basis in $\mathcal{V}_c$ and let

$$a(\mathcal{T} N_{c,i}, v_f) = -a(N_{c,i}, v_f) \quad \text{for all } v_f \in \mathcal{V}_f$$

By linearity

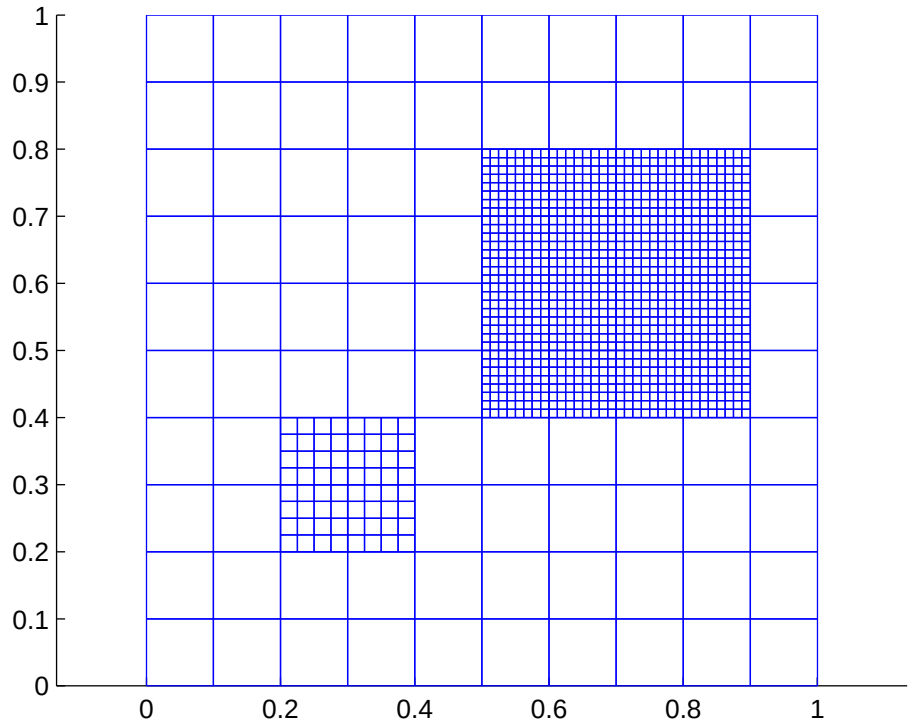
$$\mathcal{T} u_c = \sum_i u_{c,i} \mathcal{T} N_{c,i}$$

and thus we are led to computing $\mathcal{T} N_{c,i}$ for each coarse basis function $N_{c,i}$.

We define $\tilde{\mathcal{T}}$ by solving these problems approximately by

- Restricting to a localized patch problem $\text{supp}(N_{c,i}) \subset \omega_i$
- Discretizing using a fine subgrid on $\omega_i$

# Refinement and Layers




One and two layer stars. Typically homogeneous Dirichlet boundary condition are used.

## Construction of $U_{f,l}$

Recall that $u_{f,l} \in \mathcal{V}_f$ solves

$$a(u_{f,l}, v_f) = l(v_f) \quad \text{for all } v_f \in \mathcal{V}_f$$

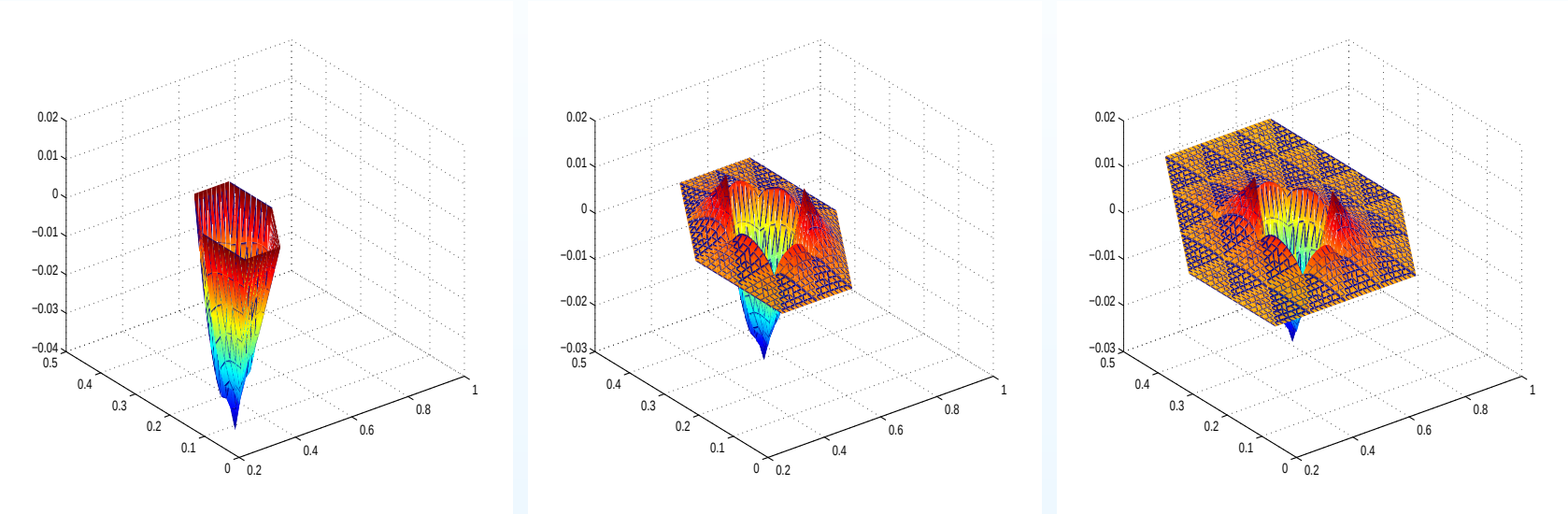
Using a partition of unity $\varphi_i$ we can split the right hand side as follows $l(v_f) = \sum_i l(\varphi_i v_f)$ to get,

$$u_{f,l} = \sum_i u_{f,l,i}$$

$$a(u_{f,l,i}, v_f) = l(\varphi_i v_f)$$

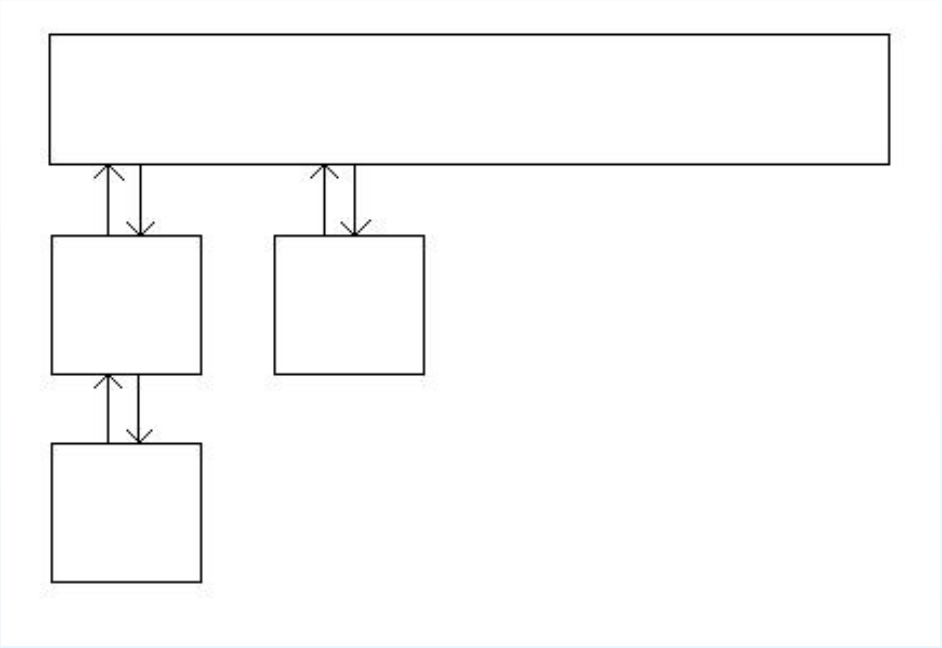
Again we find an approximation by restricting to patches and discretizing the subgrid.

# Localized Fine Scale Solution




Typical local solutions $U_{f,i} = U_{c,i}\tilde{\mathcal{T}}N_i + U_{f,l,i}$ using $1$, $2$, and $3$ layer stars on triangles. The right hand side has support in the center and the solution is forced to be zero in coarse nodes.

# Parallel Structure



- No communication between subgrid problems
- Multiple levels possible
- Different physics in subgrid model possible. Need transfer operator.

# Adaptive VMS

The Adaptive Variational Multiscale Method (AVMS) builds on the following ingredients:

- Error estimation framework
- Adaptive strategy for tuning of critical discretization parameters

The method is designed so that:

$$\boxed{\text{error} \to 0 \text{ when } h \to 0 \text{ and } L \to \infty}$$

- A priori error estimates in progress.
- To circumvent difficulties with choosing discretization parameters $h$ and $L$ we use an adaptive algorithm based on a posteriori error estimates

# Energy Norm Error Estimate

$$\|\sqrt{a}\nabla e\|^2 \leq \sum_{i\in\mathcal{C}} C_i \|H\mathcal{R}(U_c)\|^2_{\omega_i}$$
$$+ \sum_{i\in\mathcal{F}} C_i \left( \|\sqrt{H}\Sigma(U_{f,i})\|^2_{\partial\omega_i} + \|h\mathcal{R}_i(U_{f,i})\|^2_{\omega_i} \right)$$

- The first term measures the coarse mesh error
- The second term is the normal derivative of the fine scale solutions on $\partial\omega_i$ and measures the error due to restriction to subdomains
- The third term measures the fine scale error caused by discretization

# Adaptive Strategy

$$\|\sqrt{a}\nabla e\|^2 \leq \sum_{i\in\mathcal{C}} C_i \|H\mathcal{R}(U_c)\|^2_{\omega_i}$$
$$+ \sum_{i\in\mathcal{F}} C_i \left( \|\sqrt{H}\Sigma(U_{f,i})\|^2_{\partial\omega_i} + \|h\mathcal{R}_i(U_{f,i})\|^2_{\omega_i} \right)$$

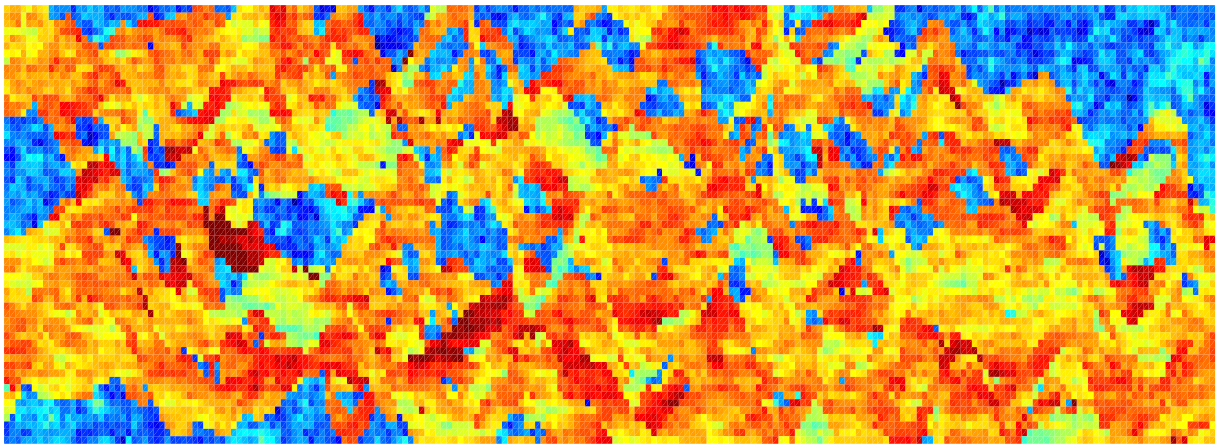
1. Compute the solution $U$.
2. Calculate residuals for each $i \in \{\mathcal{C}\ \mathcal{F}\}$. Mark large entries.
3. For marked $i \in \mathcal{C}$, let $h = H/2$ and $L = 1$ for these entries.
4. For marked $i \in \mathcal{F}$, either let $L := L + 1$ or $h := h/2$ depending on the indicators.
5. Return to $1$.

# Application to a Mixed Problem

Poisson equation on mixed form:

$$\begin{cases} \frac{1}{a}\boldsymbol{\sigma} - \nabla u = 0 & \text{in } \Omega \\ -\nabla \cdot \boldsymbol{\sigma} = f & \text{in } \Omega \\ n \cdot \boldsymbol{\sigma} = 0 & \text{on } \Gamma \end{cases}$$

where the permeability $a$ is taken from the SPE data set (upperness in log-scale),



## Splitting Based on RT-elements

We use lowest order RT basis functions together with constants.

- Let $\pi_c$ is the RT-interpolant onto $\mathcal{V}_c$ and $P_c$ be the $L^2$-projection onto $W_c$
- We define $\boldsymbol{\sigma} = \pi_c \boldsymbol{\sigma} + (I - \pi_c)\boldsymbol{\sigma}$ and thus $\boldsymbol{\sigma}_f = (I - \pi_c)\boldsymbol{\sigma} \quad \boldsymbol{\sigma}_c = \pi_c \boldsymbol{\sigma}$.
- Further we define $u = P_c u_c + (1 - P_c)u = u_c + u_f$.
- Thus we are using an $L^2$-orthogonal splitting in the scalar variable.

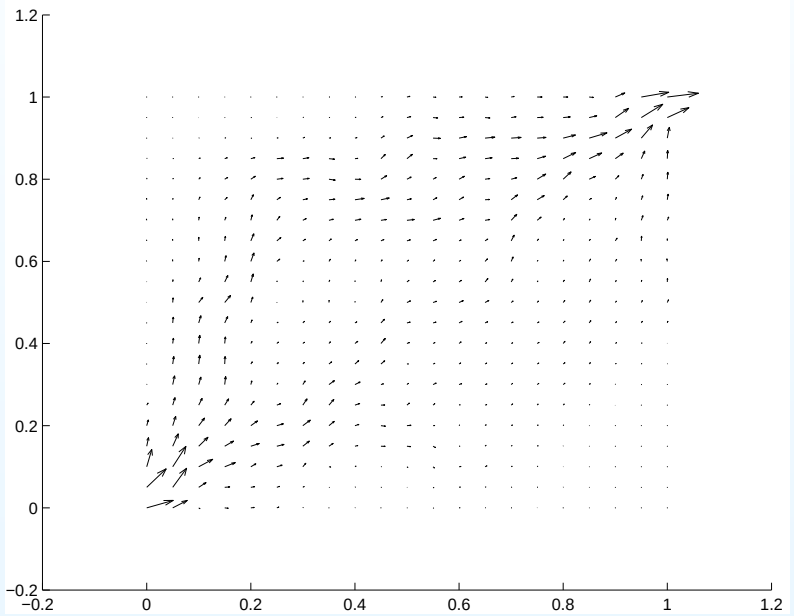
Hierarchical split for lagrangian elements leads to nodal exactness in the coarse solution while here we get exactness of average values on coarse elements.

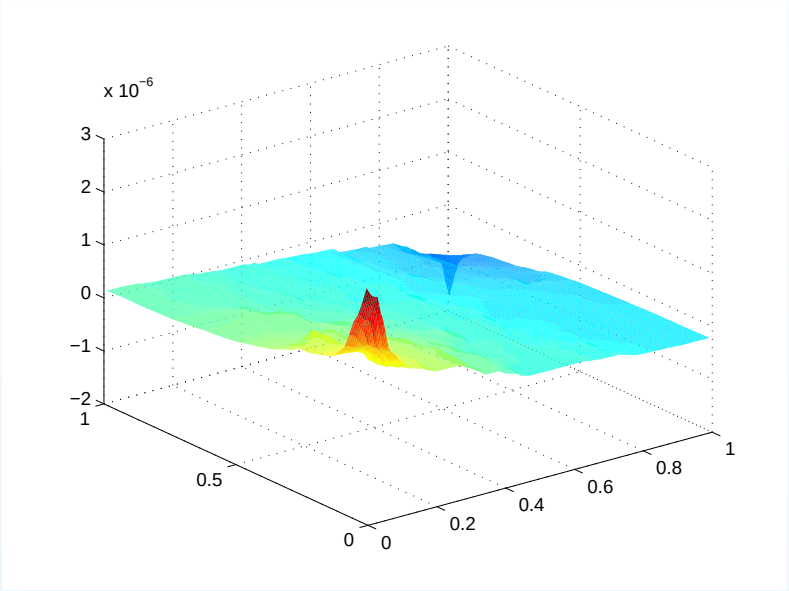
## Approximate Fine Scales

- We apply the abstract framework
- Divide the fine scale problem into contributions from the coarse scale part $\boldsymbol{\sigma}_c$ and right hand side $f$
- Let $\boldsymbol{\Sigma}_c = \sum_i \Sigma_{c,i} \boldsymbol{\phi}_i$ where $\boldsymbol{\phi}_i$ are the Raviart-Thomas basis functions. Solve the local problem driven by the basis functions (one problem for each basis function)
- Localize by restricting the problem to a patch and using homogeneous Neumann conditions
- Discretize using a suitable subgrid

# Example of Convergence

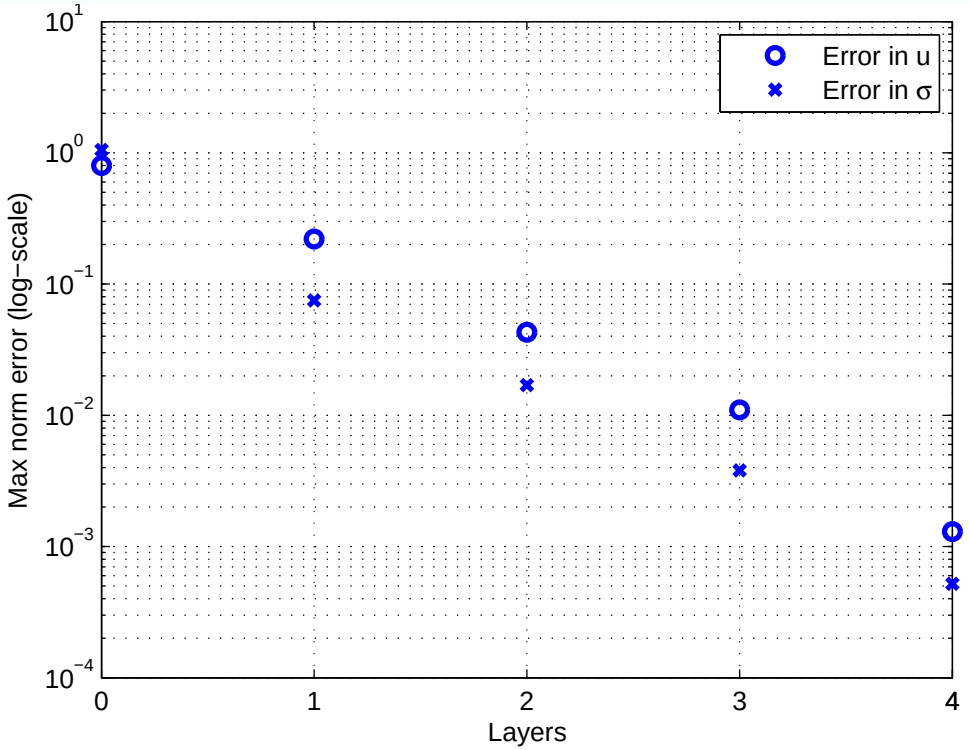
We solve the SPE problem.




To the left we see the flux and to the right the pressure. We use $220 \times 60$ elements for the reference solution, $f = 1$ in the lower left corner and $f = -1$ in the upper right corner.

# Example of convergence




$55 \times 15$ coarse elements and $h = H/4$. We get exponential decay in max norm error compared with reference solution when increasing the number of layers!

## A Posteriori Error Estimate

The following energy norm bound holds

$$\|\boldsymbol{\sigma} - \boldsymbol{\Sigma}\|_a^2 \leq C_a^2 \sum_i R_{\omega_i}^2 + R_{\partial\omega_i}^2$$

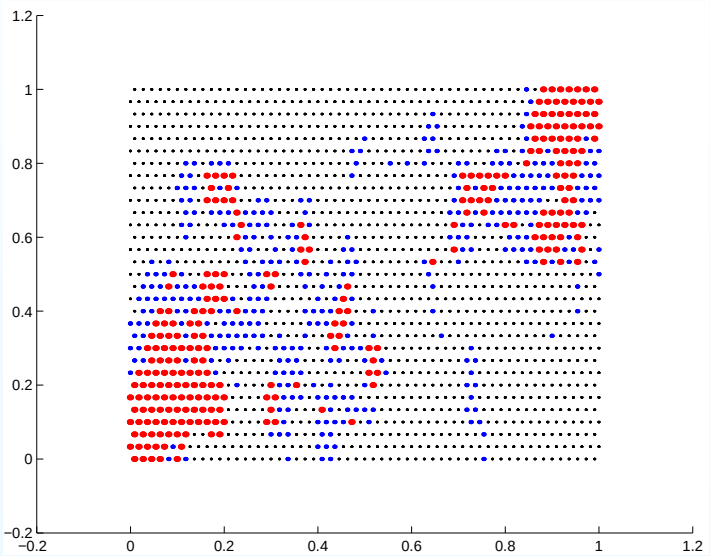
where

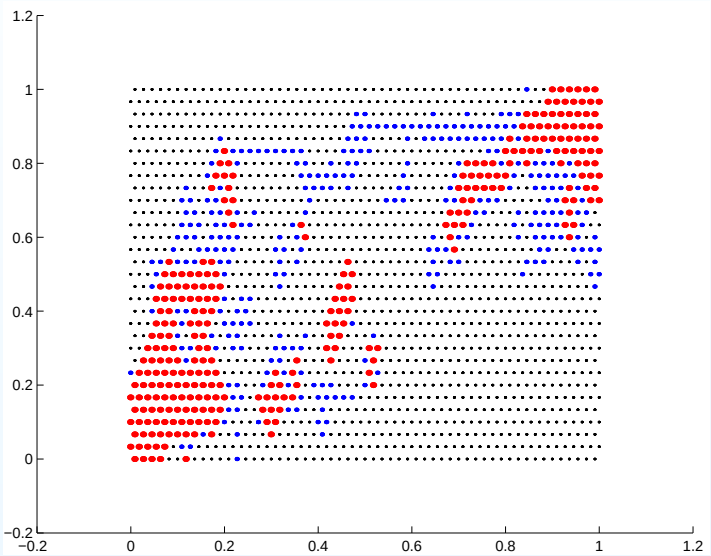
$$R_{\omega_i} = \|\frac{1}{a}(\Sigma_c^i \boldsymbol{\phi}_i + \boldsymbol{\Sigma}_{f,i}) - \nabla U_{f,i}^*\|_{\omega_i}^2 + \|\frac{h}{a}(f\psi_i + \nabla \cdot (\Sigma_c^i \boldsymbol{\phi}_i + \boldsymbol{\Sigma}_{f,i}))\|_{\omega_i}^2$$

$$R_{\partial\omega_i} = \|\frac{1}{2\sqrt{h}} U_{f,i}^*\|_{\partial\omega_i \setminus \Gamma}^2$$

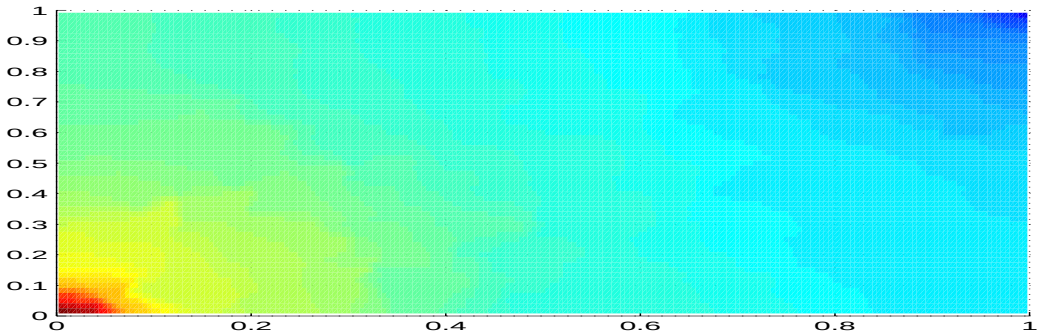
$U^*$ is a post proc version of $U$, $C_a \sim \|\sqrt{a}\|_{L^\infty(\omega_i)}$.

# Adaptivity Example: Layers and Refinements



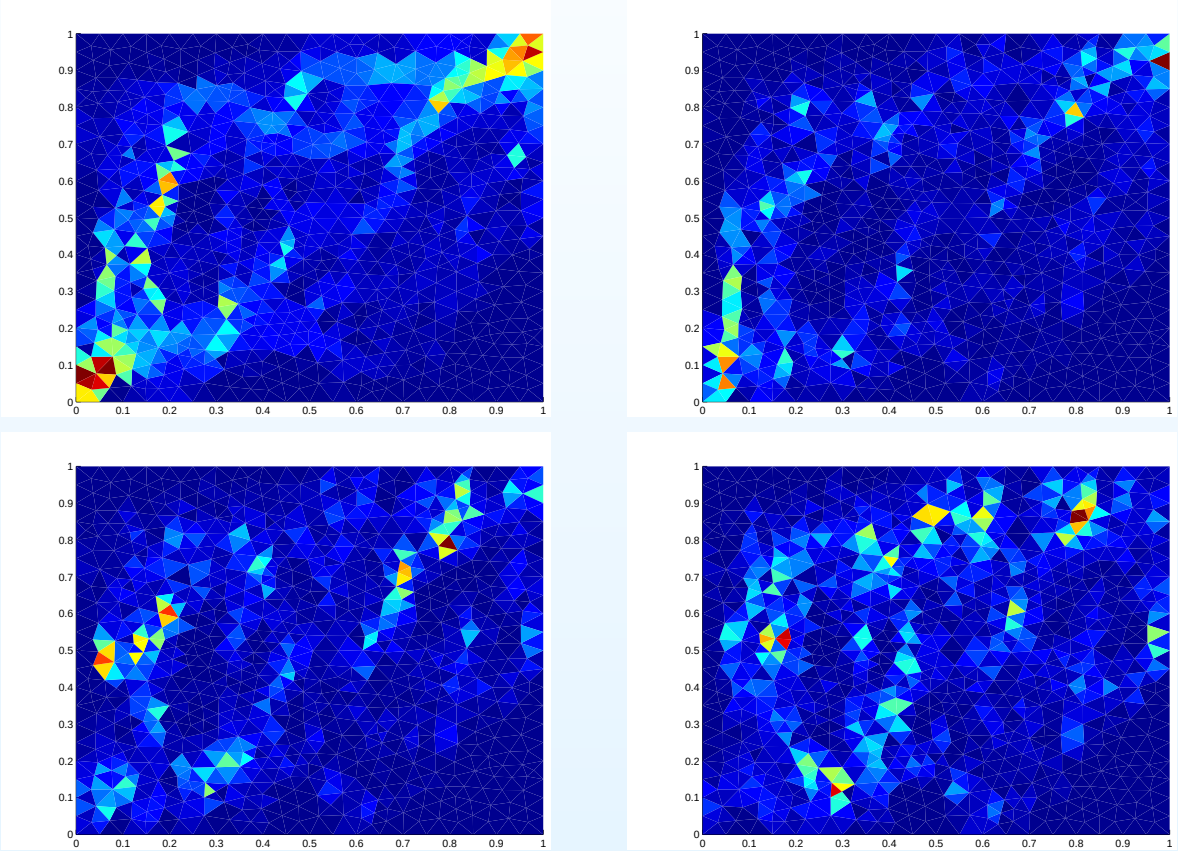




Black circle is one, blue is two and red is three layers/refinements (layers top, refs bottom). Remember,



# Relative Error in Energy Norm

Galerkin $105.6\%$ and one iteration $15.8\%$.



Two iterations $10.1\%$ and three iterations $7.6\%$.

## Conclusions from Example

- The error indicators finds critical areas
- More computational effort in these areas decreases the global error quickly
- To get an equally good approximation without adaptivity we need to use three refinements on two layer patches
- In the example above we still have $70\%$ of the patches using one layer and one refinement

## Summary

The AVMS provides:

- Systematic technique for construction of a computable approximation of the fine scale part of the solution using decoupled localized subgrid problems.
- A posteriori error estimation framework (also for goal functionals)
- Adaptive algorithms for automatic tuning of critical discretization parameters