A New Mixed Multiscale Method for Oil Reservoir Simulation

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The Model Problem

Poisson Equation on mixed form:

$$\begin{cases} \frac{1}{a}\boldsymbol{\sigma} - \nabla u = 0 & \text{in } \Omega, \\ -\nabla \cdot \boldsymbol{\sigma} = f & \text{in } \Omega, \\ n \cdot \boldsymbol{\sigma} = 0 & \text{on } \Gamma. \end{cases}$$

where $a > 0$ bounded, Ω is a domain in $\mathbf{R}^d$, $d = 1, 2, 3$, with boundary Γ, and f is a given function.

Weak form

Find $\boldsymbol{\sigma} \in \boldsymbol{V} = \{\boldsymbol{v} \in H(\text{div}; \Omega) : \boldsymbol{n} \cdot \boldsymbol{v} = 0 \text{ on } \Gamma\}$ and $u \in W = L^2(\Omega)/\mathbf{R}$ such that,

$$\begin{cases} (\frac{1}{a}\boldsymbol{\sigma}, \boldsymbol{v}) + (u, \nabla \cdot \boldsymbol{v}) = 0, \\ -(\nabla \cdot \boldsymbol{\sigma}, w) = (f, w), \end{cases}$$

for all $\boldsymbol{v} \in \boldsymbol{V}$ and $w \in W$.

Here $(\cdot, \cdot)$ denotes the $L^2(\Omega)$ scalar product for vector and scalar functions.

Applications

Elliptic problems of this kind needs to be solved in oil reservoir simulation.

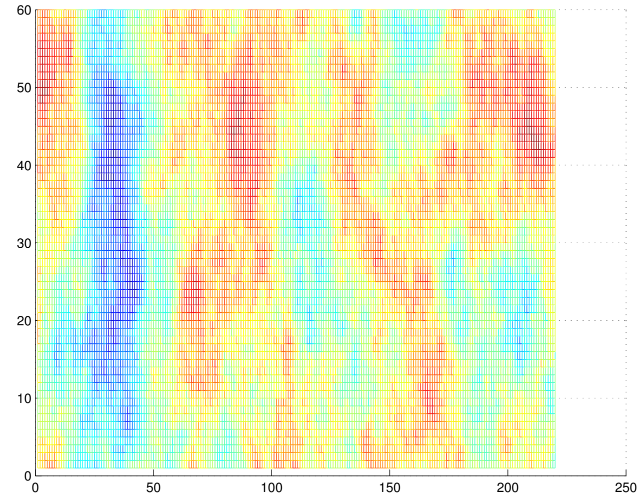


Figure 1: 2D slice of permeability (a) in oil reservoir (log scale).

Why Multiscale Method?

- If we for the moment assume a to be periodic $a = a(x/\epsilon)$ we have (Hou),

$$\|\frac{1}{\sqrt{a}}(\boldsymbol{\sigma} - \boldsymbol{\Sigma})\| \leq C\frac{h}{\epsilon}.$$

- $h > \epsilon$ will give unreliable results even with exact quadrature.
- $h < \epsilon$ will be to computationally expensive to solve on a single mesh.
- Parallelized local problems must be solved.

Other Related Methods

- Multiscale Related (Nielsen-Holden-Tveito, Durlofsky et. al., Gautier-Blunt-Christie)
- Multiscale FEM (Hou-Wu, Aarnes-Lie et. al., Jenny-Lee-Tchelepi)
- Variational Multiscale Method (Hughes et. al., Arbogast, Larson-Målqvist)

Coarse and Fine Scales

We introduce spaces $\boldsymbol{V}_c \oplus \boldsymbol{V}_f = \boldsymbol{V}$ and $W_c \oplus W_f = W$.

- $\boldsymbol{V}_c$ is a finite dimensional approximation of $H(\text{div};\Omega)$. (finite element space e.g. Raviart-Thomas)
- W_c is an approximation of $L^2(\Omega)$. (e.g. piecewise constants).
- The degrees of freedom in these spaces should be possible to handle on a single computer.

Coarse and Fine Scales

Find $\boldsymbol{\sigma}_c \in \boldsymbol{V}_c$, $\boldsymbol{\sigma}_f \in \boldsymbol{V}_f$, $u_c \in W_c$, and $u_f \in W_f$ such that,

$$\begin{cases} (\frac{1}{a}\boldsymbol{\sigma}_c, \boldsymbol{v}_c) + (\frac{1}{a}\boldsymbol{\sigma}_f, \boldsymbol{v}_c) + (u_c, \nabla \cdot \boldsymbol{v}_c) + (u_f, \nabla \cdot \boldsymbol{v}_c) = 0 \\ -(\nabla \cdot \boldsymbol{\sigma}_c, w_c) - (\nabla \cdot \boldsymbol{\sigma}_f, w_c) = (f, w_c) \\ (\frac{1}{a}\boldsymbol{\sigma}_f, \boldsymbol{v}_f) + (u_f, \nabla \cdot \boldsymbol{v}_f) = -(\frac{1}{a}\boldsymbol{\sigma}_c, \boldsymbol{v}_f) - (u_c, \nabla \cdot \boldsymbol{v}_f) \\ -(\nabla \cdot \boldsymbol{\sigma}_f, w_f) = (f, w_f) + (\nabla \cdot \boldsymbol{\sigma}_c, w_f) \end{cases}$$

for all $\boldsymbol{v}_c \in \boldsymbol{V}_c$, $\boldsymbol{v}_f \in \boldsymbol{V}_f$, $w_c \in W_c$, and $w_f \in W_f$.

Coarse and Fine Scales

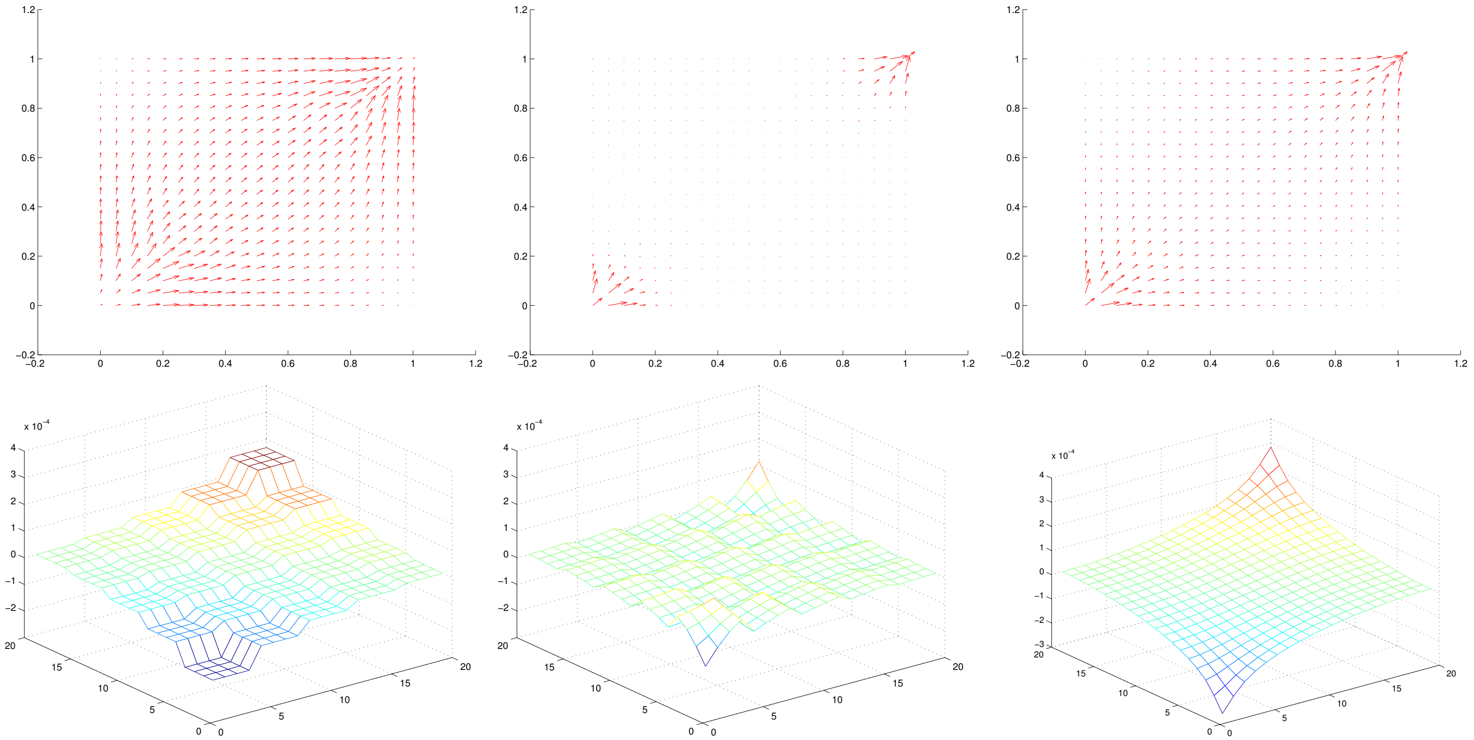


Figure 2: $\boldsymbol{\sigma}$ and u on coarse and fine scale.

Orthogonality

We use an hierarchical basis of Raviart-Thomas element for the flux and the piecewise constants for the pressure.

$$(w_c, \nabla \cdot \boldsymbol{v}_f) = \sum_K (w_c, \nabla \cdot \boldsymbol{v}_f)_K = \sum_K w_c^K \int_{\partial K} \boldsymbol{n} \cdot \boldsymbol{v}_f \, dx = 0$$

where w_c^K is the constant at coarse element K,

$$(w_f, \nabla \cdot \boldsymbol{v}_c) = \sum_K (w_f, \nabla \cdot \boldsymbol{v}_c)_K = \sum_K \nabla \cdot \boldsymbol{v}_c^K \int_K w_f \, dx = 0$$

Coarse and Fine Scales

Find $\boldsymbol{\sigma}_c \in \boldsymbol{V}_c$, $\boldsymbol{\sigma}_f \in \boldsymbol{V}_f$, $u_c \in W_c$, and $u_f \in W_f$ such that,

$$\begin{cases} (\frac{1}{a}\boldsymbol{\sigma}_c, \boldsymbol{v}_c) + (\frac{1}{a}\boldsymbol{\sigma}_f, \boldsymbol{v}_c) + (u_c, \nabla \cdot \boldsymbol{v}_c) + (u_f, \nabla \cdot \boldsymbol{v}_c) = 0 \\ -(\nabla \cdot \boldsymbol{\sigma}_c, w_c) - (\nabla \cdot \boldsymbol{\sigma}_f, w_c) = (f, w_c) \\ (\frac{1}{a}\boldsymbol{\sigma}_f, \boldsymbol{v}_f) + (u_f, \nabla \cdot \boldsymbol{v}_f) = -(\frac{1}{a}\boldsymbol{\sigma}_c, \boldsymbol{v}_f) - (u_c, \nabla \cdot \boldsymbol{v}_f) \\ -(\nabla \cdot \boldsymbol{\sigma}_f, w_f) = (f, w_f) + (\nabla \cdot \boldsymbol{\sigma}_c, w_f) \end{cases}$$

for all $\boldsymbol{v}_c \in \boldsymbol{V}_c$, $\boldsymbol{v}_f \in \boldsymbol{V}_f$, $w_c \in W_c$, and $w_f \in W_f$.

Modified set of Equations

Find $\boldsymbol{\sigma}_c \in \boldsymbol{V}_c$, $\boldsymbol{\sigma}_f \in \boldsymbol{V}_f$, $u_c \in W_c$, and $u_f \in W_f$ such that,

$$\begin{cases} (\frac{1}{a}\boldsymbol{\sigma}_c, \boldsymbol{v}_c) + (\frac{1}{a}\boldsymbol{\sigma}_f, \boldsymbol{v}_c) + (u_c, \nabla \cdot \boldsymbol{v}_c) = 0 \\ -(\nabla \cdot \boldsymbol{\sigma}_c, w_c) = (f, w_c) \\ (\frac{1}{a}\boldsymbol{\sigma}_f, \boldsymbol{v}_f) + (u_f, \nabla \cdot \boldsymbol{v}_f) = -(\frac{1}{a}\boldsymbol{\sigma}_c, \boldsymbol{v}_f) \\ -(\nabla \cdot \boldsymbol{\sigma}_f, w_f) = (f, w_f) \end{cases}$$

for all $\boldsymbol{v}_c \in \boldsymbol{V}_c$, $\boldsymbol{v}_f \in \boldsymbol{V}_f$, $w_c \in W_c$, and $w_f \in W_f$.

Decoupling of Fine Scale Equations

We start by introducing two partitions of unity, $\sum_i \boldsymbol{\phi}_i = \boldsymbol{I}$ and $\sum_i \psi_i = 1$ where $\boldsymbol{I}$ is the identity matrix, $\boldsymbol{\phi}_i \in \boldsymbol{V}_c$ coarse Raviart-Thomas base function, and $\psi_i \in W_c$ coarse piecewise constant base functions.

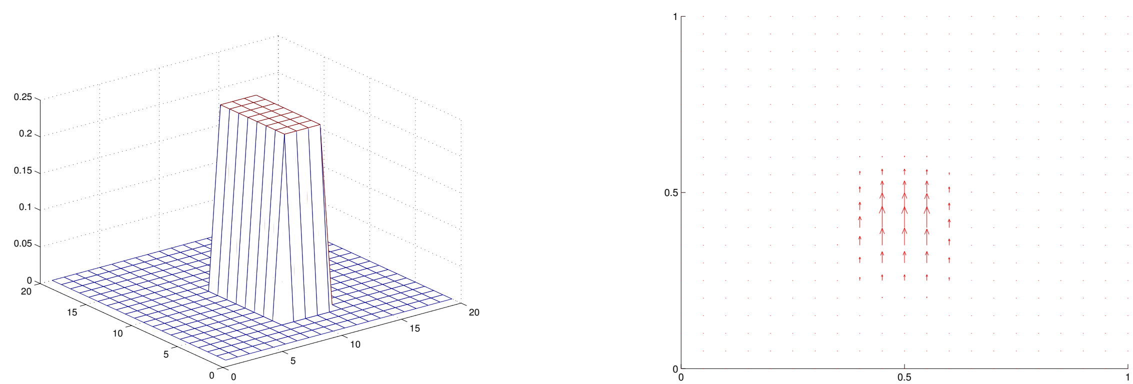


Figure 3: ψ_i, and $\boldsymbol{\phi}_i$.

Decoupling of Fine Scale Equations

We introduce $\boldsymbol{\sigma}_{f,i} \in \boldsymbol{V}_f$ and $u_{f,i} \in W_f$ such that $\boldsymbol{\sigma}_c = \sum_i \boldsymbol{\sigma}_c^i \phi_i$, $\boldsymbol{\sigma}_c^i \in \mathrm{R}$, u_c, $\boldsymbol{\sigma}_f = \sum_i \boldsymbol{\sigma}_{f,i}$, and $u_f = \sum_i u_{f,i}$ solves:

$$\left\{ \begin{aligned} (\tfrac{1}{a}\boldsymbol{\sigma}_c, \boldsymbol{v}_c) + (\tfrac{1}{a}\boldsymbol{\sigma}_f, \boldsymbol{v}_c) + (u_c, \nabla \cdot \boldsymbol{v}_c) &= 0, \\ -(\nabla \cdot \boldsymbol{\sigma}_c, w_c) &= (f, w_c), \\ (\tfrac{1}{a}\boldsymbol{\sigma}_{f,i}, \boldsymbol{v}_f) + (u_{f,i}, \nabla \cdot \boldsymbol{v}_f) &= -(\tfrac{1}{a}\boldsymbol{\sigma}_c^i \phi_i, \boldsymbol{v}_f), \\ -(\nabla \cdot \boldsymbol{\sigma}_{f,i}, w_f) &= (f, w_f \psi_i), \end{aligned} \right.$$

for all $\boldsymbol{v}_c \in \boldsymbol{V}_c$, $\boldsymbol{v}_f \in \boldsymbol{V}_f$, $w_c \in W_c$, and $w_f \in W_f$.

Local Solutions

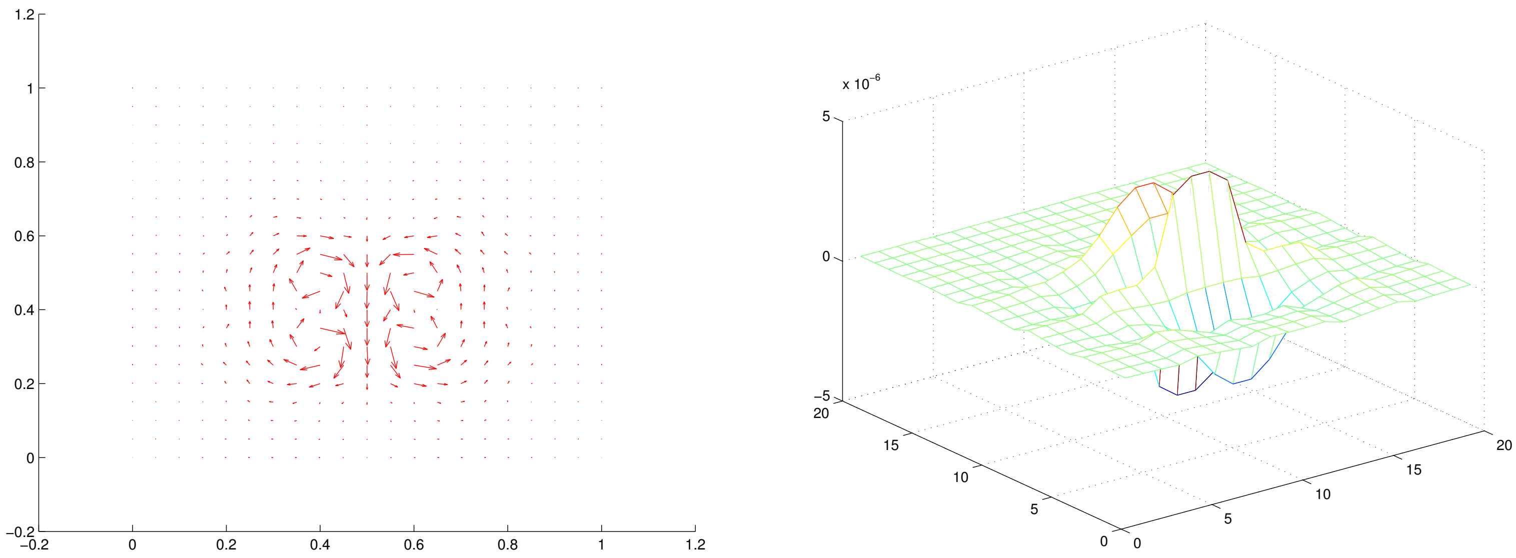


Figure 4: The local solutions $\boldsymbol{\sigma}_{f,i}$ and $u_{f,i}$

In this simple example $a = 1$.

Motivation for Introducing Patches

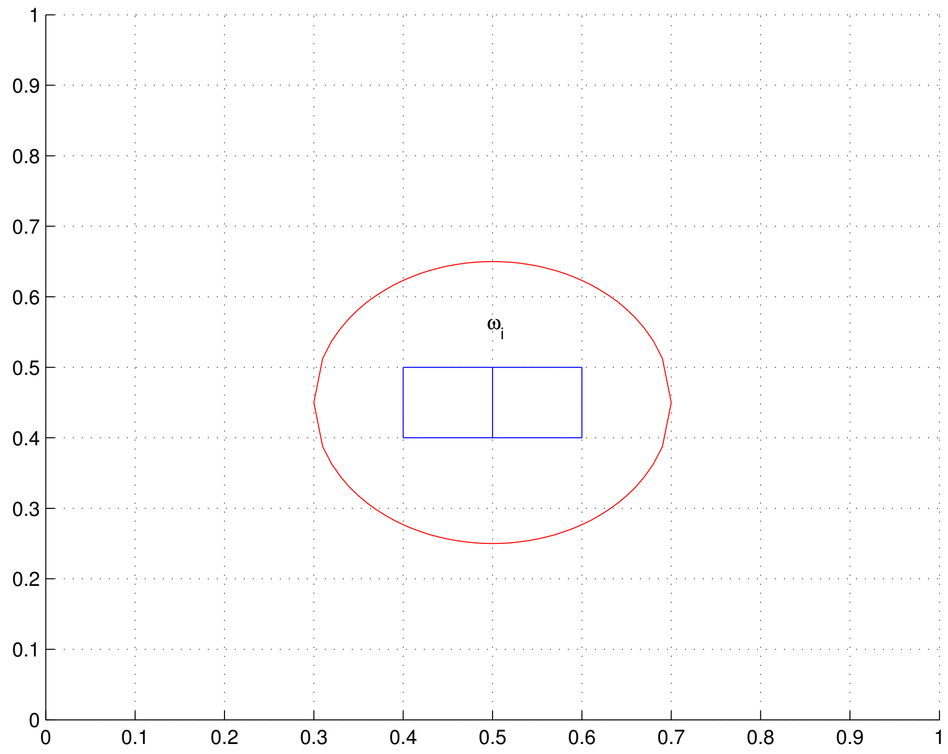
- The right hand side has support on $supp(\boldsymbol{\phi}_i) = supp(\psi_i)$.
- The equations are solved in a slice space where solutions decay rapidly.

$$\int_E \boldsymbol{n} \cdot \boldsymbol{\sigma}_{f,i} \, dx = 0$$

and

$$\int_K u_{f,i} \, dx = 0.$$

The Patch



The patch ω_i typically consists of coarse elements but could have any geometry.

Solving Local Neumann Problems

Find $\boldsymbol{\Sigma}_c = \sum_i \boldsymbol{\Sigma}_c^i \boldsymbol{\phi}_i \in \boldsymbol{V}_H$, $\boldsymbol{\Sigma}_{f,i} \in \boldsymbol{V}_h(\omega_i)$, $U_c \in W_H$, and $U_{f,i} \in W_h(\omega_i)$ such that

$$\begin{cases} (\frac{1}{a}\boldsymbol{\Sigma}_c, \boldsymbol{v}_c) + (\frac{1}{a}\boldsymbol{\Sigma}_f, \boldsymbol{v}_c) + (U_c, \nabla \cdot \boldsymbol{v}_c) = 0, \\ -(\nabla \cdot \boldsymbol{\Sigma}_c, w_c) = (f, w_c), \\ (\frac{1}{a}\boldsymbol{\Sigma}_{f,i}, \boldsymbol{v}_f) + (U_{f,i}, \nabla \cdot \boldsymbol{v}_f) = -(\frac{1}{a}\boldsymbol{\Sigma}_c^i \boldsymbol{\phi}_i, \boldsymbol{v}_f), \\ -(\nabla \cdot \boldsymbol{\Sigma}_{f,i}, w_f) = (f, w_f \psi_i), \end{cases}$$

for all $\boldsymbol{v}_c \in \boldsymbol{V}_H$, $\boldsymbol{v}_f \in \boldsymbol{V}_h(\omega_i)$, $w_c \in W_H$, and $w_f \in W_h(\omega_i)$.

Example of Local Solutions U

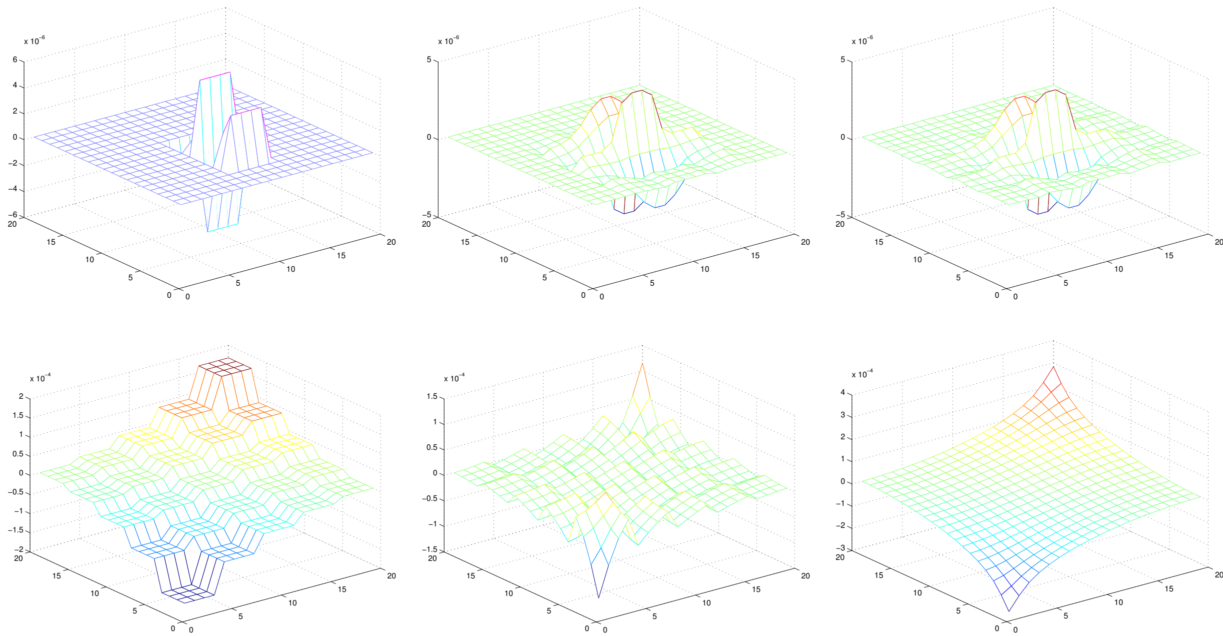


Figure 5: Above: 1, 2, and 3 layer patches, below: U_c, U_f, and, U using 3 layers of coarse elements.

Example of Local Solutions Σ

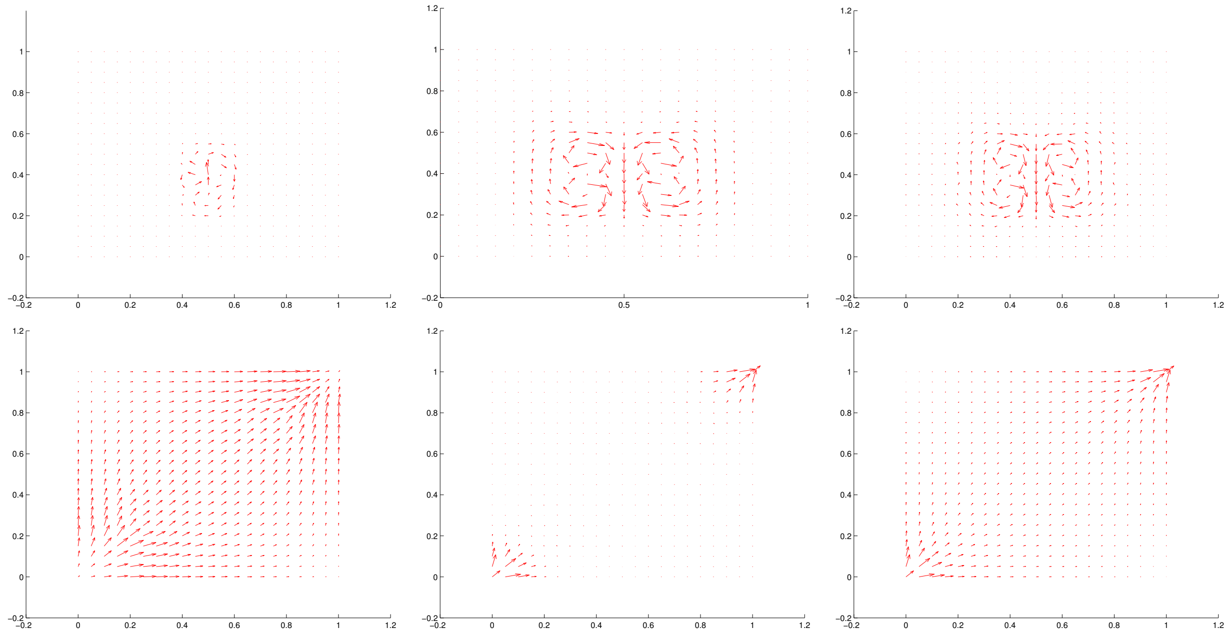


Figure 6: Above: 1, 2, and 3 layer patches, below: Σ_c, Σ_f, and, Σ using 3 layers of coarse elements.

Is the Method Conservative?

If we assume that we use the same local mesh size in all patches we can proceed with the following calculation,

$$
\begin{aligned}
-(\nabla \cdot \mathbf{\Sigma}, w_f) &= -(\nabla \cdot \mathbf{\Sigma}_f, w_f) \\
&= \sum_i -(\nabla \cdot \mathbf{\Sigma}_{f,i}, w_f) \\
&= \sum_i (f\psi_i, w_f) = (f, w_f),
\end{aligned}
$$

where w_f is piecewise constant on the fine mesh.

Energy Norm Estimate $\|v\|_a^2 = (\frac{1}{a}\boldsymbol{v}, \boldsymbol{v})$

Next we present an estimate of the error.

$$\|\boldsymbol{\sigma} - \boldsymbol{\Sigma}\|_a^2 \leq \sum_i C_a \|\frac{1}{a}(\boldsymbol{\Sigma}_c^i \boldsymbol{\phi}_i + \boldsymbol{\Sigma}_{f,i}) - \nabla U_{f,i}^*\|_{\omega_i}^2$$

$$+ \sum_i C_a \|h(f\psi_i + \nabla \cdot (\boldsymbol{\Sigma}_c^i \boldsymbol{\phi}_i + \boldsymbol{\Sigma}_{f,i}))\|_{\omega_i}^2.$$

$$+ \sum_i C_a \|\frac{1}{2\sqrt{h}} U_{f,i}^*\|_{\partial\omega_i \setminus \Gamma}^2$$

U^* is a postprocessed version of U.

Adaptive Strategy

- Calculate $\boldsymbol{\Sigma}$.
- Calculate the error indicators on each patch,

$$X_i(h) = \|\frac{1}{a}(\boldsymbol{\Sigma}_c^i \boldsymbol{\phi}_i + \boldsymbol{\Sigma}_{f,i}) - \nabla U^*_{f,i}\|^2_{\omega_i}$$

$$Y_i(h) = \|h(f\psi_i + \nabla \cdot (\boldsymbol{\Sigma}_c^i \boldsymbol{\phi}_i + \boldsymbol{\Sigma}_{f,i}))\|^2_{\omega_i}$$

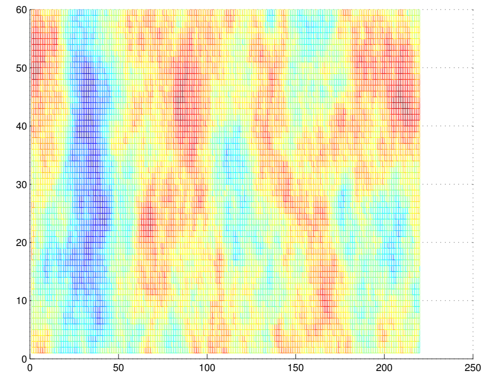
$$Z_i(L) = \|\frac{1}{2\sqrt{h}} U^*_{f,i}\|^2_{\partial\omega_i \setminus \Gamma}$$

Adaptive Strategy

- If indicators $X_i(h)$ or $Y_i(h)$ are big on a patch we decrease h.
- If indicator $Z_i(L)$ is big we increase the size of the patch.
- Go back to the first step or stop if the solution is good enough.

Numerical Examples

We use the SPE data of an oil reservoir. In the figure we see the top layer of the Tarbert formation.



We let $f = 1$ in the lower left corner and $f = -1$ in the upper right corner. $\max a / \min a = 2.9e2$

Numerical Examples

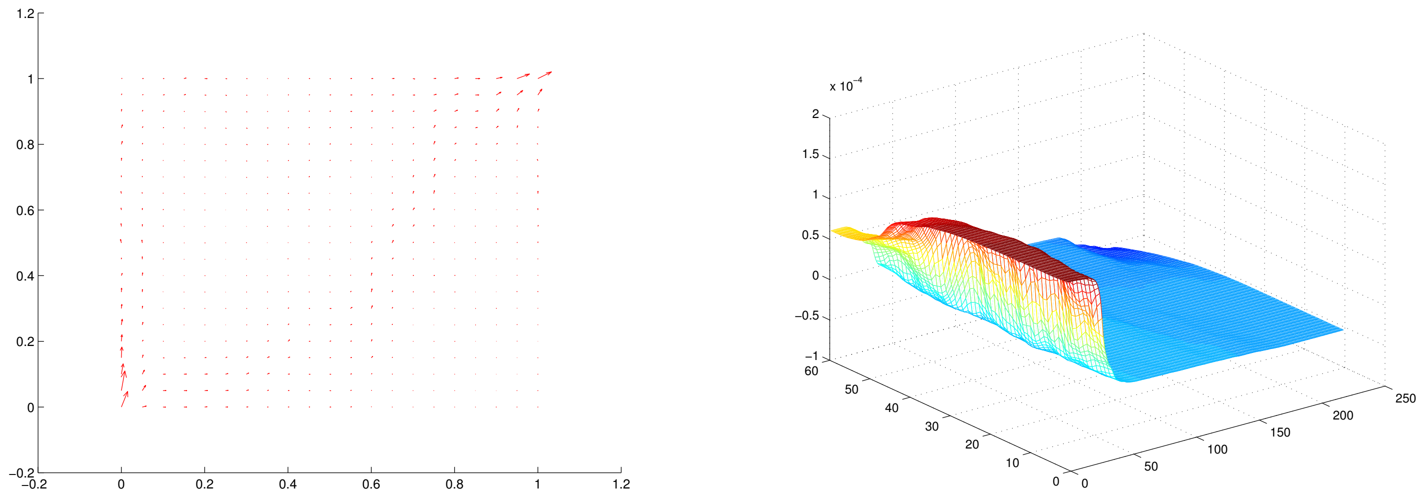


Figure 7: To the left we see the flux and to the right the pressure. We use 220×60 elements for the reference solution.

Numerical Examples

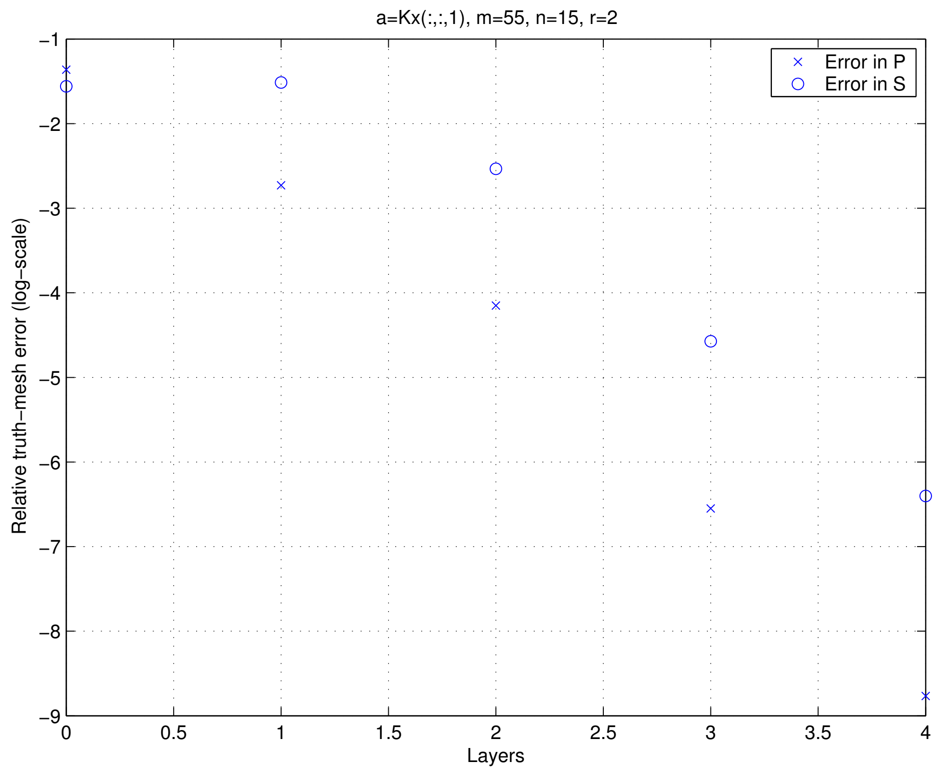
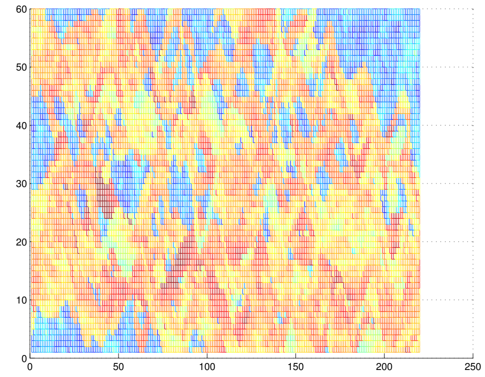


Figure 8: 55×15 coarse elements and $h = H/4$.

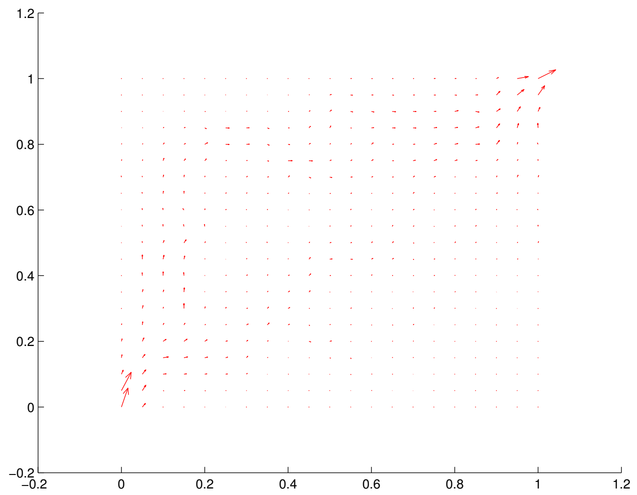
Numerical Examples

In the figure we see the bottom layer of the Upper Ness formation.



We let $f = 1$ in the lower left corner and $f = -1$ in the upper right corner. $\max a / \min a = 1.8e4$.

Numerical Examples



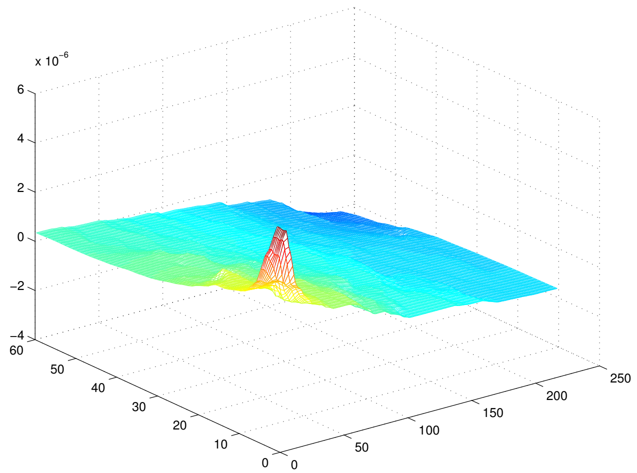


Figure 9: To the left we see the flux and to the right the pressure. We use 220×60 elements for the reference solution.

Numerical Examples

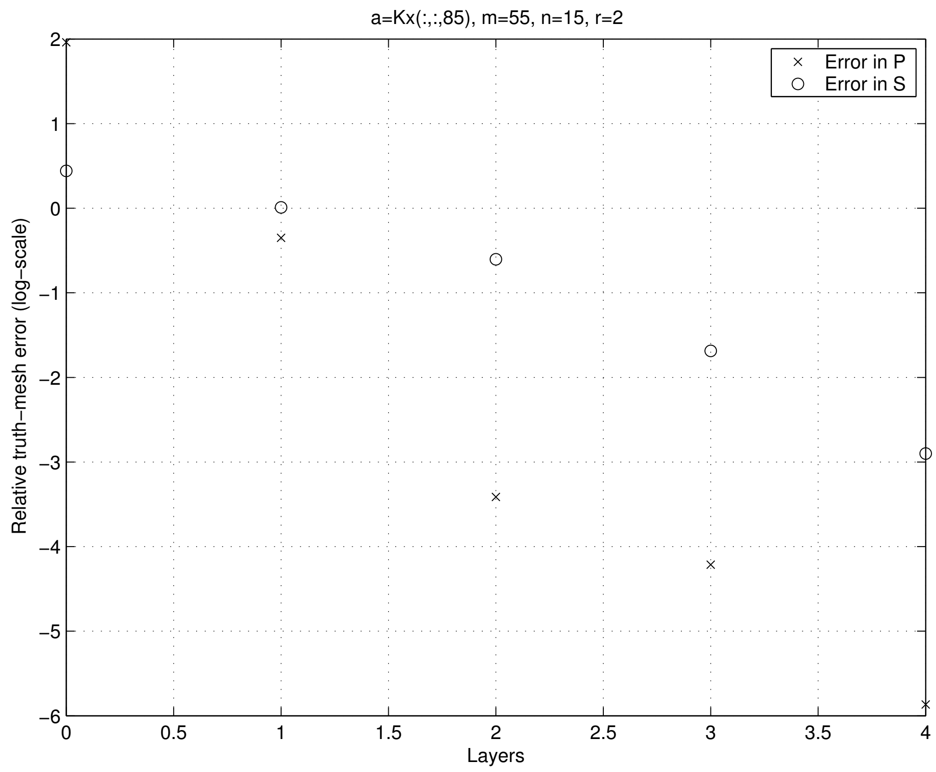


Figure 10: 55×15 coarse elements and $h = H/4$.

Applications

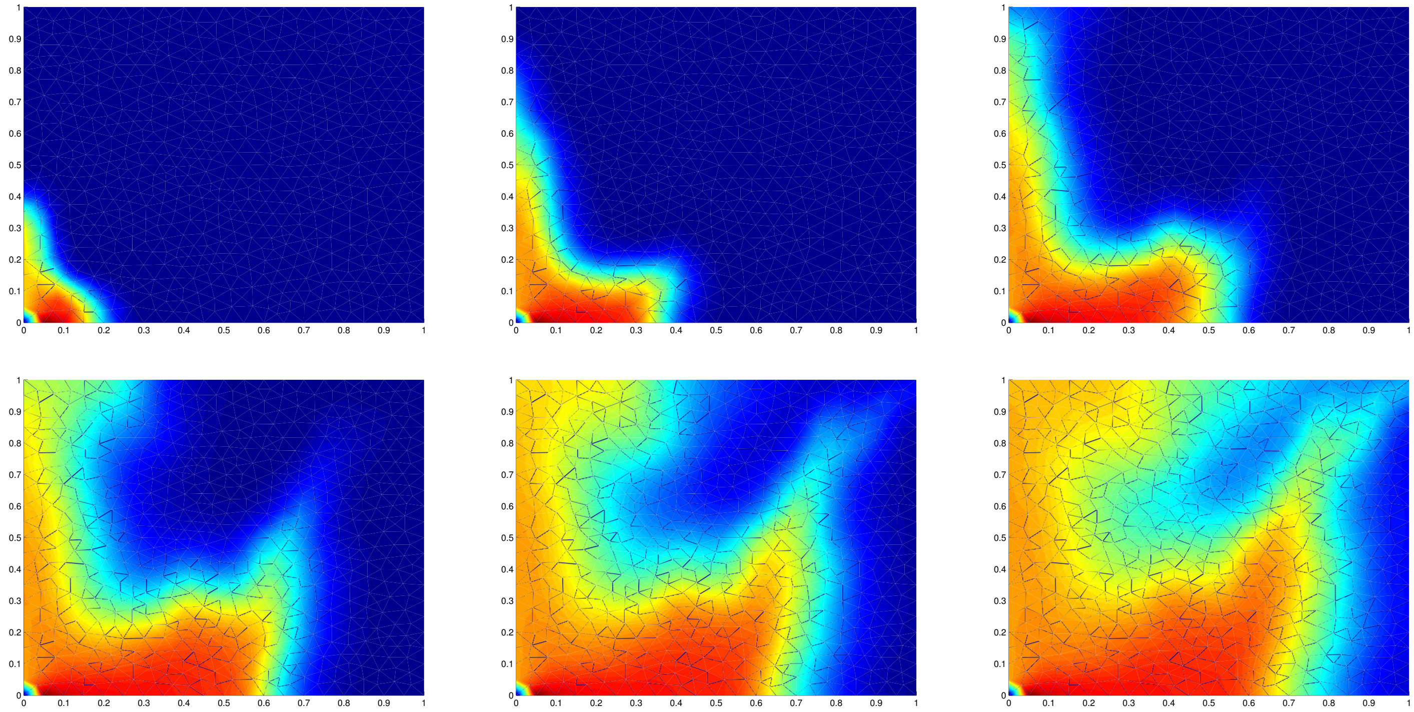
Given a good approximation of $\boldsymbol{\sigma}$ we can solve the following equation to simulate the water concentration in the well.

$$\left\{ \begin{aligned} \dot{c} + \nabla \cdot (\boldsymbol{\sigma} c) - \epsilon \triangle c = g \quad & \text{in } \Omega \times (0, T], \\ \partial_n c = 0 \quad & \text{on } \Gamma, \\ c = c_0 \quad & \text{for } t = 0, \end{aligned} \right.$$

We use cg1-cg1 with sd to solve the equation since ϵ is very small.

Applications

Water concentration at different times.



σ taken from top layer in Tarbert formation.

Outlook

- Numerical tests of the adaptive algorithm.
- Implementation in 3D.
- More scales.
- Multiscale approach for the transport problem.