" Complex geometry owes a lot to convex geometry, but it never pays anything back." (B. K.)

NOTES ON COMPLEX AND CONVEX GEOMETRY.

BO BERNDTSSON

1. Plan of the course, ambitious version

1. Convexity in $\mathbb{R}^n$. The Brunn-Minkowski theorem and Prekopa's theorem. The Legendre transform. The space of convex functions with the Mabuchi-Semmes metric. The Prekopa-Leindler theorem. Convex-concave functions and the von-Neumann min-max theorem.

2. Holomorphic vector bundles and their curvature. Positivity. Sheaves and direct images.

3. Positivity of direct images I (the case of trivial fibrations). Comparison with Brunn-Minkowski. The Legendre transform of a metric on a line bundle. The Bergman kernel and the inverse Legendre transform.

4. Some applications to complex analysis. Moser-Trudinger inequalities, the Suita conjecture, Kiselman's minimum principle, interpolation and the Santalo inequality.

5. The space of Kähler metrics. Applications of the positivity theorem to Kähler geometry: Geodesics and some uniqueness theorems.

6. Positivity of direct images II (non trivial fibrations). The Kodaira-Spencer class and the Gauss-Manin connection. Weil-Petersson metrics and the period map.

7. More applications (?)

2. Convexity in $\mathbb{R}^n$.

Let A_0 and A_1 be two *convex bodies* in $\mathbb{R}^n$, i e compact convex sets with non empty interior. Their *Minkowski sum* is then defined as

$$A_0 + A_1 = \{a_0 + a_1; a_j \in A_j, j = 0, 1\}. \tag{2.1}$$

For any measurable set in $\mathbb{R}^n$ we let $|A|$ denote its Lebesgue measure. The main theorem in convex geometry is

Theorem 2.1. *(The Brunn-Minkowski Theorem.)*

$$|A_0 + A_1|^{1/n} \geq |A_0|^{1/n} + |A_1|^{1/n}.$$

The B-M theorem was first proved by Brunn in 1887 for $n = 2$ and later generalized to arbitrary dimensions by Minkowski in 1896. It was further generalized by Lyusternik in 1935.

Theorem 2.2. *(Lyusternik) The same inequality holds for arbitrary nonempty compact sets.*

Remark: Notice that the assumption that A and B be nonempty is important: Since $A+\emptyset=\emptyset$ the B-M inequality can not hold if B is empty. It was pointed out to me by Christer Borell that this seemingly silly remark is actually quite important, see the proof of Theorem 2.8. □

We begin by giving a classical illustration of these results. Take $A_0 = A$ a compact set with say smooth boundary and take $A_1 = sB$, where B is the unit ball and s is a small number. Let $f(s) := |A + sB|$. Then

$$f(s) := |A + sB| = |A| + s|\partial A| + o(s)$$

(here $|\partial A|$ is the $n-1$-dimensional volume of the boundary of A). The B-M-theorem implies that

$$(d/ds)|_{s=0} f^{1/n} \geq |B|^{1/n},$$

so

$$\frac{|\partial A|}{|A|^{1-1/n}} \geq n|B|^{1/n}.$$

If $A = B$ we have equality here, since $B + sB = (1+s)B$ when B is convex. Thus

$$n|B|^{1/n} = \frac{|\partial B|}{|B|^{1-1/n}}$$

and we get

$$\frac{|\partial A|}{|A|^{1-1/n}} \geq \frac{|\partial B|}{|B|^{1-1/n}}.$$

This is the *isoperimetric inequality*, saying that of all sets with a given volume, the surface area is the smallest for a ball. Note that this follows from a very special case of the B-M theorem. We could define $f(s)$ in the same way, with B' a different convex body instead of B, and then define the 'surface area' of A as $f'(0)$. All such generalized surface area also satisfy an isoperimetric inequality, where the (arbitrary) convex body B' is the minimizer. This follows from exactly the same argument since we only used that B was convex.

We will now give an alternative formulation of the B-M theorem, and start with the following obvious consequence. Put $A_t = tA_1 + (1-t)A_0$ for t between 0 and 1. Then

$$t \to |A_t|^{1/n}$$

is concave. It follows from this that

$$|A_t| \geq \min(|A_0|, |A_1|). \tag{2.2}$$

Exercise Prove that (2.2) implies the B-M theorem. □

Theorem 2.3. *The function $t \to |A_t|$ is a polynomial (of degree n) for t between 0 and 1.*

This is a nontrivial fact that will be proved later. It is the starting point for the *Alexandrov-Fenchel* theorem, a far reaching generalisation of B-M. Let now $\mathcal{A}$ be a convex body in $\mathbb{R}^{n+1}$. For t in $\mathbb{R}$ we let $\mathcal{A}_t = \{x \in \mathbb{R}^n; (t,x) \in \mathcal{A}\}$ be the corresponding slice of $\mathcal{A}$, of course we will only care about t:s such that $\mathcal{A}_t$ is a convex body.

Theorem 2.4. *The function $t \to |\mathcal{A}_t|^{1/n}$ is concave on the interval where it is non zero.*

This theorem is equivalent to B-M: First, given A_0 and A_1 we may construct a convex body $\mathcal{A}$ in $\mathbb{R}^{n+1}$ such that $\mathcal{A}_t = A_t = tA_1 + (1-t)A_0$ for t between 0 and 1. Therefore Theorem 1.4 implies B-M. On the other hand, given $\mathcal{A}$ and two slices, say $\mathcal{A}_0$ and $\mathcal{A}_1$ we have that $t\mathcal{A}_1 + (1-t)\mathcal{A}_0 \subset \mathcal{A}_t$ if $\mathcal{A}$ is convex. Hence B-M implies that

$$|\mathcal{A}_t|^{1/n} \geq t|\mathcal{A}_1|^{1/n} + (1-t)|\mathcal{A}_0|^{1/n},$$

so Theorem 1.4 follows from B-M. The next obvious corollary is again equivalent to B-M, since it implies (1.2).

Corollary 2.5. $t \to \log|\mathcal{A}_t|$ *is concave.*

Remark: Even though Theorem 2.4 and its corollary are 'equivalent' to Theorem 2.1 in the sense that each of the statements is an easy consequence of the other, they are *philosophically* quite different. If we aim to generalize them, Theorem 2.1 suggests looking at situations where we have a notion of 'addition', like lattices or more general groups. On the other hand, Theorem 2.4 has a meaning as soon as we have a notion of 'convexity'. like e g holomorphic convexity or pseudoconvexity. In these notes we shall follow this latter route. □

The following function version of the corollary is fundamental for everything that follows.

Theorem 2.6. *(Prekopa's Theorem) Let* $\phi(t,x)$ *be a convex function in* $\mathbb{R}^{n+1}$. *Let*

$$\tilde{\phi}(t) = -\log \int_{\mathbb{R}^n} e^{-\phi(t,x)} dx, \tag{2.3}$$

or equivalently

$$e^{-\tilde{\phi}(t)} = \int_{\mathbb{R}^n} e^{-\phi(t,x)} dx. \tag{2.4}$$

Then $\tilde{\phi}$ *is convex or identically equal to* $-\infty$.

Prekopa's theorem is a generalization of the B-M theorem. To see this, it is convenient to allow convex functions to attain the value $+\infty$. Then convexity does no longer imply continuity (as it does for finite valued functions o open sets), and to fix ideas we will instead require our convex functions to be lower semicontinuous. As an example, if A is a convex body, we define its *indicator function* 1_A to be 0 on A and ∞ outside A. If we apply Prekopa's theorem to $\phi = 1_{\mathcal{A}}$ it is easily seen that we get Corolloray 1.5. In the proof of Prekopa's theorem that we give next, largely following Brascamp and Lieb, we will however first assume that ϕ is finite valued and even smooth. (The general case follows, since we can write a general convex ϕ as an increasing limit of nice ϕ:s.) The proof goes at follows:

Note first that we may assume that $n = 1$. This is most easily seen from the definition in formula (2.4). If we first carry out the integration with respect to x_n, and if we know the theorem for $n = 1$, this gives us a convex function $\hat{\phi}$ on depending on $(t, x_1, ...x_n)$ defined by

$$e^{-\hat{\phi}(t,x_1,...x_{n-1})} = \int_{\mathbb{R}} e^{-\phi(t,x)} dx_n.$$

Then we just iterate, integrating with respect to x_{n-1} and so on. We will compute the second derivative of $\tilde{\phi}$ and want to prove that it is nonnegative. For simplicity we choose $t = 0$ and we may assume (adding a constant to ϕ that $\tilde{\phi}(0) = 0$, and (adding a linear function of t to ϕ) that $\tilde{\phi}'(0) = 0$. This means that

$$\int_{\mathbb{R}^n} e^{-\phi(0,x)} dx = 1$$

and

$$\int_{\mathbb{R}^n} \dot{\phi}(0,x) e^{-\phi(0,x)} dx = 0.$$

We use dots to indicate derivatives with respect to time and doubledots for second order time derivatives. A direct computation gives

$$\tilde{\phi}''(0) = \int_{\mathbb{R}^n} \left(\ddot{\phi}(0,x) - \dot{\phi}^2(0,x) \right) e^{-\phi(0,x)} dx. \tag{2.5}$$

The main ingredient in the proof is the following lemma. It is known as the Brascamp-Lieb Theorem and it can be viewed as a real-variable analog of Hörmander's L^2-estimates for $\bar{\partial}$.

Lemma 2.7. *Let ψ be a smooth strictly convex function on the real line with $e^{-\psi}$ integrable on $\mathbb{R}$, and let u be a function such that*

$$\int_{\mathbb{R}} u^2 e^{-\psi} dx < \infty,$$

and

$$\int_{\mathbb{R}} u e^{-\psi} dx = 0. \tag{2.6}$$

Then

$$\int_{\mathbb{R}} u^2 e^{-\psi} dx \leq \int_{\mathbb{R}} \frac{(u')^2}{\psi''} e^{-\psi} dx. \tag{2.7}$$

Proof. We first prove a dual estimate. Let

$$d^\psi v = e^\psi (d/dx)(v e^{-\psi});$$

it is the negative of the formal adjoint of d/dx for the weighted L^2 scalar product defined by $e^{-\psi}$. This means that

$$\int (du/dx) v e^{-\psi} dx = - \int u d^\psi v e^{-\psi} dx,$$

if v has compact support, which follows from integration by parts.

Let v be smooth of compact support. Then we claim that

$$\int \psi'' v^2 e^{-\psi} dx + \int (dv/dx)^2 e^{-\psi} dx = \int (d^\psi v)^2 e^{-\psi} dx.$$

In particular

$$\int \psi'' v^2 e^{-\psi} dx \leq \int (d^\psi v)^2 e^{-\psi} dx. \tag{2.8}$$

To prove this we will use the fundamental commutator formula

$$(d/dx) d^\psi v + \psi'' v = d^\psi (d/dx) v$$

which will reappear in many quises later. Its proof follows from noting that

$$d^\psi v = v' - \psi' v.$$

Then

$$\int (d^\psi v)^2 e^{-\psi} dx = -\int ((d/dx)d^\psi v) v e^{-\psi} dx = -\int (d^\psi (d/dx) v) v e^{-\psi} dx + \int \psi'' v^2 e^{-\psi} dx$$

$$= \int \psi'' v^2 e^{-\psi} dx + \int (dv/dx)^2 e^{-\psi} dx.$$

To prove (2.7) we next solve

$$d^\psi v = u,$$

by

$$v(x) e^{-\psi} = \int_{-\infty}^x u e^{-\psi}.$$

We may assume that u has compact support (check !) and it then follows from (2.6) that v also has compact support. Then

$$\int u^2 e^{-\psi} dx = \int (d^\psi v) u e^{-\psi} dx = -\int v u' e^{-\psi} dx.$$

By Cauchy's inequality

$$\int u^2 e^{-\psi} dx \leq \left(\int \psi'' v^2 e^{-\psi} dx \int (u')^2/\psi'' e^{-\psi} dx \right)^{1/2}.$$

By (2.8) this is dominated by

$$\left(\int (d^\psi v)^2 e^{-\psi} dx \int (u')^2/\psi'' e^{-\psi} dx \right)^{1/2} = \left(\int u^2 e^{-\psi} dx \int (u')^2/\psi'' e^{-\psi} dx \right)^{1/2}$$

so (2.7) follows. □

From here we get Prekopa's theorem. By (2.5) and the Brascamp-Lieb estimates we get

$$\tilde\phi''(0) \geq \int_{\mathbb{R}^n} \left(\ddot\phi(0,x) - (\phi''_{t,x}(0,x))^2/\phi''_{x,x} \right) e^{-\phi(0,x)} dx =$$

$$\int_{\mathbb{R}^n} \left(\phi''_{x,x} \ddot\phi(0,x) - (\phi''_{t,x})^2(0,x) \right) / \phi''_{x,x} e^{-\phi(0,x)} dx.$$

We then note that the numerator

$$\phi''_{x,x} \ddot\phi - (\phi''_{t,x})^2$$

is just the determinant of the Hessian of ϕ (with respect to x and t). Since ϕ is convex this is nonnegative, and we are done. □

Remark With a little bit more work one can carry out the same proof for general n without using induction. One then essentially has to replace d^ψ by (the negative of) the formal adjoint of d, the exterior derivative of a function on $\mathbb{R}^n$. (The main complication that arises is to solve the equation $d^\psi v = u$ with v of compact support, if u satisfies (2.6).) One then gets the following bound for the second derivative of $\tilde{\phi}$

$$\tilde{\phi}'' \geq \int_{\mathbb{R}^n} \frac{MA_{t,x}(\phi)}{MA_x(\phi)} e^{-\phi} dx.$$

Here MA(ψ) stands for the Monge-Ampere operator applied to ψ, i e the determinant of the Hessian of ψ, and the indices mean that we take the Monge-Ampere with respect to x or both x and t, respectively. The quotient

$$\frac{MA_{t,x}(\phi)}{MA_x(\phi)}$$

will play an important role in the next section.

Remark: The above proof is by no means the simplest approach to Prekopa or B-M. Nevertheless it is an interesting proof since it connects in many other directions. Below we give a very simple proof of Prekopa's thm in one variable that I learnt from Bo'az Klartag.

Theorem 2.8. *Let f, g and h be nonnegative functions on the real line satisfying*

$$h((x+y)/2) \geq \min(f(x), g(y))$$

for all x and y in $\mathbb{R}$. Assume moreover that $\max(f) = \max(g) = M$. *Then*

$$\int h dx \geq (\int f dx + \int g dx)/2 \geq \sqrt{\int f dx \int g dx}.$$

Proof. First note that for any $t \geq 0$

$$(1/2)\{f > t\} + (1/2)\{g > t\} \subset \{h > t\},$$

since if $f(x) > t$ and $g(y) > t$ then our assumption implies $h((x+y)/2) > t$. If $t < M$ both sets in the left hand side are nonempty, so by one dimensional Brunn-Minkowski (see below)

$$2|\{h > t\}| \geq |\{f > t\}| + |\{g > t\}|. \tag{2.9}$$

Now use the classical fact that for any nonnegative function k

$$\int_0^\infty |\{k > t\}| dt = \int \int_{k(x) > t} dx dt = \int k(x) dx.$$

Applying this to f, g and h in the previous inequality we get

$$2 \int h dx \geq 2 \int_0^M |\{h > t\}| dt \geq \int_0^\infty |\{f > t\}| dt + \int_0^\infty |\{g > t\}| dt = \int f dx + \int g dx.$$

□

In this proof we have used the Brunn-Minkowski theorem in one variable. Let us now apply Theorem 2.8 to $f = e^{-\phi_0}$, $g = e^{-\phi_1}$, where we moreover assume that $\min(\phi_0) = \min(\phi_1)$. If we assume ϕ_0 and ϕ_1 to be convex, their sublevel sets, i e the superlevel sets of f and g are intervals. In this case, the Brunn-Minkowski theorem is trivial. Thus we get a proof of Prekopa's theorem by letting $h = e^{-\phi_{1/2}}$, if $\phi_t(x)$ is convex with respect to t and x jointly. (Our extra assumption that the minima of ϕ_j are equal can be dispensed with by adding a linear function of

t.) In fact, Theorem 2.8 is much stronger; it also implies the *Prekopa-Leindler Theorem*, that we will encounter in the next section.

We conclude this section with a cute inequality that follows from Prekopa's (add ref!).

Theorem 2.9. *Let ϕ be convex on $\mathbb{R}^n$ and assume*

$$\int e^{-\phi} dx = 1.$$

Let

$$\hat{\phi} = \int \phi e^{-\phi} dx.$$

Then

$$\int (\phi - \hat{\phi})^2 e^{-\phi} dx \leq n.$$

Proof. Let $\Phi(t, x) := t\phi(x/t)$ for $t > 0$. This is a convex function of (t, x) for any convex ϕ. Indeed, this is clearly so if ϕ is affine, and therefore holds in general since any convex function is the supremum of a family of affine functions. By Prekopa

$$f(t) := \log \int e^{-\Phi(t,x)} dx$$

is concave. But, changing varaibles $x = ty$ in the integral,

$$f(t) = \log t^n + \int e^{-t\phi(y)} dy.$$

The theorem follows from $f''(1) \leq 0$. □

Notice that if ϕ is homogenous of order 1, $\Phi(t, x) = \phi(x)$ so f is constant then. Hence equality holds for any function that is homogenous of order 1.

3. Affine strucures on the space of convex bodies and convex functions.

We start by looking more closely at the notion of Minkowski sum. This operation introduces a kind of affine structure on the space of convex bodies. We have already seen that there is a natural imbedding of the space of convex bodies into CVX, the space of convex functions on $\mathbb{R}^n$, namely

$$A \to 1_A.$$

We have a natural affine structure on CVX simply by adding two convex function. This however does not correspond to Minkowski addition since

$$1_{A_0} + 1_{A_1} = 1_{A_0 \cap A_1} \neq 1_{A_0 + A_1}.$$

This can be remedied by appealing to **the most important concept in convex geometry**.

Definition 3.1. Let ϕ be a function on $\mathbb{R}^n$. Then its **Legendre transform** is the function

$$\phi^*(p) := \sup_{x \in \mathbb{R}^n} p \cdot x - \phi(x). \tag{3.1}$$

Notice that, as the supremum of a family of convex (in fact, affine) functions, the Legendre transform is always convex (and lower semicontinuous). It is also defined for functions defined only on a subset of $\mathbb{R}^n$, since we can extend such functions to global functions by ∞ on the complement of the set where it is defined. Equivalently, we take sup only over the domain of definition of ϕ. It turns out that on convex functions, the Legendre transform is involutive:

Theorem 3.2. $\phi^{**} = \phi^\circ$, *the supremum of all finite valued convex functions* $\psi \leq \phi$.

Proof. By the Hahn-Banach Theorem ϕ° is also the supremum of all affine functions smaller that ϕ,

$$\phi^\circ(x) = \sup_{\{(p,a); p\cdot y + a \leq \phi(y), y \in \mathbb{R}^n\}} p \cdot x + a.$$

When is $p \cdot y + a \leq \phi(y)$ for all y? This is the case exactly when

$$-a \geq \phi^*(p).$$

Hence

$$\sup_{\psi \leq \phi} \psi(x) = \sup_{a \leq -\phi^*(p)} p \cdot x + a = \sup_p p \cdot x - \phi^*(p) = \phi^{**}(x).$$

□

As an example of this we take $\phi = 1_A$ for a convex body A. Then

$$1_A^*(p) = \sup_{x \in A} p \cdot x =: H_A(p).$$

The function H_A is the support function of A and it follows from the theorem that its Legendre transform is 1_A. Now note that

$$H_{A_0 + A_1}(p) = \sup_{x_0 \in A_0, x_1 \in A_1} p \cdot (x_0 + x_1) = H_{A_0}(p) + H_{A_1}(p),$$

so Minkowski addition is reflected as the addition of support functions. Thus we get a new affine structure on CVX as

$$\phi_0 \oplus \phi_1 := (\phi_0^* + \phi_1^*)^*,$$

and this affine structure extends Minkowski addition under the embedding $A \to 1_A$.

This leads up to the problem of describing $\phi_0 \oplus \phi_1$ directly.

Proposition 3.3.

$$\phi_0 \oplus \phi_1(x) = \inf_{x_0 + x_1 = x} \phi_0(x_0) + \phi_1(x_1) := \phi_0 \star \phi_1,$$

which is called the infimal convolution of ϕ_0 *and* ϕ_1.

Proof. Call the right hand side in the proposition ψ. Then

$$\psi^*(p) = \sup_x \sup_{x_0 + x_1 = x} p \cdot (x_0 + x_1) - \phi_0(x_0) - \phi_1(x_1) = \phi_0^*(p) + \phi_1^*(p).$$

Accepting for a moment that ψ is convex, the proposition follows by taking Legendre transforms again. □

The missing piece in the proof follows from the following fact, the *minimum principle* for convex functions.

Proposition 3.4. *Let $\Phi(x,y)$ be convex on $\mathbb{R}^n \times \mathbb{R}^m$. Then*

$$\inf_y \Phi(x,y)$$

is convex. In particular, the infimal convolution of two convex functions

$$\phi_0 \star \phi_1(x) = \inf_y \phi_0(x-y) + \phi_1(y)$$

is convex.

Proof. A function $F(z)$ is convex if and only if its epigraph $\{(z,t); t > F(z)\}$ is convex. The epigraph of $\inf_y \Phi(x,y)$ is the image of the epigraph of Φ under the linear map $(x,y) \to x$, thus convex. □

Remark: It is worth noticing that the minimum principle also follows from Prekopa's theorem. Indeed, Prekopa's theorem implies that

$$x \to -\log \int e^{-p\Phi(x,y)-|y|^2} dy$$

is convex for all $p > 0$. We get the minimum principle if we divide by p and then let p tend to ∞. This proof is certainly more complicated than the one based on epigraphs, but we will see later that it generalizes nicely to the complex setting. □

Remark: The formula for the infimal convolution as

$$\phi_0 \star \phi_1(x) = \inf_y \phi_0(x-y) + \phi_1(y),$$

suggests an analogy with the ordinary convolution. This can be substantially extended to include an interesting analogy between the Legendre transform and the Fourier-Laplace transform. Let $f_{0,1} = e^{-\phi_{0,1}}$. Then of course addition of convex functions corresponds to multiplication of f_0 and f_1. We can think of $e^{\sup -\phi}$ as an approximation of $\int f dx$. Then, formally, the Fourier-Laplace transform

$$\int e^{p\cdot x - \phi(x)} dx$$

is approximated by the (exponential of the) Legendre transform e^{ϕ^*}. Similarily, the convolution

$$\int f_0(x-y) f_1(y) dy$$

corresponds to the infimal convolution of ϕ_0 and ϕ_1. Proposition 3.2 reflects the classical fact that the Fourier-Laplace-tranform of a product is the convolution of the Fourier-Laplace transforms. □

Somewhat more generally we can look at the linear segment

$$\psi_t = (1-t)\phi_1^* + t\phi_0^*$$

betwen the Legendre transforms of two functions. One can then verify in a similar way that

$$\psi_t^*(x) = \inf_{(1-t)x_0 + tx_1 = x} (1-t)\phi_0(x_0) + t\phi_1(x_1) := I_t(\phi_0, \phi_1). \tag{3.2}$$

Hence I_t defines an interpolating family between two convex functions, which corresponds to the naive interpolation (convex combination) under the Legendre transform.

Exercise: Prove (3.1) and check that if $\phi_j = 1_{A_j}$ for $j = 1, 2$, then $I_t(\phi_0, \phi_1) = 1_{A_t}$. □

We shall now go one step further and show that these two interpolating families, the 'naive' and the 'sophisticated' ones, can be seen as geodesics for two different metrics on the space of convex functions. For this we need to restrict our attention to the space CVX^2, by which we mean the twice continuously differentiable strictly convex functions. This is an open subset of a Banach space (C^2), so it is a Banach manifold. The tangent space of C^2 is of course C^2 itself (at any point), but in order to get a nice tangent space for CVX^2 we *define* it to be C^2_c, functions in C^2 of compact support. We the define a Hilbert norm on the tangent space to be just the L^2-norm

$$\|\chi\|^2 := \int_{\mathbb{R}^n} \chi^2 dx.$$

Since this norm is independent of the point in CVX^2 it is at least intuitively reasonable that geodesics for this metric are just linear segments $(1-t)\phi_0 + t\phi_1$. To see what the 'sophisticated' paths correspond to we need to take a closer look at the Legendre transform.

Recall that

$$\phi^*(p) = \sup_x p \cdot x - \phi(x).$$

If ϕ is in CVX^2 and the sup is attained, it must be attained at a unique point x satisfying $p = \partial\phi(x) := p(x)$.

Proposition 3.5. *The image of $\mathbb{R}^n$ under $x \to \partial\phi(x)$ is an open set. It is precisely the set where the supremum in (3.1) is attained and it is the interior of the set where ϕ^* is finite.*

Proof. The first statement is clear since the derivative of the map $x \to \partial\phi(x)$ is invertible. We have just seen that if the supremum, for a given p is attained, then p lies in the image $R(\partial\phi)$ of this map. Conversely, if $p_0 = \partial\phi(x_0)$, concavity of $x \to p_0 \cdot x - \phi(x)$ implies that the supremum is attained at x_0. Then cl;early $\phi^*(p_0)$ is finite. Since the image is open, p_0 must in fact lie in the interior of the set where ϕ^* is finite. Conversely, assume p_0 lies in the interior of the set where ϕ^* is finite. We may assume that $p_0 = 0$. Then

$$\sup_x p \cdot x - \phi(x) < \infty$$

for $|p|$ smaller than some positive ϵ. Since this is a convex function of p, hence continuous, we get a uniform bound

$$\sup_x p \cdot x - \phi(x) \le C$$

for $|p| \le \epsilon/2$. Taking sup over such p we find

$$\phi(x) \ge -C + \epsilon|x|/2.$$

Hence ϕ tends to infinity at infinity so it must have a minimum where the derivative vanishes. □

Since ϕ is strictly convex, the hessian is invertible, so this map is locally inverible. Call the inverse $x = x(p)$. Then locally

$$\phi^*(p) = p \cdot x(p) - \phi(x(p))$$

so ϕ^* is at least of class C^1. We have

$$\phi^*(p) + \phi(x) = p \cdot x \tag{3.3}$$

when $p = \partial\phi(x)$, and only then. If we repeat the same argument for ϕ^* we see that (3.3) holds when $x = \partial\phi^*(p)$. Therefore $x = \partial\phi^*(p)$ implies that $p = \partial\phi(x)$ so $\partial\phi \circ \partial\phi^*(p) = p$. Hence $\partial\phi$ and $\partial\phi^*$ are inverse transformations, so ϕ^* is also of class C^2 and strictly convex, at least in the image of $\mathbb{R}^n$ under $x \to \partial\phi(x)$, i e in the interior of the set where it is finite. Next we record a strking consequence of this.

Corollary 3.6. *The image $R(\partial\phi)$ is a convex set and if $\phi_j \in CVX^2$ for $j = 1, 2$, then*

$$R(\partial\phi_0) + R(\partial\phi_1) = R(\partial(\phi_0 + \phi_1))$$

.

Proof. The first statement follows since the set where the convex function ϕ^* is finite is convex. The second statement follows since the Legendre transform of $\phi_0 + \phi_1$ equals $\inf_{x_0+x_1=x} \phi_0^*(x_0) + \phi_1^*(x_1)$. Therefore the set where $(\phi_0 + \phi_1)^*$ is finite is the Minkowski sum of the corresponding sets for ϕ_0 and ϕ_1. □

Before continuing we give a few more interesting consequences of Propositions 3.4 and 3.5.

3.1. **Interlude: Minkowski's second inequality and the Alexandrov-Fenchel's theorem.** Let us start with the following observation:

Proposition 3.7. *Let A be a convex body and ϕ be a strictly convex function of class C^2 with $R(\partial\phi) = A^\circ$. Then*

$$|A| = \int_{\mathbb{R}^n} MA(\phi)(x)dx.$$

Proof. This follows from changing variables

$$|A| = \int_{A^\circ} dp = [p = \partial\phi(x)] = \int_{R^n} MA(\phi)dx.$$

□

Corollary 3.8. *Let A_1, ... A_n be convex bodies. Then*

$$p(t) := |t_1 A_1 + ... t_n A_n|$$

is a homogenous polynomial of degree n in t if $t_j \geq 0$.

Proof. We may assume that A_j are smoothly bounded. Then we can find smooth convex functions ϕ_j, with $R(\partial\phi_j) = A_j^\circ$. For this it suffices to take smooth strictly convex functions in A_j

that tend to infinity at the boundary, extend them by ∞ outside of A_j, and take their Legendre transforms. Then by Proposition 3.6 and Corollary 3.5

$$|t_1A_1 + ...t_nA_n| = \int_{\mathbb{R}^n} MA(t_1\phi_1 + ...t_n\phi_n)dx. \tag{3.4}$$

This is a homogenous polynomial of degree n in t. □

Definition 3.9. The *mixed volume* $V(A_1, ...A_n)$ of A_1, $...A_n$ is the coefficent of $t_1...t_n$ in the polynomial $p(t)$. It is denoted $V(A_1, ...A_n)$.

To get a feeling for what the mixed volumes are and how they behave we write (3.4) as

$$|t_1A_1 + ...t_nA_n| = \int_{\mathbb{R}^n} \det(t_1H(\phi_1) + ...t_nH(\phi_n))dx, \tag{3.5}$$

where $H(\phi_j)$ is the Hessian of ϕ_j.

Definition 3.10. Let for $j = 1, ...n$ H_j be $n\times n$ matrices. Then the *mixed discrimant* $D(H_1, ...H_n)$ of this n-tuple of matrices is the coefficient of $t_1...t_n$ in the polynomial

$$q(t) = \det(t_1H_1 + ...t_nH_n).$$

Thus the mixed volume is an integral of the mixed discriminant of the Hessians of the convex functions ϕ_j. For calculations with mixed discriminants it is convenient to use a little bit of exterior algebra.

Lemma 3.11. *Let $e_1, ...e_n$ be a basis for $\mathbb{R}^n$ and let $e_1^*, ...e_n^*$ be the dual basis. Associate to any linear map H from $\mathbb{R}^n$ to itself, the form*

$$\hat{H} := \sum H_{j,k} e_j \wedge e_k^*,$$

where $(H_{j,k})$ is the matrix representing H in the basis e_j. Then

$$\hat{H}^n = n! \det(H)dV,$$

where $dV = e_1 \wedge e_1^*...e_n \wedge e_n^*$.

(It can easily be checked that $\hat{H}$ does not depend on the choice of basis and neither does dV (consequently).) This can be seen by diagonalising the matrix H. (Don't worry that this is not always possible. We will use only symmetric matrices, like hessians, and besides, the set of matrices that can be diagonalised is dense in the set of all matrices.) From the lemma we see

Proposition 3.12. *We have*

1.$D(H_1, ...H_n)dV = \hat{H}_1 \wedge ...\hat{H}_n$,

2. The mixed discriminant is symmetric in the arguments H_j and linear with respect to each of them.

3. $D(H, , , , H) = n! \det(H)$.

Formula (3.5) now shows that the mixed volume have similar properties.

Proposition 3.13. *The mixed volume satisfies*

1. $V(A_1, ...A_n)$ is symmetric in the arguments A_j and linear (under Minkowski summation) in each of them.

2. $V(A) := V(A, ...A) = n!|A|$.

We now use formula (3.5) to compute the volume of $A + tB$. The result is known as Steiner's formula.

Proposition 3.14.

$$V(A + tB) = \sum_0^n t^k \binom{n}{k} V_k(B, A),$$

where

$$V_k(B, A) = V(B, ..B, A, ...A),$$

and we take B k times and A $n - k$ times.

Remark: When B is the unit ball, $A + tB$ equals A_t, the set of points with distance at most t to A. Thus we get, for convex bodies, a power series expansion of the volume of A_t. Remarkably, this expansion has only finitely many terms, so the volume function is in particular real analytic, regardless of how smooth the boundary of A is. It is worthwhile looking at the example of a convex polytope to understand this, e g a triangle in $\mathbb{R}^2$. The coefficents, $V_k(B, A)$, are called *queermasses*. In the polytope case they are related to volumes of faces of the polytope of dimension k. □

Now note that it follows from the B-M theorem that if

$$r(t) = V(A + tB) = V((1 - t)A + t(A + B))$$

$r^{1/n}$ is concave. This implies that its second derivative at 0 is nonpositive, or in other words that

$$r(0)r''(0) \leq (1 - (1/n))r'(0)^2.$$

Using Steiner's formula we get

Theorem 3.15. *(Minkowski's second inequality):*

$$V_1(B, A)^2 \geq V(A)V_2(B, A).$$

This has a nice interpretation: The quadratic form

$$Q(x, y) := V(xA + yB, xA + yB, A, ...A)$$

is indefinite. This follows if we note that the previous theorem says that Q:s determinant is nonpositive. The final theorem of this section is a far reaching generalization of these facts.

Theorem 3.16. *(The Alexandrov-Fenchel theorem.) Let A , B and $A_3, ...A_n$ be convex bodies. Then*

$$V(B, A, A_3...A_n)^2 \geq V(A, A, A_3....A_n)V(B, B, A_3...A_n).$$

Equivalently, the quadratic form

$$Q(x, y) := V(xA + yB, xA + yB, A_3, ...A_n)$$

is indefinite.

We will not give the proof of this theorem. The original proof was based on approximation by polyhedrons and combinatorics. A later proof by Alexandrov, based on ideas by Hilbert to prove Minkowski's second inequality, uses the convex functions ϕ_j to construct certain elliptic operators and estimate their eigenvalues. It is also worth mentioning that related arguments have been used by Khovanski and Teissier to prove a generalization of *Hodge's index theorem*. Maybe we will come back to that later.

3.2. **Back to business: The space of convex functions as a Riemannian manifold.** Recall that CVX^2 is the space of C^2-smooth strictly convex functions. It is an open subset of the space of all C^2-functions, so it is a Frechet manifold. We define formally its tangent space at any point as C^2_c, the space of compactly supported C^2-functions. Then, for any ϕ in the space, and any tangent vector χ, $\phi + s\chi$ stays in the space if s is small. Let us now for simplicity consider the subspace Ω of functions ϕ such that $R(\partial)$ is all of $\mathbb{R}^n$. (Need to elaborate on this). Then as we have seen the Legendre transform of ϕ lies in the same space, so the Legendre transform is a map from Ω to itself, which we call L, so that $\phi^* = L(\phi)$. We shall now compute its differential.

Proposition 3.17.

$$dL_\phi.\chi = \eta$$

where $\eta(p) = -\chi(x)$ *for* $p = \partial\phi$. *In other words*

$$\eta(p) = -\chi(\partial\phi^*(p)), \quad \eta(\partial\phi(x)) = -\chi(x).$$

Proof. First, the formula makes sense. χ is by definition a compactly supported function, so $\eta(p) = \chi \circ \partial\phi^*(p)$ is also compactly supported since the gradient map is a diffeomorphism from $\mathbb{R}^n$ to itself. (Hmm, C^1 or C^2? Maybe C^∞ is better?) By definition

$$dL_\phi.\chi = (d/ds)|_{s=0} L(\phi + s\chi).$$

Now recall that by (3.3)

$$L(\phi)(\partial\phi(x)) + \phi(x) = x \cdot \partial\phi(x)$$

for all x. Similarily

$$L(\phi + s\chi)(\partial\phi(x) + s\partial\chi(x)) + \phi(x) = x \cdot \partial\phi(x) + sx \cdot \partial\chi(x).$$

Identifying terms of order 1 in s we get

$$dL_\phi.\chi(\partial\phi(x)) + \partial L(\phi)(\partial\phi(x)) \cdot \partial\chi(x) + \chi(x) = x \cdot \partial\chi(x).$$

But $L(\phi) = \phi^*$ and $\partial\phi^* \circ \partial\phi(x) = x$. Hence

$$dL_\phi.\chi(\partial\phi(x)) + \chi(x) = 0$$

so we are done. □

Now consider the Riemannian metric on Ω introduced earlier

$$\|\chi\|^2 = \int_{\mathbb{R}^n} \chi^2 dx. \tag{3.6}$$

Changing variables in the integral by $x = \partial\phi^*(p)$, and noting that dx transforms to $\mathrm{MA}(\phi^*)dp$, we find that

$$\|\chi\|^2 = \int_{\mathbb{R}^n} (-\chi(\partial\phi^*(p))^2 MA(\phi^*)dp = \int_{\mathbb{R}^n} (dL_\phi.\chi)^2 MA(\phi^*)dp.$$

By Proposition 3.13 this means that the Legendre transform is an isometry between our 'trivial' Riemannian metric and a new metric

$$\|\eta\|^2_{M-S} = \int_{\mathbb{R}^n} (\eta)^2 MA(\phi^*)dp.$$

This is the *Mabuchi-Semmes metric*. It was discovered independently by T Mabuchi (in a similar complex setting that we will get to later) and S Semmes. Since curves $t \to t\phi_1 + (1-t)\phi_0$ are geodesics for the trivial metric, the curves

$$\psi_t^*(x) = \inf_{(1-t)x_0 + tx_1 = x} (1-t)\phi_0(x_0) + t\phi_1(x_1) := I_t(\phi_0, \phi_1).$$

from formula (3.2) are geodesics for the Mabuchi-Semmes metric. Amazingly, one can show that the geodesic equation for the Mabuchi-Semmes metric is

$$MA_{t,x}(\phi(t,x)) = 0.$$

Notice that this is precisely what came up in connection with our dicussion of Prekopa's theorem; the lower bound for the second derivative of $\tilde{\phi}$ was expressed as an integral involving $MA_{t,x}(\phi(t,x))$.

By construction the Mabuchi-Semmes metric for convex functions is just the trivial metric after Legendre transformation, so it is in particular flat. Later on we will define an analogous metric on spaces of plurisubharmonic functions, or positively curved metrics on line bundles (this is in fact what Mabuchi did). In that case there is no obvious counterpart of the Legendre transform to linearize the metric, and in fact the Mabuchi-Semmes metric has strictly negative curvature in the complex setting. This is one very major difference between the real and complex cases that will be of central importance in the sequel.

4. Beginning of discussion of the complex case. Basic notions and failure of trivial analogies of Prekopa's theorem.

Going back to the remark after Corollary 2.5 we will now first investigate one conceivable generalization of Prekopa's theorem. Recall that a complex counterpart of a convex function of one real variable is a subharmonic function of a complex variable. By definition, a smooth function u of one complex varaible is subharmonic if $\Delta u \geq 0$. More generally, a not necessarily smooth u is subharmonic if it is upper semicontinuous, locally in L^1 and satisfies $\Delta u \geq 0$ in the sense of distributions.

A function u of several complex variables is *plurisubharmonic* if it is upper semcontinuous, locally in L^1, and is subharmonic along any complex line. For smooth functions, this means that the matrix

$$(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k})$$

is positively semidefinite. (This follows from the chain rule.) Similarily, a general function is plurisubharmonic (psh) if

$$\sum \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} a_j \bar{a}_k \geq 0$$

for any constant vector a in $\mathbb{C}^n$. In analogy with Prekopa's theorem one might now conjecture that the following statement would hold:

Let $\phi(t, z)$ be a plurisubharmonic function in $\mathbb{C}_t \times \mathbb{C}^n_z$. Define

$$\tilde{\phi}(t) = -\log \int_{\mathbb{C}^n} e^{-\phi(t,z)} d\lambda(z)$$

(where $d\lambda$ is Lebsgue measure). Then $\tilde{\phi}$ is subharmonic.

This is however not true.

Example 1. (**Kiselman** Let $\phi(t, z) = |t - \bar{z}|^2 - |t|^2$, with t and z in $\mathbb{C}$. Then $\tilde{\phi}(t) = -|t|^2 + c$, so in particular it is not subharmonic. □

Proof. Since

$$\phi(t, z) = |z|^2 - 2\mathrm{Re}\, tz$$

ϕ is psh.

$$\int_{\mathbb{C}} e^{-\phi(t,z)} d\lambda(z) = Ce^{|t|^2}.$$

□

It turns out that in order to get a good analogy to Prekopa's theorem one needs to consider not only integrals of $e^{-\phi}$, but rather the L^2-norms induced by ϕ on spaces of holomorphic functions

$$\|h\|_t^2 := \int_{\mathbb{C}^n} |h|^2 e^{-\phi(t,\cdot)} d\lambda(z).$$

At first this may look like a very bad idea since not even the function $h = 1$ gives good 'convexity' properties. The explanation of this is that we should instead consider these norms as metrics on a vector bundle, and it turns out that this metric has positive curvature. This does not mean that the norms of individual functions behaves particualarily nicely, but together they do.

To explain this more carefully we shall first introduce basic notions of complex vector bundles.

4.1. **Holomorphic vector bundles and their curvature.** The local model for a (holomorphic) vector bundle is just a product

$$E = \Omega \times W$$

where Ω is an open set in $\mathbb{C}^n$ and W is a complex vector space. We then have a natural projection π from E to the 'base' Ω and the inverse image of any point in Ω is the complex vector space W. More generally, a *complex vector bundle* over a complex manifold Y is a smooth manifold E, together with a projection map $\pi : E \to Y$ such that the fiber over any point $y \in Y$, $E_y := \pi^{-1}(y)$ is complex vector space. Importantly, one also demands that this picture is *locally trivial*. This means that Y has a covering of open sets U_i, such that $\pi^{-1}(U_i)$ is isomorphic to the local model above, $U_i \times W$, where W is one fixed vector space. More precisely, there are maps

$$f_i : \pi^{-1}(U_i) \to U_i \times W$$

that are fiber preserving and respect the vector space structure of each fiber. Then

$$G_{ij} := f_j \circ f_i^{-1}$$

maps $U_i \cap U_j \times W$ to itself, is fiber preserving and linear on each $\{y\} \times W$. This means that

$$G_{ij}(z, \xi) = (z, g_{ij}(z)\xi),$$

where $g_{ij}(z)$ is a linear isomorphism of W for each z. It is easily checked that these *transition functions* satisfy the *cocycle conditions*, $g_{ij}^{-1} = g_{ji}$ and $g_{ij}g_{jk}g_{ki} = I$, the identity map. Conversely, given such a collection of g_{ij}:s satisfying the cocycle condition, one can show that they define a vector bundle by making the natural identifications in the disjoint union of $U_i \times W$. We say that E is a *holomorphic* bundle if E is a complex manifold and π and f_i are holomorphic. Equivalently $g_{ij}(z)$ are holomorphic functions of z.

Definition 4.1. A section of E over an open set in Y, U, is a holomorphic map from U to E such that $\pi \circ s = id_U$. A frame over U is a collection of sections $e_1, ...e_r$ that form a basis for $\pi^{-1}(z)$ at each point z. The dimension r is called the rank of E.

One usually requires that the model vector space is of finite dimension, indeed a frame as we have defined it only exists then. Later on however we shall consider also vector bundles of infinite rank. If we have a local frame it gives a natural identification of $\pi^{-1}(U) := E_U$ with $U \times \mathbb{C}^r$. We shall mostly discuss local properties of E. Then we can think of E as our local model, but it is important to have all concepts defined so they are independent of the choice of frame.

Given a frame, any section can be written $s(z) = \sum s_j(z)e_j(z)$. The space of all smooth sections over U is denoted $\Gamma(U, E)$ and the space of holomorphic sections is denoted $H^0(U, E)$. We can also consider the space of smooth forms of degree p, $\Gamma^p(U, E)$. Locally they are obtained as before $\eta = \sum \eta_j e_j$, but the coefficients η_j are now p-forms instead of functions. The next important concept is a *connection*. This is intutively a way to define differentiation of a section along a vector field. Naively, one writes, if V is a vector field and s a section,

$$V(s) = \sum V(s_j)e_j + s_j V(e_j).$$

The problem is to give a meaning to $V(e_j)$. Since it should again be a section we must have

$$V(e_j) = \sum \theta_{kj}(V)e_k.$$

To choose a connection means to choose $\theta = (\theta_{jk})$ in a coherent way,

Definition 4.2. A connection on E is a map

$$(V, s) \to D_V s$$

where V is a vector field and s a section which satisfies the product rule

$$D_V(fs) = V(f)s + fD_V s$$

for any smooth function f. Moreover we require that $D_{fV_1+gV_2} = fD_{V_1} + gD_{V_2}$ if f and g are smooth functions and V_j are two vector fields.

Given a connection, we can argue as before (this time a little bit less naively) and write (given a local frame)

$$D_V s = \sum V(s_j)e_j + \sum s_j \theta_{kj}(V)e_k,$$

where $\sum_k \theta_{kj}(V)e_k = D_V e_j$. By the linearity condition in the definition of a connection, $\theta(V)$ is linear in V, so θ can be thought of as a matrix of 1-forms. This way we can define

$$s \to Ds$$

as a map from sections to E-valued 1-forms, i e as a variant of the exterior derivative d. Given a frame we get

$$Ds = ds + \theta s$$

or $D = d + \theta$.

It now becomes natural to let D act on the space Γ^p of E-valued p-forms. Then, given frame, we get $D\eta = d\eta + \theta \wedge \eta$. Having done this, it makes sense to consider the operator D^2.

Proposition 4.3. *The operator D^2 maps Γ^p to Γ^{p+2} for any p and satisfies $D^2 f\eta = fD^2\eta$ if f is smooth. In terms of a frame*

$$D^2\eta = (d\theta + \theta \wedge \theta) \wedge \eta.$$

Proof.

$$D^2\eta = (d+\theta)(d\eta + \theta \wedge \eta) = d\theta \wedge \eta - \theta \wedge d\eta + \theta \wedge d\eta + \theta \wedge \theta \wedge \eta = (d\theta + \theta \wedge \theta) \wedge \eta.$$

□

Definition 4.4. $D^2 := \Theta^D$ is the curvature of the connection D. Given a frame it is a matrix of 2-forms, invariantly it is a 2-form with values in the endomorphisms of E.

So far everything we have done makes sense on a general complex vector bundle, but now it is time to bring the complex structure into play.

Definition 4.5. Let E be a holomorphic vector bundle over a complex manifold Y, with a connection D. Then D is *holomorphic* if Ds is of bidegree $(1,0)$ for any holomorphic section s.

Let us elaborate on this a little bit. A connection D on a holomorphic vector bundle can be decomposed according to bidegree as

$$D = D^{1,0} + D^{0,1} := D' + D'',$$

meaning that $D's$ is the $(1,0)$ part of Ds for any smooth section s. In terms of a holomorphic frame we then have $D' = \partial + \theta_{1,0}$ and $D'' = \bar\partial + \theta_{0,1}$, if we decompose the connection matrix θ according to bidegree. If s is holomorphic $D''s = \theta_{0,1}s$. Hence D is a holomorphic connection if and only if $\theta_{0,1} = 0$, for any (or equivalently some) holomorphic frame. Notice that then the curvature becomes

$$\Theta = (\partial\theta + \theta \wedge \theta) + \bar\partial\theta. \tag{4.1}$$

We shall now see that if we introduce one more condition on the connection, then the first parenthesis vanishes, so $\Theta = \bar\partial\theta$.

Definition 4.6. A (hermitian) metric on E is a scalarproduct on each fiber E_z such that

$$\langle s(z), s(z)\rangle_z$$

is smooth for each smooth section s.

In terms of a holomorphic frame e_j, the metric gets represented by a hermitian matrix $h = (h_{j,k})$, so that

$$\langle s(z), t(z)\rangle_z = \sum h_{j,k} s_k \bar{t}_j = t^* hs,$$

where the last equation means that we think of s and t as column vectors.

Definition 4.7. The connection is compatible with the metric $\langle\cdot,\cdot\rangle$ if

$$d\langle s, t\rangle = \langle Ds, t\rangle + \langle s, Dt\rangle.$$

(Explain how the bracket is extended to forms.)

Proposition 4.8. *Given a hermitian metric on a holomorphic vector bundle there is exactly one connection D that is both holomorphic and compatible with the metric. Given a frame and a hermitian matrix that represents the metric in this frame*

$$\theta = h^{-1}\partial h.$$

Proof. We have

$$d\langle s, t\rangle = (dt)^* hs + t^* hds + t^* dhs,$$

and

$$\langle Ds, t\rangle + \langle s, Dt\rangle = t^* h(ds + \theta s) + (dt + \theta t)^* hs.$$

These two expressions are equal if and only if

$$\theta^* h + h\theta = dh.$$

Decomposing after bidegree this means that $h\theta = \partial h$ and $\theta^* h = \bar{\partial} h$ if D is holomorphic. Here the second equation is just the hermitian transpose of the first, so θ defines a connection that is metric compatible and holomorphic if and only if $\theta = h^{-1}\partial h$. □

Proposition 4.9. *If D is metric compatible, then the curvature Θ is antihermitian in the sense that*

$$\langle \Theta s, t\rangle + \langle s, \Theta t\rangle = 0.$$

Proof. We have

$$0 = d^2\langle s, t\rangle = \langle \Theta s, t\rangle + \langle s, \Theta t\rangle + \langle Ds, Dt\rangle - \langle Ds, Dt\rangle.$$

□

Proposition 4.10. *If D is both metric compatible and holomorphic, then*

$$\Theta = \bar{\partial}\theta = \bar{\partial}(h^{-1}\partial h).$$

Proof. By (4.1) Θ has no component of bidegree $(0, 2)$ if D is holomorphic. If D is also metric compatible, the previous proposition shows that it has no $(2, 0)$ component either. Hence (4.1) implies that $\Theta = \bar{\partial}\theta$. □

The uniquely determined connection of Proposition 4.8 is called the *Chern connection* of our bundle, and from now on we shall only dicuss this connection.

Definition 4.11. Let E be a holomorphic vector bundle with a hermitian metric (a hermitian holomorphic vector bundle). We say that E has positive curvature (in the sense of Griffiths) if the $(1,1)$-form

$$\langle i\Theta s, s\rangle \geq 0$$

for any local section s. Similarily, E has negative curvature if

$$\langle i\Theta s, s\rangle \leq 0.$$

As an example, let us look at the case when the rank r is equal to one, so that E is a *line bundle*. Then h is just a positive function, and we write (locally) $h = e^{-\phi}$. Then $\Theta = -\bar{\partial}\partial\phi = \partial\bar{\partial}\phi$ and we see that E is positively curved precisely when ϕ is plurisubharmonic. Hence we can think of positive curvature as a vector bundle analog of plurisubharmonicity.

Now let E^* be the dual bundle of E, i e the vector bundle whose fiber over each point is the dual of the fiber of E. The local model for E^* is then $U \times W^*$, and the transitions functions for E^* (see section 4.1), $g^*_{i,j} = g^{-1}_{i,j}$ are the inverses of the transition functions of E. We have a natural pairing (s, ξ) between sections of E and sections of E^*, and if both s and ξ are holomorphic sections, then (s, ξ) is a holomorphic function.

Theorem 4.12. *If s and ξ are sections of E and E^* resepctively, then*

$$(\Theta^E s, \xi) = -(s, \Theta^{E^*}\xi),$$

so that $\Theta^{E^*} = -(\Theta^E)^t$. It follows that E has positive curvature if and only if E^* has negative curvature.

4.2. **Subbundles and Griffiths' curvature formula.** Let E be a holomorphic hermitian vector bundle and let F be a subbundle of E. The F inherits the metric of E and therefore also has a Chern connection. We shall first investigate how the Chern connections of E and F are related to each other. First note that from the hermitian metric on E we get for each point z in the base a projection map

$$p^F_z : E_z \to F_z,$$

the orthogonal projection. It is not holomorphic as z varies, but together these maps make up a smooth bundle map from E to F. Similarily we let $p^\perp_z$ be the orthogonal projection on the orthogonal complement of F_z in E_z, and the two bundle maps then sum to the identity.

Proposition 4.13. *Let D^E and D^F be the Chern connections of E and F respectively. Then*

1. $D^F = p^F \circ D^E$.

2. $s \to \beta(s) := p^\perp \circ D^E(s)$ satisfies $\beta(fs) = f\beta(s)$ if s is a smooth section of F. Hence $\beta(s) = 0$ at z if $s = 0$ at z, so β defines a linear map from F to its orthogonal complement in E.

Proof. 1. Since $D^E fs = dfs + fD^E s$, it follows that $p^F \circ D$ satisfies the same thing if s is a section of F. Hence $p^F \circ D$ is a connection. Moreover, if s is holomorphic, the fact that $D^E s$ is of bidegree $(1,0)$ implies that D^F is also of bidegree $(1,0)$, so our connection is holomorphic. Finally, if s and t are sections of F, then

$$d\langle s, t\rangle = \langle D^E s, t\rangle + \langle s, D^E t\rangle = \langle p^F \circ D^E s, t\rangle + \langle s, p^F \circ D^E t\rangle.$$

Hence $p^F \circ D^E$ is a holomorphic connection that is compatible with the metric and therefore must equal D^F.

2. $\beta(fs) = p^\perp(dfs + fD^E s) = f\beta(s)$ if s is a section of F. □

Next we give Griffiths formula for $\partial\bar\partial$ of the norm of a section.

Theorem 4.14. *Let s be a holomorphic section of E. Then*

$$i\partial\bar\partial\|s\|^2 = -\langle i\Theta s, s\rangle + \langle Ds, Ds\rangle. \tag{4.2}$$

It follows that if E has negative curvature and s is holomorphic, then $\|s\|^2$ and $\log\|s\|^2$ are plurisubharmonic.

Proof. We have

$$\partial\langle s, s\rangle = \langle D's, s\rangle,$$

since $D''s = 0$. Hence

$$\bar\partial\partial\langle s, s\rangle = \langle D''D's, s\rangle - \langle D's, D's\rangle = \langle \Theta s, s\rangle - \langle Ds, Ds\rangle.$$

Since $\bar\partial\partial = \partial\bar\partial$, this gives formula (4.2). Clearly this implies that $\|s\|^2$ is plurisubharmonic if s is holomorphic and the curvature is negative. For the last claim one can either estimate $i\partial\bar\partial\log\|s\|^2$ using (4.2), or note that the the first claim applied to $s' = e^h s$ where h is holomorphic implies that $e^{2\mathrm{Re}\, h}\|s\|^2$ is plurisubharmonic for each holomorphic h. This implies that $\log\|s\|^2$ is plurisubharmonic. □

Using this we can now compute the curvature of a holomorphic subbundle.

Theorem 4.15. *If F is a holomorphic subbundle of a hermitian holomorphic bundle E, then*

$$\Theta^F = p^F \circ \Theta^E - \beta^*\beta.$$

Proof. Using the previous theorem for E and F we see that

$$\langle \Theta^F s, s\rangle - \langle \Theta^E s, s\rangle = \|D^F s\|^2 - \|D^E s\|^2$$

if s is a holomorphic section of F. By Proposition 4.12 and Pythagoras theorem we get

$$\|D^F s\|^2 - \|D^E s\|^2 = \|p^F \circ D^E s\|^2 - \|p^F \circ D^E s\|^2 - \|p^\perp \circ D^E s\|^2 = -\|\beta s\|^2.$$

□

The quadratic form $-\|\beta s\|^2$ is called *the second fundamental form* of F in E. In the complex case this is always negative semidefinite. In case E has curvature zero, we see that the curvature of the subbundle F only comes from the second fundamental form. As an example of this, consider a holomorphic submanifold Y of an open set in $\mathbb{C}^n$. Then take E to be the (holomorphic) tangent space of $\mathbb{C}^n$, restricted to Y, with the Euclidean metric. This bundle is certainly flat. The curvature of the (holomorphic) tangent bundle of Y, F, is therefore always seminegative and equal to zero only if $\beta = 0$. Thus submanifolds of $\mathbb{C}^n$ are always negatively curved. This is no longer true for submanifolds of $\mathbb{P}^n$ since there is a certain amount of positivity in the tangent bundle of $\mathbb{P}^n$, so the submanifold can be negatively curved only if β is large. This corresponds to the submanifold being of sufficiently high degree.

We finally give a converse to the last part of Theorem 4.13.

Theorem 4.16. *If $\|s\|^2$ is plurisubharmonic for any local holomorphic section of E, then E has negative curvature.*

Proof. We first claim that for any z fixed in the base Y and ξ in E_z, we can find a local holomorphic section of E, s, such that $s(z) = \xi$ and $Ds = 0$ at z. Indeed, near z, E can be thought of as $U \times W$, where U is a neighbourhood of z and W is our model vector space. Then ξ is a vector in W and the local holomorphic section we are looking for must have the form $s(\zeta) = \xi + \sum V_k(\zeta_k - z_k)$ if we think of z and ζ also as local coordinates. Then at z, $Ds = \theta\xi + \sum V_k d\zeta_k$ and we just need to choose V_k so that this is zero. Explicitly, if $\theta = \sum \theta^k d\zeta_k$, we take

$$V_k = -\theta^k \xi.$$

For such an s we now get at z that

$$i\partial\bar{\partial}\|s\|^2 = -i\langle \Theta\xi, \xi\rangle.$$

If $\|s\|^2$ is plurisubharmonic then the left hans side is positive, so the curvature must be negative. □

Altogether, E has negative curvature if and only if $\|s\|^2$ is a plurisubharmonic function for any holomorphic section s. By Theorem 4.14 this is also equivalent to $\log \|s\|^2$ being plurisubharmonic for all holomorphic sections.

5. First version of complex Prekopa.

Let D be a domain in $\mathbb{C}$ and Ω a domain in $\mathbb{C}^n$. Let $\phi(t, z)$ be a plurisubharmonic function in $D \times \Omega$. For each t in D we put

$$A_t^2 = \{h \in H(\Omega); \int_\Omega |h|^2 e^{-\phi(t,\cdot)} d\lambda < \infty\}.$$

This is for any t a Hilbert space of holomorphic functions. Let us now make the additional assumption (not very natural!) that Ω is bounded and that ϕ is smooth up to the boundary on $D \times \bar{\Omega}$. Then all $A_t^2 = A^2$ are identical as vector spaces and we can form a globally trivial vector bundle over D as $F := D \times A^2$. The L^2-norms however vary with t and we get a trivial bundle with non trivial hermitean metric.

Theorem 5.1. *(First complex Prekopa teorem) The curvature of F is positive.*

In this section we will sketch a proof of this, assuming first that ϕ is smooth and strictly plurisubharmonic in z for each t fixed. First we will embed F in a larger vector bundle E, which is also trivial. The fibers of E are

$$L_t^2 = \{f \in L^2_{loc}; \int_\Omega |f|^2 e^{-\phi(t,\cdot)} d\lambda < \infty\}.$$

Given our hypothesis on ϕ these spaces are also all identical as vector spaces, but their (implied) Hilbert norms vary with t. Notice that E is actually a holomorphic vector bundle, even though it's modeled on a vector space of very 'non holomorphic' objects. In fact, that the bundle is holomorphic means just that the transition functions are holomorphic, and here we have no transition functions! A smooth section of E is a map $t \to f_t$ which is smooth as a map from D to L^2, and the section is holomorphic if its dependence on t is holomorphic.

Proposition 5.2. *The Chern connection of E is the (bounded) operator defined by*

$$Df = d_t f - (\partial_t \phi) f,$$

where d_t and ∂_t are exterior derivatives with respect to t for z fixed.

Proof. It is clear that Df defines a connection. It is moreover holomorphic, since if f is holomorphic in t, then Df is of bidegree $(1,0)$. Finally, differentiating under the integral sign, we see that D is compatible with the metric. □

We see from this formula, that the role of the connection form θ is played by the operator multiplication by $-\partial_t \phi$. Therefore the curvature is $\Theta^E = -\bar{\partial}_t \partial_t \phi = \partial \bar{\partial}_t \phi$, more precisely the operator defined as multiplication by this form. Therefore the curvature of E is positive as soon as ϕ is subharmonic with respect to t.

To compute the curvature of F we use Theorem 4.14, and the first step is to compute β. By definition, if h is a smooth section of E, i e a holomorphic function in A^2 depending smoothly on t,

$$\beta(h) = p^{\perp}(d_t h - (\partial_t \phi) h) = -p^{\perp}(\partial_t \phi h) =: u_t.$$

For each t fixed, u_t solves the $\bar{\partial}$-equation

$$\bar{\partial}_z u_t = \bar{\partial}_z(\partial_t \phi h) = h \bar{\partial}_z \partial_t \phi_t := f.$$

To estimate the curvature of F we must now estimate $\|\beta(h)\|_t^2$. For this we use that by definition $u_t = \beta(h)$ is orthogonal to the space of holomorphic functions. This means that it is the minimal solution to the $\bar{\partial}$-equation $\bar{\partial} u = f$ (see Corollary 4.5 below), and we can apply Hörmander's L^2-estimate for the $\bar{\partial}$-equation. This step of the proof is analogous to the use of the Brascamp-Lieb inequality in the proof of Prekopa's theorem.

5.1. **Hörmander's L^2-estimate for $\bar{\partial}$.** In this subsection we state Hörmander's famous L^2-estimate for the $\bar{\partial}$-equation for a domain Ω in $\mathbb{C}^n$. This theorem concerns the existence of solutions to the equation

$$\bar{\partial} u = f = \sum f_k d\bar{z}_k.$$

As is often the case with differential equations, the solvability of the equation is closely related to estimates for solutions and it is this aspect of the theorem that will be most important for us.

Theorem 5.3. *Let Ω be a pseudoconvex domain in $\mathbb{C}^n$ and let ψ be a smooth strictly plurisubharmonic function in Ω, so that $(\psi_{j,\bar{k}}) > 0$, or equivalently*

$$i\partial\bar{\partial}\psi = i \sum \psi_{j,\bar{k}} dz_j \wedge d\bar{z}_k > 0.$$

Let f be a differential form of bidegree $(0,1)$ satisfying $\bar{\partial} f = 0$. Then the equation $\bar{\partial} u = f$ has a solution that satisfies the estimate

$$\int_\Omega |u|^2 e^{-\psi} d\lambda \le \int_\Omega \|f\|^2_{i\partial\bar{\partial}\psi} e^{-\psi} d\lambda.$$

In this theorem we have used the plurisubharmonic function ψ to define a Kähler metric with Kähler form $\omega_\psi = i\partial\bar\partial\psi$ to define a norm on the space of $(0,1)$-*forms*. Explicitly, this norm is

$$\|f\|^2_{i\partial\bar\partial\psi} = \sum \psi^{j,\bar k} f_j \bar f_k,$$

where $(\psi^{j,\bar k})$ is the transpose of the inverse of the complex hessian $(\psi_{j,\bar k})$. This is the dual of the natural norm

$$\|v\|^2 := \sum \psi_{j,\bar k} v_j \bar v_k$$

on *vectors* v, under the pairing $(v,f) = \sum v_j f_j$. The aspect of Hörmander's theorem that we will use is the following corollary.

Corollary 5.4. *Let u be a function in Ω satisfying*

$$\int u\bar h e^{-\psi} d\lambda = 0$$

for all holomorphic functions h (in the naturally associated L^2-space. Then

$$\int_\Omega |u|^2 e^{-\psi} d\lambda \le \int_\Omega \|\bar\partial u\|^2_{i\partial\bar\partial\psi} e^{-\psi} d\lambda,$$

provided the left hand side is finite.

Proof. Let $f := \bar\partial u$. Then by definition u solves $\bar\partial u = f$. Any other solution with finite L^2-norm can be written $u' = u + h$ where h is holomorphic. Since u is orthogonal to the space of holomorphic functions, Pythagoras theorem implies that the L^2-norm of u is the smallest among all solutions to this $\bar\partial$-equation. Therefore the corollary follows from the previous theorem. □

The corollary is an instance of a so called *Poincaré inequality*, i e an estimate of a function u in terms of $P(u)$, where P is a differential operator, assuming the function is orthogonal to the null space of P. Notice the strong similarity with the Brascamp-Lieb inequality, Lemma 2.4.

5.2. **Conclusion of the proof of Theorem 5.1.** By Theorem 4.15 we have if $t \to h_t$ is a smooth section of F,

$$\langle \Theta^F h_t, h_t\rangle_t = \langle \Theta^E h_t, h_t\rangle_t - \|\beta(h_t)\|^2_t = \int \ddot\phi_{t,\bar t} |h|^2 e^{-\phi_t} d\lambda - \int_\Omega |\beta(h_t)|^2 d\lambda. \tag{5.1}$$

But

$$\bar\partial_z \beta(h_t) = \bar\partial_z(\dot h_t - h_t\dot\phi_t) = -h_t \bar\partial_z \dot\phi_t$$

and $\beta(h_t)$ is by definition ortogonal to all holomorphic functions (in L^2). We can therefore apply Hörmander's theorem, or rather its Corollary 5.4, and get

$$\int_\Omega |\beta(h_t)|^2 e^{-\phi_t} d\lambda \le \int_\Omega |h|^2 \|\bar\partial\dot\phi_t\|^2_{i\partial\bar\partial\phi_t} e^{-\phi_t}.$$

Combining with (5.1) we find

$$\langle \Theta^F h_t, h_t\rangle_t \ge \int \left(\ddot\phi_{t,\bar t} - \|\bar\partial\dot\phi_t\|^2_{i\partial\bar\partial\phi_t}\right) |h|^2 e^{-\phi_t} d\lambda. \tag{5.2}$$

The proof of Theorem 5.1 now follows from

Lemma 5.5. *Let $\phi(t, z)$ be plurisubharmonic with respect to (t, z) and stritly plurisubharmonic in z. Let*

$$c(\phi) := \ddot{\phi}_{t,\bar{t}} - \|\bar{\partial}\dot{\phi}_t\|^2_{i\partial\bar{\partial}\phi_t}.$$

Then

$$c(\phi) = MA_{t,z}(\phi)/MA_z(\phi).$$

In particular, $c(\phi) \geq 0$, with equality only if $MA_{t,z}(\phi) = 0$.

Notice that from (5.2) we even get an explicit estimate for the curvature. The curvature is by definition an operator on the vector space that is the fiber of the bundle. In our case the fiber is A^2 and we estimate the curvature operator by the Topelitz operator defined by the *symbol* $c(\phi)$.

Definition 5.6. Let Ω be a domain in $\mathbb{C}^n$ and let ψ be a weight function in Ω. Let A^2_ψ be the space of all holomorphic functions in Ω such that

$$\|h\|^2 := \int_\Omega |h|^2 e^{-\psi} d\lambda < \infty.$$

If χ is a bounded function in Ω, the Toeplitz operator with symbol χ, T_χ is the operator on A^2_ψ defined by

$$\langle T_\chi h, h' \rangle = \int_\Omega \chi h \bar{h}' e^{-\psi} d\lambda.$$

At the price of introducing a somewhat abstract notion, we can even get an exact *formula* for the curvature.

Definition 5.7. Let f be a $\bar{\partial}$-closed $(0, 1)$-form in a pseudoconvex domain Ω, and let ψ be a smooth strictly plurisubharmonic function in Ω. Let u be the solution to $\bar{\partial}v = f$ of minimal norm in $L^2(\Omega, e^{-\psi})$. Then

$$e_\psi(f) := \int_\Omega \|f\|^2_{i\partial\bar{\partial}\psi} e^{-\psi} d\lambda - \int_\Omega |u|^2 e^{-\psi} d\lambda.$$

Moreover, if h lies in A^2_ψ and χ is a smooth real valued function , we put

$$E_\psi(h, \chi) := e(h\bar{\partial}\chi).$$

Notice that by Hörmander's theorem, $e(f)$ and $E(h, \chi)$ are always nonnegative. With these definitions we get

Theorem 5.8. *With the same notations and assumptions as in Theorem 5.1 we have*

$$\langle \Theta^F h, h \rangle = \langle T_{c(\phi)} h, h \rangle + E(h, \dot{\phi}_t).$$

Notice that $E(h, \chi)$ is a quadratic form in both h and χ, so it is something like a tensor with four indices. It can be seen as a sort of curvature tensor, defined on a vector bundle with fiber A^2_ψ over the space of all plurisubharmonic weights ψ. This space resembles the space of convex functions from section 3, and we can think of χ as lying in its tangent space. In this way we have decomposed the curvature of F as a sum of two terms. The second term comes from the bundle over the space of all plurisubharmonic weights, and the first term depends on the curve $t \to \phi_t$ in this space that we are looking at. This picture will be elaborated later when we consider similar formulas with Ω replaced by a compact manifold. At this point we just indicate an intriguing problem that arises in this connection, which will be (partially) clarified later when we deal with compact manifolds instead of $\mathbb{C}^n$.

We said that $E_\psi(h, \chi)$ behaves like a curvature tensor on a vector bundle with fiber A^2_ψ over the space, say Ψ, of all plurisubharmonic weights ψ. Remember that with our assumptions all $A^2_\psi = A^2$ are identical as vector spaces, but their norms depend on ψ. The bundle is just $\mathcal{E} := \Psi \times A^2$, and the metric is the 'tautological' metric

$$\|h\|^2_\psi = \int_\Omega |h|^2 e^{-\psi} d\lambda,$$

if h belongs to the fiber $\mathcal{E}_\psi$. However, a metric does not in itself determine a connection or curvature. For that we needed also to assume that the connection was compatible with the complex structure, and Ψ has no natural complex structure. So, where does the curvature come from? The answer to this is, roughly, that there is a complex manifold $\mathcal{M}$ with a projection map $p : \mathcal{M} \to \Psi$. We then pull back our bundle to the bundle $\mathcal{M} \times A^2$, which is now a holomorphic bundle, and E_ψ is now the curvature of the Chern connection on that bundle.

5.3. **Interpretations of the complex Prekopa theorem.** Recall that the conclusion of (the real) Prekopa's theorem was that certain functions

$$-\log \int e^{-\phi(t,\cdot)}$$

are convex. This leads to many interesting geometric inequalities. With the complex version the situation is different; that a vector bundle has positive curvature does not imply that the norms of its holomorphic sections have any properties of convexity or plurisubharmonicity. Let us look at this a bit more closely, and start with the case of line bundles.

A holomorphic section s of a line bundle is locally, given a frame e, just a local scalar valued holomorphic function h, $s = he$. Its norm is

$$\|s\|^2 = |h|^2 e^{-\phi},$$

where $e^{-\phi} = \|e\|^2$. Thus

$$i\partial\bar{\partial}(-\log \|s\|^2) = i\partial\bar{\partial}\phi,$$

where $h \neq 0$. If the curvature is positive, $i\partial\bar{\partial}\phi \geq 0$, so $-\log \|s\|^2$ is plurisubharmonic, at least where $h \neq 0$. Thus, if our vector bundle E were of rank 1, it would follow that

$$-\log \int_\Omega |h|^2 e^{\phi(t,\cdot)}$$

were plurisubharmonic and we would have statements parallell to the real case, but this is of course not the case. To see what happens for bundles of higher rank, let us look at direct sums of line bundles. By this we mean that we have a local frame of ortogonal holomorphic sections, e_j. A local section can now be written $s = \sum h_j e_j$ and its norm is

$$\|s\|^2 = \sum |h_j|^2 e^{-\phi_j},$$

where $e^{-\phi_j} = \|e_j\|^2$. Now there is no simple formula for $i\partial\bar\partial(-\log\|s\|^2)$, and it is certainly not the case that it is always positive - not even if all $\phi_j = 0$.

There are two different ways of getting explicit convexity statements from Theorem 5.1. First, we can try to cook up other bundles of rank 1 from our bundle E. Second, we can look instead at the dual bundle E^*. By Theorem 4.12, E^* has *negative* curvature, and by Theorem 4.14, this means that $\log\|t\|^2$ (without the minus sign!) is plurisubharmonic for any holomorphic section t of the dual bundle. Both these routes give interesting statements, but the second one is probably the most powerful. Since E^* is negative if and only if E is positive, the positivity of E is actually equivalent to saying that $\log\|t\|^2$ is plurisubharmonic for any holomorphic section t of the dual bundle, so we have not lost anything.

We first look at a special class of domains.

Definition 5.9. A domain Ω in $\mathbb{C}^n$ is balanced if $z \in \Omega$ implies $\lambda z \in \Omega$ for any λ in $\mathbb{C}$ of norm at most 1. Ω is S^1-invariant if the same thing holds for any λ of norm equal to 1. A function ψ is S^1-invariant if $\psi(\lambda z) = \psi(z)$ for any $\lambda = e^{i\alpha}$ in $\mathbb{C}$ of norm equal to 1.

Theorem 5.10. *Let Ω be a balanced domain in $\mathbb{C}^n$ and let $\phi(t,z)$ be plurisubharmonic in $D\times\Omega$ and S^1-invariant in z for any t in D. Then*

$$-\log\int_\Omega e^{-\phi(t,\cdot)}d\lambda$$

is subharmonic in t (or identically equal to $-\infty$).

Proof. We may assume that Ω is bounded since any balanced Ω can be exhausted by an increasing sequence of bounded balanced domains, and decreasing limits of plurisubharmonic functions are plurisubharmonic. By a similar token, we may assume that ϕ is smooth, so we can apply Theorem 5.1. The fibers of our vector bundle E consist of holomorphic functions on Ω. Let for any natural number k, E_k be the subbundle of E of homogenuos polynomials of degree k. Since, if $h \in E_k$ and $g \in E_m$,

$$\int_\Omega h(z)\bar g(z)e^{-\phi_t(z)}d\lambda = \int_\Omega h(e^{i\alpha}z)\bar g(e^{i\alpha}z)e^{-\phi_t(z)}d\lambda = e^{(k-m)i\alpha}\int_\Omega h(z)\bar g(z)e^{-\phi_t(z)}d\lambda,$$

for any α we see that $E_k \perp E_m$ if $k \neq m$. This means that E is the orthogonal sum of the holomorphic subbundles E_k and it follows that all of them have positive curvature. In particular E_0 has positive curvature. This is a line bundle and the constant function 1 is a local frame. Consequently $-\log\|1\|^2$ is a plurisubharmonic, which is the statement of the theorem. □

This may appear to be a weaker statement than Prekopa's theorem, since we require extra hypotheses on S^1-invariance etc. This is however not the case, since the class of plurisubharmonic functions is much wider than the space of convex functions. We shall see later that Prekopa's theorem follows from a variant of the previous theorem where ϕ is not only S^1-invariant, but even invariant under the full torus action $z \to (e^{i\alpha_1}z, ...e^{i\alpha_n}z_n)$.

REFERENCES

DEPARTMENT OF MATHEMATICS, CHALMERS UNIVERSITY OF TECHNOLOGY, S-412 96 GÖTEBORG, SWEDEN
E-mail address: bob@chalmers.se