L^2-methods for the $\bar{\partial}$-equation

Bo Berndtsson

KASS UNIVERSITY PRESS - JOHANNEBERG - MASTHUGGET - SISJÖN

1995

Contents

Preface

These are the lecture notes from a course given at CTH. The main purpose of the course was to introduce the basic ideas of the weighted L^2-estimates for the $\bar{\partial}$-equation in domains in $\mathbb{C}^n$, and then go on to the analogous invariant formalism on complex manifolds. Here is an overview of the content:

In the first chapter we treat the $\bar{\partial}$-equation for $(0,1)$-forms in domains in $\mathbb{C}^n$, following Hörmander [1]. This approach is technically more complicated than the one used in Hörmander's book [2], but it is probably conceptually easier to understand. The main technical difficulty is the proof of the approximation lemma in Section 1.5. After having proved the main existence theorem using this method, we also show how the use of the approximation lemma can be avoided, following [2].

In Chapter 2 we set up and solve the $\bar{\partial}$-Neumann problem. The presentation differs from e.g. Folland-Kohn [6] in that we establish solvability without proving regularity first. Again, the main point is the approximation lemma from Chapter 1, Section 1.5. After that we discuss regularity very briefly, using a fundamental theorem of Kohn-Nirenberg, [7], that we do not prove.

Chapter 3 is devoted to the $\bar{\partial}$-equation on complex manifolds. We treat only the case of Kähler manifolds, and the first object is to set up the Kähler identities. We do this in a pedestrian way, using calculations in normal coordinates. Then we prove the Lefschetz decomposition of differential forms which is later used for the so-called "Hard Lefschetz theorem". But, the most important formula in this chapter is the Nakano identity, Theorem 3.7.3. This formula implies the fundamental identity, Theorem 1.4.2, and its generalizations to vector bundles over Kähler manifolds. This far, Chapter 3 consists basically of linear algebra, but then we use these formulas to prove vanishing theorems, i.e., existence theorems for the $\bar{\partial}$-equation. Apart from the case of compact manifolds, we treat non-compact manifolds with a complete Kähler metric, basically following Demailly [5].

There is nothing original in this presentation (except for the errors, and perhaps not even all of them). These notes were written to serve as an easy reference for myself. Maybe they can serve the same purpose for someone else.

Finally I would like to thank Yumi Karlsson for helping to type the manuscript.

Chapter 1

The $\bar\partial$-equation for $(0,1)$-forms in domains in $\mathbb{C}^n$

If u is a function defined in a domain Ω in $\mathbb{C}^n$, the differential $\bar\partial u$ is defined by

$$\bar\partial u = \sum_1^n \frac{\partial u}{\partial \bar z_j} d\bar z_j$$

where

$$\frac{\partial u}{\partial \bar z_j} = \frac{1}{2}\Big[\frac{\partial u}{\partial x_j} + i\frac{\partial u}{\partial y_j}\Big].$$

In general, a $(0,1)$-form f is a formal combination

$$f = \sum_1^n f_j d\bar z_j$$

where the f_j:s are functions. The equation

$$\bar\partial u = f \tag{1.1}$$

is thus just a compact way of writing the system of differential equations

$$\frac{\partial u}{\partial \bar z_j} = f_j \qquad j = 1, \dots, n.$$

For the equation (1.1) to be solvable it is necessary that f satisfy the compatibility conditions

$$\frac{\partial f_j}{\partial \bar z_k} = \frac{\partial f_k}{\partial \bar z_j} \qquad 1 \le j,\, k \le n. \tag{1.2}$$

If we introduce

$$\begin{aligned} \bar\partial f &= \sum_1^n \bar\partial f_j \wedge d\bar z_j = \sum_{j,k=1}^n \frac{\partial f_j}{\partial \bar z_k} d\bar z_k \wedge d\bar z_j = \\ &= \sum_{k<j} \big(\frac{\partial f_j}{\partial \bar z_k} - \frac{\partial f_k}{\partial \bar z_j}\big) d\bar z_k \wedge d\bar z_j, \end{aligned}$$

the equations (1.2) can be written

$$\bar\partial f = 0. \tag{1.3}$$

The $\bar\partial$-problem is thus to solve the equation $\bar\partial u = f$ where $\bar\partial f = 0$.

The same problem can be posed when f is a differential form of higher degree. In the first chapters, however, we will consider only the case when f is a $(0,1)$-form since it shows the basic ideas in their simplest form. The general problem will appear in Chapter 3 when we consider the $\bar\partial$-equation on complex manifolds.

[12pt,a4paper]report

times,a4wide,cmbb,amssymb,dot-theorem [12pt,a4wide,amssymb,cmbb,dot-theorem,leqno]report

1.1 Formulation of the main result

The main result of this first chapter is Hörmander's weighted L^2-estimate for the $\bar\partial$-equation. To state it we first need to introduce a few basic concepts.

Let $\phi \to [-\infty, \infty)$ be an (extended-) realvalued function in Ω. We say that ϕ is *plurisubharmonic* if ϕ is upper semicontinuous, and has the property that for each $a \in \Omega$ and each $w \in \mathbb{C}^n$, the function $\zeta \to \phi(a + \zeta w)$ is a subharmonic function of the complex variable ζ, for ζ near 0. In other words, we require that the restriction of ϕ to each complex line is subharmonic. In case ϕ is smooth (of class C^2) we can check this by computing the Laplacian with respect to ζ:

$$\frac{\partial^2}{\partial\zeta\partial\bar\zeta}\phi(a+\zeta w) = \sum \phi_{j\bar k}(a+\zeta w) w_j \bar w_k.$$

Here we have used the notation $\phi_{j\bar k} = \frac{\partial^2}{\partial z_j \partial \bar z_k}\phi$.

Since a smooth function of one complex variable is subharmonic if and only if its Laplacian is nonnegative, we see that ϕ is plurisubharmonic if and only if the matrix $(\phi_{j\bar k})$ is positively semidefinite. We also say that ϕ is strictly plurisubharmonic if this matrix is positively definite.

Plurisubharmonicity can be thought of as a (or one possible) complex notion of convexity for functions. The corresponding concept for domains is *pseudoconvexity.*

To define it we shall first assume that our domain Ω has smooth boundary. Then Ω can be given as

$$\Omega = \{z \in \mathbb{C}^n; \rho(z) < 0\},$$

where ρ is a smooth ("defining") function satisfying $d\rho \neq 0$ on $\partial\Omega$. Let p be a point on $\partial\Omega$. The tangent plane to $\partial\Omega$ at p is the set of vectors a such that $d\rho|_p.a = 0$. Here, if $a = \alpha + i\beta$,

$$d\rho|_p.a = \sum \alpha_j \frac{\partial\rho}{\partial x_j}(p) + \beta_j \frac{\partial\rho}{\partial y_j}(p).$$

This can be written as

$$2\Re\partial\rho|_p.a = 2\Re\sum a_j \frac{\partial\rho}{\partial z_j}(p).$$

The *complex tangent plane* of $\partial\Omega$ at p is now defined as

$$T_p^{(1,0)} = \{a; \sum a_j \frac{\partial\rho}{\partial z_j}(p) = 0\}.$$

Note that $T_p^{(1,0)}$ is a $(n-1)$-dimensional complex subspace of $\mathbb{C}^n$ which is contained in the (real)tangent plane. Clearly, this property also determines $T_p^{(1,0)}$ uniquely.

Definition: Ω is pseudoconvex if for all $p \in \partial\Omega$ the quadratic form

$$L(p,\rho)(a) = \sum \rho_{j\bar{k}}(p) a_j \bar{a}_k,$$

defined for $a \in T_p^{(1,0)}$, is positively semidefinite.

Note that, in particular, if we can choose ρ plurisubharmonic, the domain must of course be pseudoconvex.

It may seem that this definition depends on the choice of defining function ρ but in reality it does not. Namely, if $\tilde{\rho}$ is another choice of defining function, then $\tilde{\rho}$ can be written as $\tilde{\rho} = g\rho$, where $g > 0$ on $\partial\Omega$. Remembering the definition of $T_p^{(1,0)}$ we see that

$$L(p,\tilde{\rho})(a) = gL(p,\rho)(a),$$

so the definition of pseudoconvexity is indeed independent of the choice of ρ.

It is worth remarking that when $n = 1$, $T^{(1,0)} = \{0\}$. Therefore any domain in $\mathbb{C}$ is pseudoconvex (just like any domain in $\mathbb{R}$ is convex).

One can prove that a smoothly bounded domain in $\mathbb{C}^n$ is pseudoconvex in the sense we just have described if and only if there is some smooth and plurisubharmonic function ψ defined in Ω which tends to ∞ at the boundary. (Such a ψ is called an *exhaustion function.*) This property makes sense whether the boundary is smooth or not, and can be taken as the general definition of pseudoconvexity.

Note that this second definition implies that any pseudoconvex domain Ω can be written as an increasing union of relatively compact subdomains, Ω_k that are (strictly) pseudoconvex and have smooth boundaries. This follows since we can take

$$\Omega_k = \{\psi < C_k\},$$

where C_k is a sequence that tends to ∞ sufficiently rapidly, for by Sard's theorem these domains will be smoothly bounded for almost all choices of C_k.

We are now ready to state one version of the main theorem of this chapter.

Theorem 1.1.1 *Let Ω be a pseudoconvex domain in $\mathbb{C}^n$, and let ϕ be smooth and strictly plurisubharmonic in Ω. Suppose f is a $(0,1)$-form with coefficients in L^2_{loc}, satisfying $\bar{\partial} f = 0$, in the sense of distributions. Then there is a solution, u, to the equation $\bar{\partial} u = f$, satisfying the estimate*

$$\int_\Omega |u|^2 e^{-\phi} \leq \int_\Omega \sum \phi^{j\bar{k}} f_j \bar{f}_k e^{-\phi},$$

provided the right hand side is finite. Here $(\phi^{j\bar{k}}) = (\phi_{j\bar{k}})^{-1}$.

1.2 The one-dimensional case.

Throughout this section we shall identify functions and $(0,1)$-forms, so we make no distinction between f and $f d\bar{z}$. Let us first repeat what Theorem 1.1.1 says in the case when $n = 1$. Then Ω is allowed to be any domain in $\mathbb{C}$, and ϕ is any function satisfying

$$\frac{\partial^2}{\partial z \partial \bar{z}} \phi = \Delta\phi > 0.$$

The compatibility condition $\bar\partial f = 0$ is also always satisfied (*here* we have to consider f as a $(0,1)$-form!), and the conclusion is that we can solve

$$\frac{\partial u}{\partial \bar z} = f$$

with a function u satisfying

$$\int |u|^2 e^{-\phi} \leq \int \frac{|f|^2}{\Delta\phi} e^{-\phi}.$$

Even this one variable case is a very precise and useful result, and it is quite surprising that it was discovered in several variables first. Moreover, the proof when $n = 1$ is considerably more elementary than the general case, and we shall therefore treat it separatedly.

We begin by giving the problem a dual formulation. Remember that, interpreted in the sense of distributions, the equation $\frac{\partial}{\partial \bar z} u = f$ means precisely that

$$-\int u \frac{\partial}{\partial \bar z}\alpha = \int f\alpha \tag{1.4}$$

for all $\alpha \in C^2_c(\Omega)$. To introduce the weighted L^2-norms of the theorem we substitute for α, $\bar\alpha e^{-\phi}$. The equality (1.4) then says

$$\int u\overline{\bar\partial^*_\phi \alpha} e^{-\phi} = \int f\bar\alpha e^{-\phi}, \tag{1.5}$$

where

$$\bar\partial^*_\phi \alpha =: -e^{\phi}\frac{\partial}{\partial z}(e^{-\phi}\alpha),$$

is the *formal adjoint* of the $\bar\partial$-operator with respect to our weighted scalar product

$$< f, g >_\phi = \int f\bar g e^{-\phi}.$$

Proposition 1.2.1 *Given f there exists a solution, u, to $\frac{\partial}{\partial \bar z}u = f$ satisfying*

$$\int |u|^2 e^{-\phi} \leq C, \tag{1.6}$$

if and only if the estimate

$$|\int f\,\bar\alpha e^{-\phi}|^2 \leq C\int |\bar\partial^*_\phi\alpha|^2 e^{-\phi} \tag{1.7}$$

holds for all $\alpha \in C^2_c(\Omega)$. On the other hand, for a given function $\mu > 0$, (1.7) holds for all f satisfying

$$\int \frac{|f|^2}{\mu} e^{-\phi} \leq C \tag{1.8}$$

if and only if

$$\int \mu|\alpha|^2 e^{-\phi} \leq \int |\bar\partial^*_\phi\alpha|^2 e^{-\phi}, \tag{1.9}$$

holds for all $\alpha \in C^2_c(\Omega)$.

Proof: It is clear that if (1.5), and (1.6) hold, then (1.7) follows. Suppose conversely that the inequality (1.7) is true. Let

$$E = \{\bar\partial^*_\phi\alpha; \alpha \in C^2_c(\Omega)\},$$

and consider E as a subspace of

$$L^2(e^{-\phi}) = \{g \in L^2_{loc}; \int |g|^2 e^{-\phi} < \infty\}.$$

Define an antilinear functional on E by

$$L(\bar\partial_\phi^* \alpha) = \int f \bar\alpha e^{-\phi}.$$

The inequality (1.7) then says that L is (well defined and) of norm not exceeding C. By Hahn-Banach's extension theorem L can be extended to an antilinear form on all of $L^2(e^{-\phi})$, with the same norm. The Riesz representation theorem then implies that there is some element,u, in $L^2(e^{-\phi})$, with norm less than C, such that

$$L(g) = \int u \bar g e^{-\phi},$$

for all $g \in L^2(e^{-\phi})$. Choosing $g = \bar\partial_\phi^* \alpha$, we see that

$$\int u \overline{\bar\partial_\phi^* \alpha} e^{-\phi} = \int f \bar\alpha e^{-\phi},$$

so u solves $\frac{\partial}{\partial \bar z} u = f$.

The first part of the proposition is therefore proved. The second part is obvious.

To complete the proof of Hörmander's theorem in the one-dimensional case it is therfore enough to prove an inequality of the form (1.7). This will be accomplished by the following integral identity.

Proposition 1.2.2 *Let Ω be a domain in $\mathbb{C}$ and let $\phi \in C^2(\Omega)$. Let $\alpha \in D_{0,1}(\Omega)$. Then*

$$\int \Delta\phi |\alpha|^2 e^{-\phi} + \int |\frac{\partial}{\partial \bar z} \alpha|^2 e^{-\phi} = \int |\bar\partial_\phi^* \alpha|^2 e^{-\phi} \tag{1.10}$$

Proof: Since α has compact support we can integrate by parts and get

$$\int |\bar\partial_\phi^* \alpha|^2 e^{-\phi} = \int \bar\partial \bar\partial_\phi^* \alpha \cdot \bar\alpha e^{-\phi}.$$

Next note that

$$\bar\partial_\phi^* \alpha = -\frac{\partial}{\partial z} \alpha + \phi_z \alpha,$$

so that

$$\bar\partial \bar\partial_\phi^* \alpha = -\Delta \alpha + \phi_z \frac{\partial}{\partial \bar z} \alpha + \Delta\phi \alpha = \bar\partial_\phi^* \frac{\partial}{\partial \bar z} \alpha + \Delta\phi\alpha.$$

Hence

$$\int |\bar\partial_\phi^* \alpha|^2 e^{-\phi} = \int \Delta\phi |\alpha|^2 e^{-\phi} + \int |\frac{\partial}{\partial \bar z} \alpha|^2 e^{-\phi}$$

and the proof is complete.

Combining the last two propositions we now immediately conclude

Theorem 1.2.3 *Let Ω be a domain in $\mathbb{C}$ and suppose $\phi \in C^2(\Omega)$ satisfies $\Delta\phi \geq 0$. Then, for any f in $L^2_{loc}\Omega$ there is a solution u to $\frac{\partial}{\partial \bar z} u = f$ satisfying*

$$\int |u|^2 e^{-\phi} \leq \int \frac{|f|^2}{\Delta\phi} e^{-\phi}.$$

1.3 Dual formulation in higher dimensions

Now we turn to the case of dimensions larger than 1. Denote by $D_{(0,1)}$ the class of $(0,1)$-forms whose coefficients are, say, of class C^2 with compact support in Ω. If f and α are $(0,1)$-forms we denote by $f \cdot \bar{\alpha}$ their pointwise scalar product, i e

$$f \cdot \bar{\alpha} = \sum f_j \bar{\alpha}_j.$$

The equation $\bar{\partial} u = f$, in the sense of distributions, means that

$$\int f \cdot \alpha = -\int u \sum \frac{\partial \alpha_j}{\partial \bar{z}_j}, \tag{1.11}$$

for all $\alpha \in D_{(0,1)}$. Just like in the one-dimensional case we replace α by $\bar{\alpha} e^{-\phi}$ (where ϕ is a C^2-function which will later be chosen to be plurisubharmonic). The condition (1.11) is then equivalent to

$$\int f \cdot \bar{\alpha} e^{-\phi} = \int u \overline{\bar{\partial}_\phi^* \alpha} e^{-\phi} \tag{1.12}$$

for all $\alpha \in D_{(0,1)}$, where

$$\bar{\partial}_\phi^* \alpha = -e^{\phi} \sum \frac{\partial}{\partial z_j} (e^{-\phi} \alpha_j).$$

Assume now that we can find a solution, u, to $\bar{\partial} u = f$, satisfying

$$\int |u|^2 e^{-\phi} \leq C.$$

Then (1.12) implies

$$|\int f \cdot \bar{\alpha} e^{-\phi}|^2 \leq C \int |\bar{\partial}_\phi^* \alpha|^2 e^{-\phi}.$$

The next proposition says that the converse of this also holds.

Proposition 1.3.1 *There is a solution, u, to the equation $\bar{\partial} u = f$ satisfying*

$$\int |u|^2 e^{-\phi} \leq C. \tag{1.13}$$

if and only if the inequality

$$|\int f \cdot \bar{\alpha} e^{-\phi}|^2 \leq C \int |\bar{\partial}_\phi^* \alpha|^2 e^{-\phi}. \tag{1.14}$$

holds for all $\alpha \in D_{(0,1)}$.

Proof: It only remains to prove that (1.14) implies that there is a solution to the $\bar{\partial}$-equation satisfying (1.13). This is done precisely as in the one-dimensional case (cf Proposition 1.2.1).

To prove inequality (1.14) one might first try to prove an inequality of the form

$$\int |\alpha|^2 e^{-\phi} \leq C \int |\bar{\partial}_\phi^* \alpha|^2 e^{-\phi}.$$

The main problem in higher dimensions (as compared to the one-dimensional case), is that no such inequality can hold. Indeed, if it did, then by Proposition 1.3.1 , we would be able to solve $\bar{\partial} u = f$, even when f does not satisfy the compatibility condition $\bar{\partial} f = 0$. Thus we must somehow feed this information, $\bar{\partial} f = 0$, into the method. This requires a little bit more of functional analysis.

First we introduce the weighted Hilbert spaces

$$L^2(\Omega, e^{-\phi}) = \{u \in L^2_{loc}; \int |u|^2 e^{-\phi} < \infty\},$$

and

$$L^2_{(0,1)}(\Omega, e^{-\phi}) = \{f = \sum f_j dz_j; f_j \in L^2_{loc}, \int |f|^2 e^{-\phi} < \infty\}.$$

(Here of course $|f|^2 = \sum |f_j|^2$.)

In the sequel, as long as the domain Ω under consideration is kept fixed, we will write simply $L^2(e^{-\phi})$ etc, for the weighted L^2-spaces. We also let

$$N = \{f \in L^2_{(0,1)}(e^{-\phi}); \bar\partial f = 0\}.$$

Here the condition $\bar\partial f = 0$ means that

$$\frac{\partial f_j}{\partial \bar z_k} = \frac{\partial f_k}{\partial \bar z_j}$$

in the sense of distributions. It follows that N is a closed subspace of $L^2_{(0,1)}(e^{-\phi})$.

We can then extend the definition of $\bar\partial$ by allowing it to act on any $u \in L^2(e^{-\phi})$ such that $\bar\partial u$ (in the sense of distributions) lies in $L^2_{(0,1)}(e^{-\phi})$. This way we get a densely defined operator

$$T : L^2(e^{-\phi}) \to L^2_{(0,1)}(e^{-\phi}).$$

T has an adjoint

$$T^* : L^2_{(0,1)}(e^{-\phi}) \to L^2(e^{-\phi})$$

defined by

$$< u, T^*\alpha >_{L^2(e^{-\phi})} = < Tu, \alpha >_{L^2_{(0,1)}(e^{-\phi})} .$$

This means that $\alpha \in Dom(T^*)$ and $T^*\alpha = v$ if and only if

$$< u, v >_{L^2(e^{-\phi})} = < Tu, \alpha >_{L^2_{(0,1)}(e^{-\phi})} .$$

for all u in the domain of T. Recall that α lies in the domain of T^* if and only if the inequality

$$| < Tu, \alpha >_{L^2_{(0,1)}(e^{-\phi})} | \leq C||u||_{L^2(e^{-\phi})}$$

holds for all u in the domain of T. Observe that if α lies in the domain of T^*, then

$$T^*\alpha = \bar\partial^*_\phi \alpha.$$

(The difference between T^* and $\bar\partial^*_\phi$ is that T^* has a specified domain. Thus we may apply $\bar\partial^*_\phi$ to forms that are not in the domain of T^*)

Let us now return to our testform α in $D_{(0,1)}$. Clearly $\alpha \in L^2_{(0,1)}(e^{-\phi})$, so we can decompose

$$\alpha = \alpha^1 + \alpha^2,$$

where α^1 lies in N and α^2 is orthogonal to N. This implies in particular that α^2 is orthogonal to any form Tu, so we see that α^2 lies in the domain of T^* and $T^*\alpha^2 = 0$. Since clearly α lies in the domain of T^* (why?), it follows that $T^*\alpha = T^*\alpha^1$.

In the proofs below, we will need a simple generalization of Cauchy's inequality when we estimate pointwise scalar products. It says that if $\mu = (\mu_{j\bar k})$ is any positively definite hermitean matrix, and $(\mu^{j\bar k})$ denotes the inverse matrix, then

$$|f \cdot \bar\alpha|^2 \leq \sum \mu^{j\bar k} f_j \bar f_k \sum \mu_{j\bar k} \alpha_j \bar\alpha_k.$$

This is easily seen since we may diagonalize μ by a unitary transformation. We are now ready to give the dual formulation of the $\bar\partial$-problem.

Proposition 1.3.2 *Let $\mu = (\mu_{j\bar{k}})$ be a continuous function defined in Ω whose values are hermitean $n \times n$ matrices. Assume that μ is uniformly bounded and uniformly positive definite on Ω. Suppose that for any α in Dom(T^*) $\cap N$ it holds*

$$\int \sum \mu_{j\bar{k}} \alpha_j \bar{\alpha}_k e^{-\phi} \leq \int |T^* \alpha|^2 e^{-\phi}.$$

Then, for any $f \in L^2_{(0,1)}(e^{-\phi})$ satisfying $\bar{\partial} f = 0$, there is a solution, u to $\bar{\partial} u = f$ satisfying

$$\int |u|^2 e^{-\phi} \leq \int \sum \mu^{j\bar{k}} f_j \bar{f}_k e^{-\phi}.$$

If μ is a constant multiple of the identity matrix, the converse to this also holds.

Proof: To prove the first part, according to Proposition we need to verify the inequality

$$|\int f \cdot \bar{\alpha} e^{-\phi}|^2 \leq C \int |\bar{\partial}^*_\phi \alpha|^2 e^{-\phi}.$$

But if $f \in N$,

$$\int f \cdot \bar{\alpha} e^{-\phi} = \int f \cdot \bar{\alpha}^1 e^{-\phi},$$

and since α^1 lies in N intersected with the domain of T^*

$$|\int f \cdot \bar{\alpha} e^{-\phi}|^2 \leq \int \sum \mu_{j\bar{k}} \alpha^1_j \bar{\alpha}^1_k e^{-\phi} \int \sum \mu^{j\bar{k}} f_j \bar{f}_k e^{-\phi} \leq$$

$$\leq \int |T^* \alpha^1|^2 e^{-\phi} \int \sum \mu^{j\bar{k}} f_j \bar{f}_k e^{-\phi} = \int |T^* \alpha|^2 e^{-\phi} \int \sum \mu^{j\bar{k}} f_j \bar{f}_k e^{-\phi}.$$

The first part therefore follows from Proposition 1.3.1.

For the converse we note that if $\alpha \in N$, and the conclusion of the Proposition holds, we can write $\alpha = Tu$. Then, if moreover α lies in the domain of T^*,

$$\int |\alpha|^2 e^{-\phi} = < Tu, \alpha > = < u, T^* \alpha > \leq ||u|| ||T^* \alpha||,$$

from which the converse follows. ■

The condition that μ be uniformly bounded and positive definite is not a very serious restriction. One main feature of the conclusion of the proposition is that the constant ($= 1!$) in the estimate for u is uniform, and we shall see in section 1.6 that this permits us to treat much more general growth conditions by simple limiting arguments.

We shall finally give somewhat more general versions of Propositions 1.3.1 and 1.3.2, which allow us to vary the estimate of the solution, as well as of the right hand side, of $\bar{\partial} u = f$.

Proposition 1.3.3 *Let w be a continuous function which is uniformly bounded and uniformly positive in Ω. Then there is a solution, u, to the equation $\bar{\partial} u = f$ satisfying*

$$\int \frac{|u|^2}{w} e^{-\phi} \leq C. \tag{1.15}$$

if and only if the inequality

$$|\int f \cdot \bar{\alpha} e^{-\phi}|^2 \leq C \int |\bar{\partial}^*_\phi \alpha|^2 w e^{-\phi}. \tag{1.16}$$

holds for all $\alpha \in D_{(0,1)}$.

Proof: The proof is virtually identical to the one of Proposition 1.3.1. Assuming 1.16 holds we find that there is a function v such that

$$\int |v|^2 w e^{-\phi} \leq C,$$

and

$$\int f \cdot \bar{\alpha} e^{-\phi} = \int v \overline{\bar{\partial}_\phi^* \alpha} w e^{-\phi},$$

for all $\alpha \in D_{(0,1)}$. Letting $u = vw$ we see that u solves $\bar{\partial} u = f$, and satisfies 1.15. ■

Just like before this implies

Proposition 1.3.4 *Let $\mu = (\mu_{j\bar{k}})$ be a continuous function defined in Ω whose values are hermitean $n \times n$ matrices. Assume that μ is uniformly bounded and uniformly positive definite on Ω. Let w be a continuous function which is uniformly bounded and uniformly positive in Ω. Suppose that for any α in Dom(T^*) $\cap N$ it holds*

$$\int \sum \mu_{j\bar{k}} \alpha_j \bar{\alpha}_k e^{-\phi} \leq \int |\bar{\partial}_\phi^* \alpha|^2 w e^{-\phi}.$$

Then, for any $f \in L^2_{(0,1)}(e^{-\phi})$ satisfying $\bar{\partial} f = 0$, there is a solution, u to $\bar{\partial} u = f$ satisfying

$$\int \frac{|u|^2}{w} e^{-\phi} \leq \int \sum \mu^{j\bar{k}} f_j \bar{f}_k e^{-\phi}.$$

If μ is a constant multiple of the identity matrix, the converse to this also holds.

Proof: By Proposition 1.3.3 we just need to verify that

$$|\int f \cdot \bar{\alpha} e^{-\phi}|^2 \leq \int \sum \mu^{j\bar{k}} f_j \bar{f}_k e^{-\phi} \int |\bar{\partial}_\phi^* \alpha|^2 w e^{-\phi},$$

for all α in $D_{(0,1)}$. Following the proof of Proposition 1.3.2 we decompose $\alpha = \alpha^1 + \alpha^2$, where $\alpha^1 \in N$ and α^2 is orthogonal to N in $L^2(e^{-\phi})$. Then

$$\int f \cdot \bar{\alpha} e^{-\phi} = \int f \cdot \bar{\alpha}^1 e^{-\phi},$$

since $f \in N$. Since α^1 lies in N and in the domain of T^*, by our hypothesis

$$|\int f \cdot \bar{\alpha}^1 e^{-\phi}|^2 \leq \int \sum \mu^{j\bar{k}} f_j \bar{f}_k e^{-\phi} \int |\bar{\partial}_\phi^* \alpha^1|^2 w e^{-\phi}.$$

On the other hand $\bar{\partial}_\phi^* \alpha^1 = \bar{\partial}_\phi^* \alpha$, so the proof is complete. ■

We will have use for the last two propositions in section 1.8.

1.4 The basic (in)equality

Notice what we have gained through Proposition 1.3.2. To prove an estimate for solutions to the $\bar{\partial}$-equation it is now enough to be able to control a form α in N, i e a form satisfying $\bar{\partial}\alpha = 0$ by $\bar{\partial}_\phi^* \alpha$. But we have also lost something. Before we were dealing with forms that were smooth and had compact support. This information we have now lost, the only additional information we have on α is that $\alpha \in$Dom(T^*). The strategy is to first prove the estimate we are looking for assuming that α is smooth up to the boundary, and then remove this assumption by an approximation lemma. To do this we shall first investigate what it means for a smooth form to lie in Dom(T^*).

Let ρ be a function of class C^2 in a neighbourhood of Ω such that

$$\Omega = |z \in U; \rho(z) < 0\} \quad \text{and} \quad \nabla\rho \neq 0 \text{ on } \partial\Omega.$$

We then have

Lemma 1.4.1 *Suppose α is a $(0,1)$-form of class C^1 on $\bar{\Omega}$, and that $\alpha \in$ Dom T^*. Then*

$$\sum \alpha_j \frac{\partial\rho}{\partial z_j} = 0 \quad on \quad \partial\Omega. \tag{1.17}$$

Proof: First note that the divergence theorem on Ω takes the following form in complex notation: if $a, b \in C^1(\bar{\Omega})$ then

$$\int_\Omega b\frac{\partial a}{\partial \bar{z}_j} d\lambda = -\int_\Omega a\frac{\partial b}{\partial \bar{z}_j} d\lambda + \int_{\partial\Omega} ab\frac{\partial\rho}{\partial\bar{z}_j}\frac{dS}{|\partial\rho|}$$

Now let u be of class C^1 on $\bar{\Omega}$. Then

$$\begin{aligned} <\bar{\partial}u, \alpha> &= \int_\Omega \sum \frac{\partial u}{\partial\bar{z}_j}\bar{\alpha}_j e^{-\varphi} d\lambda = \\ &= \int_\Omega u\overline{\bar{\partial}^*_\varphi \alpha} e^{-\varphi} d\lambda + \int_{\partial\Omega} u\overline{\sum \alpha_j \frac{\partial\rho}{\partial z_j}} e^{-\varphi}\frac{dS}{|\partial\rho|} =< u, T^*\alpha > \end{aligned}$$

if $\alpha \in$ Dom T^*. By first taking u with compact support in Ω we see that

$$T^*\alpha = \bar{\partial}^*_\varphi \alpha$$

(which we already knew). But this means that

$$\int_{\partial\Omega} u\overline{\sum \alpha_j \frac{\partial\rho}{\partial z_j}} e^{-\varphi}\frac{dS}{|\partial\rho|}$$

must vanish for <u>any</u> u. Clearly this means that

$$\sum \alpha_j \frac{\partial\rho}{\partial z_j} = 0 \quad \text{on} \quad \partial\Omega.$$

■

If $n = 1$, (1.17) just means that α vanishes on $\partial\Omega$. In higher dimensions it says that the component of α in the direction of the complex normal vanishes. Actually the converse to Lemma 1 also holds; if $\alpha \cdot \partial\rho = 0$ on $\partial\Omega$ then $\alpha \in$ Dom T^*. The proof of this follows from the same calculation we have just done but it requires an approximation of a general element in Dom T by smooth functions, and since that is the object of the next section we omit it.

The following identity is basic for everything that follows.

Theorem 1.4.2 *Assume that $\alpha \in$ Dom T^* and is of class $C^2(\bar{\Omega})$. Assume also that $\rho, \varphi \in C^2(\bar{\Omega})$. Then*

$$\int_\Omega \sum \varphi_{j\bar{k}}\alpha_j\bar{\alpha}_k e^{-\varphi} + \int_\Omega \sum \left|\frac{\partial\alpha_j}{\partial\bar{z}_k}\right|^2 e^{-\varphi} + \int_{\partial\Omega} \sum \rho_{j\bar{k}}\alpha_j\bar{\alpha}_k e^{-\varphi}\frac{dS}{|\partial\rho|} = \int_\Omega |\bar{\partial}^*_\varphi\alpha|^2 e^{-\varphi} + \int_\Omega |\bar{\partial}\alpha|^2 e^{-\varphi}. \tag{1.18}$$

Here

$$|\bar\partial\alpha|^2 = \sum_{j<k}\left|\frac{\partial\alpha_j}{\partial\bar z_k} - \frac{\partial\alpha_k}{\partial\bar z_j}\right|^2.$$

Proof. Consider the expression

$$I = \int_\Omega |\bar\partial^*_\varphi\alpha|^2 e^{-\varphi}d\lambda = < \bar\partial^*_\varphi\alpha, \bar\partial^*_\varphi\alpha > .$$

Since $\alpha \in \text{Dom } T^*$

$$I = \int \alpha \cdot \overline{\bar\partial\bar\partial^*_\varphi\alpha} e^{-\varphi}.$$

Now

$$\bar\partial^*_\varphi\alpha = -\sum e^{\varphi}\frac{\partial}{\partial z_j}e^{-\varphi}\alpha_j = -\sum \delta_j\alpha_j,$$

where

$$\delta_j = e^{\varphi}\frac{\partial}{\partial z_j}e^{-\varphi} = \frac{\partial}{\partial z_j} - \frac{\partial\varphi}{\partial z_j}.$$

Thus

$$\bar\partial\bar\partial^*_\varphi = -\sum\frac{\partial}{\partial\bar z_k}\delta_j\alpha_j d\bar z_k.$$

Observe that

$$\frac{\partial}{\partial\bar z_k}\delta_j\alpha_j = \frac{\partial^2}{\partial\bar z_k\partial z_j}\alpha_j - \frac{\partial\varphi}{\partial z_j}\frac{\partial\alpha_j}{\partial\bar z_k} - \frac{\partial^2\varphi}{\partial z_j\partial\bar z_k}\alpha_j.$$

On the other hand

$$\delta_j\frac{\partial}{\partial\bar z_k}\alpha_j = \frac{\partial^2\alpha_j}{\partial z_j\partial\bar z_k} - \frac{\partial\varphi}{\partial z_j}\frac{\partial\alpha_j}{\partial\bar z_k}$$

so

$$\bar\partial\bar\partial^*_\varphi\alpha = -\sum\delta_j\frac{\partial\alpha_j}{\partial\bar z_k}d\bar z_k + \sum\varphi_{j\bar k}\alpha_j d\bar z_k.$$

Note also that if $a, b \in C^1(\Omega)$ then

$$\int_\Omega \frac{\partial a}{\partial\bar z_k}\bar b e^{-\varphi}d\lambda = -\int_\Omega a\overline{\delta_j b}e^{-\varphi}d\lambda + \int_{\partial\Omega} a\bar b e^{-\varphi}\frac{\partial\rho}{\partial\bar z_k}\frac{dS}{|\partial\rho|}. \tag{1.19}$$

Collecting we have

$$I = \int_\Omega \alpha\overline{\bar\partial\bar\partial^*_\varphi\alpha}e^{-\varphi}d\lambda =$$

$$= \int_\Omega\sum\varphi_{j\bar k}\alpha_j\bar\alpha_k e^{-\varphi}d\lambda + \int -\sum\alpha_k\overline{\delta_j\frac{\partial\alpha_j}{\partial\bar z_k}}e^{-\varphi}d\lambda = I_1 + I_2.$$

By (1.19)

$$I_2 = \int_\Omega\sum\frac{\partial\alpha_k}{\partial\bar z_j}\overline{\frac{\partial\alpha_j}{\partial\bar z_k}}e^{-\varphi}d\lambda - \int_{\partial\Omega}\sum\alpha_k\overline{\frac{\partial\alpha_j}{\partial\bar z_k}}\frac{\partial\rho}{\partial\bar z_j}e^{-\varphi}\frac{dS}{|\partial\rho|} = I_3 - I_4.$$

Let us first consider the boundary term I_4: We know that

$$\sum\alpha_j\frac{\partial\rho}{\partial z_j} = 0 \quad \text{on} \quad \partial\Omega. \tag{1.20}$$

This means that this expression still vanishes on $\partial\Omega$ if we apply a tangential operator to it. So let us apply the operator

$$\sum\alpha_k\frac{\partial}{\partial z_k}$$

((1.20) means precisely that this operator is tangential.) If we first take conjugates, we find

$$\sum \alpha_k \frac{\partial}{\partial z_k}(\bar{\alpha}_j \frac{\partial \rho}{\partial \bar{z}_j}) = 0$$

so

$$-\sum \alpha_k \overline{\frac{\partial \alpha_j}{\partial \bar{z}_k}} \frac{\partial \rho}{\partial \bar{z}_j} = \sum \alpha_k \bar{\alpha}_j \rho_{k\bar{j}}.$$

Thus

$$-I_4 = \int_\Omega \sum \rho_{jk} \alpha_j \bar{\alpha}_k e^{-\varphi} \frac{dS}{|\partial \rho|}. \tag{1.21}$$

It now remains to compute I_3. This is done by verifying the identity

$$I_3 = \int \left(\frac{-1}{2} \sum \left| \frac{\partial \alpha_k}{\partial \bar{z}_j} - \frac{\partial \alpha_j}{\partial \bar{z}_k} \right|^2 + \sum \left| \frac{\partial \alpha_k}{\partial \bar{z}_j} \right|^2 \right) e^{-\varphi} = \frac{-1}{2} \int |\bar{\partial}\alpha|^2 e^{-\varphi} + \int \sum \left| \frac{\partial \alpha_k}{\partial \bar{z}_j} \right|^2 e^{-\varphi},$$

which can be done e g by expanding the expression within the brackets in the middle term.

Collecting we find

$$I = \int_\Omega |\bar{\partial}^*_\varphi \alpha|^2 e^{-\varphi} d\lambda = I_1 + I_2 = \int_\Omega \sum \varphi_{j\bar{k}} \alpha_j \bar{\alpha}_k e^{-\varphi} d\lambda + I_3 - I_4 =$$

$$\int_\Omega \sum \varphi_{j\bar{k}} \alpha_j \bar{\alpha}_k e^{-\varphi} d\lambda + \int_\Omega \sum \left| \frac{\partial \alpha_j}{\partial \bar{z}_k} \right|^2 e^{-\varphi} d\lambda - \frac{1}{2} \int_\Omega |\bar{\partial}\alpha|^2 e^{-\varphi} + \int_{\partial\Omega} \sum \rho_{j\bar{k}} \alpha_j \bar{\alpha}_k e^{-\varphi} \frac{dS}{|\partial \rho|}.$$

This is precisely the formula in the theorem. ■

Assume now that Ω is pseudoconvex. Then it follows from Theorem 2 that

$$\int_\Omega \sum \varphi_{j\bar{k}} \alpha_j \bar{\alpha}_k e^{-\varphi} d\lambda \leq \int_\Omega |\bar{\partial}^*_\varphi \alpha|^2 e^{-\varphi}$$

if α is sufficiently smooth, $\alpha \in \mathrm{Dom}\, T^*$ and $\bar{\partial}\alpha = 0$. If moreover φ is plurisubharmonic, this is precisely the kind of inequality we need to apply Proposition 1.3.2 . The problem that remains to take care of is to avoid the assumption that α be smooth up to the boundary. This is the object of the next section.

1.5 Approximation of L^2-forms by smooth forms

The basic tool for the approximation is of course convolution with a smooth approximation to the identity. Since the first part of the argument has nothing to do with the complex structure, we will start our discussion in $\mathbb{R}^N$.

Let $\varphi \geq 0$ be in $C^\infty_c(\mathbb{R}^N)$ and assume

$$\int_{\mathbb{R}^N} \varphi(x) d\lambda = 1.$$

Let also

$$\varphi_\epsilon(x) = \frac{1}{\epsilon^N} \varphi(\frac{x}{\epsilon}).$$

If $v \in L^2_{\rm loc}$, we consider

$$\varphi_\epsilon * v(x) = \int v(x-y)\varphi_\epsilon(y)d\lambda(y) = \int v(x-\epsilon y)\varphi(y)dy.$$

Then it is well known that

$$\varphi_\epsilon * v \to v \quad \text{in} \quad L^2_{\rm loc}, \quad \text{as} \quad \epsilon \to 0.$$

Moreover, $\varphi_\epsilon * v$ is a smooth function.

Lemma 1.5.1 *Let $a \in C^1$ and $v \in L^2$ and assume v has compact support. Define*

$$A_\epsilon = a\frac{\partial}{\partial x_i}(v * \varphi_\epsilon) - (a\frac{\partial}{\partial x_i}v) * \varphi_\epsilon.$$

Then $A_\epsilon \to 0$ in L^2 as $\epsilon \to 0$.

Proof. If $v \in C_c^\infty(\mathbb{R}^N)$, the result is clear since we even have uniform convergence. We will show that for some C

$$\|A_\epsilon\|_{L^2} \leq C\|v\|_{L^2}, \tag{1.22}$$

which gives the claim in general since we can approximate L^2-functions by smooth ones.

$$\begin{aligned} A_\epsilon(x) &= a(x)\int \frac{\partial}{\partial x_i}v(x-\epsilon y)\varphi(y)d\lambda - \int a(x-\epsilon y)\frac{\partial}{\partial x_i}v(x-\epsilon y)\varphi(y)d\lambda \\ &= -\int [a(x) - a(x-\epsilon y)]\frac{\partial}{\partial y_i}v(x-\epsilon y)/\epsilon\,\varphi(y)d\lambda = \\ &= \int \frac{\partial}{\partial x_i}a(x-\epsilon y)v(x-\epsilon y)\varphi(y)d\lambda + \\ &+ \int \frac{a(x)-a(x-\epsilon y)}{\epsilon}v(x-\epsilon y)\frac{\partial\varphi}{\partial y_i}d\lambda = A_1 + A_2. \end{aligned}$$

Since $a \in C^1$, $\partial a/\partial x_i$ is uniformly bounded on compacts. Hence

$$|A_1| \leq C\left|\int v(x-\epsilon y)\varphi(y)d\lambda\right|$$

where C can be taken uniform for all v with support in a fix compact. Hence

$$\|A_1\|_{L^2} \leq C\|v\|_{L^2}.$$

On the other hand

$$\left|\frac{a(x)-a(x-\epsilon y)}{\epsilon}\right| \leq C'|y|$$

where C' also is uniform for x and y in compact sets. Therefore A_2 can be estimated in the same way.

We will also have use for a more general version of the lemma. Let ψ be a function satisfying the same conditions as φ but defined on $\mathbb{R}^{N-1}$. We can use ψ to define a partial regularisation by

$$\psi_\epsilon \wedge v = \int_{\mathbb{R}^{N-1}} v(x'-\epsilon y', x_N)\psi(y')d\lambda \tag{1.23}$$

where we write $x = (x', x_N)$. We note for later use that if we define A_ϵ using this operation instead of convolution with φ_ϵ Lemma 1 still holds, provided we only consider derivatives $\partial/\partial x_i$ where $i < N$.

Next, we consider a linear differential operator of first order with variable coefficients of the form

$$v = (v_1, \ldots, v_m) \to: Lv = \sum_{i,k} a_i^k \frac{\partial}{\partial x_i} v_k, \tag{1.24}$$

where we assume that $a_i^k \in C^2$. We can then use Lemma 1 to prove the following important

Proposition 1.5.2 *Assume that $v \in L^2$ and that moreover Lv (in the sense of distributions) also lies in L^2_{loc}. Then*

$$L(v * \varphi_\epsilon) \to Lv \quad as \quad \epsilon \to 0 \quad in \quad L^2_{\text{loc}}.$$

It also holds that

$$L(v \wedge \psi_\epsilon) \to Lv \quad as\ \epsilon \to 0\ in\ L^2_{\text{loc}},$$

provided a_N^k is constant.

The important feature of the proposition is that we assume only that $Lv \in L^2_{\text{loc}}$, not that this holds for all the derivatives of v separately. We will have use for this later on when we deal with forms satisfying e.g. $\bar\partial^* \alpha \in L^2$, whereas we know nothing about other derivatives of α.

Proof. We may assume that v has compact support since otherwise we can multiply v by a smooth function with compact support. Since $Lv \in L^2$ we have

$$\sum (a_i^k \frac{\partial}{\partial x_i} v_k) * \varphi_\epsilon \to Lv \quad \text{as} \quad \epsilon \to 0.$$

Now Lemma 1.5.1 tells us that

$$a_i^k \frac{\partial}{\partial x_i}(v_k * \varphi_\epsilon) - (a_i^k \frac{\partial}{\partial x_i} v_k) * \varphi_\epsilon \to 0 \quad \text{as} \quad \epsilon \to 0. \tag{1.25}$$

Hence

$$L(v * \varphi_\epsilon) \to Lv$$

so we are done. To see the second statement, we note that according to the comment after Lemma 1.5.1, (1.25) still holds if we change convolution with φ_ϵ to convolution with ψ_ϵ with respect to x', provided that $i < N$. But if $i = N$, (1.25) is trivial since we have assumed a_N^k are constant. Hence the same proof works. ■

Before proceeding, we also note that the proposition is till valid if we add lower order terms

$$\sum b_k v_k$$

to the definition of L, where say b_k are continuous.

Let us now return to our forms living in a domain Ω in $\mathbb{C}^n$. We assume that Ω is smoothly bounded and is relatively compact. Denote

$$E = L^2_{(0,1)}(\Omega, \varphi) \cap \text{Dom}\, \bar\partial \cap \text{Dom}\, T^*.$$

Proposition 1.5.3 *Assume $\alpha \in E$. Then there are $\alpha^\nu \in E \cap C^\infty_{(0,1)}(\bar\Omega)$ $\quad k = 1, 2, \ldots$ such that*

$$|\!|\!|\alpha^\nu - \alpha|\!|\!| =: \|\alpha^\nu - \alpha\| + \|T^*(\alpha^\nu - \alpha)\| + \|\bar\partial(\alpha^\nu - \alpha)\|$$

tends to zero as ν goes to infinity.

This is the main result of this section and its proof requires some more preparations. The first step is to localize the problem.

Lemma 1.5.4 *Assume $\alpha \in E$ and $\chi \in C^\infty(\bar\Omega)$. Then $\chi\alpha \in E$.*

Proof. This is easy. It is obvious that $\chi\alpha \in L^2$ and that $\bar\partial\chi\alpha \in L^2$. If $u \in \text{Dom } T$ then

$$< Tu, \chi\alpha >=< \chi Tu, \alpha >=< T\chi u, \alpha > - < u\bar\partial\chi, \alpha > .$$

But

$$\begin{aligned} | < T\chi u, \alpha > | &\leq \|\chi u\| \|T^*\alpha\| \quad \text{and} \\ | < u\bar\partial\chi, \alpha > | &\leq C\|u\|\|\alpha\|. \end{aligned}$$

Hence

$$\| < Tu, \chi\alpha > | \leq C\|u\|$$

where C is independent of u, which shows that $\chi\alpha \in \text{Dom } T^*$. ∎

Take a partition of unity $\{\chi_j\}$ in $\bar\Omega$ so that

$$\alpha = \sum \chi_j\alpha.$$

It is enough to prove Proposition 1.5.3 for all $\chi_j\alpha$. If χ_j has compact support in Ω, the proof is very simple since we just approximate $\chi_j\alpha$ with

$$(\chi_j\alpha) * \varphi_\epsilon$$

and apply the first part of Proposition 1.5.2 (in a very simple form since all the coefficients of first order terms in $\bar\partial$ and T^* are constant). So consider a χ_j whose support intersects $\partial\Omega$, and write for simplicity α for $\chi_j\alpha$. That is, we have reduced the problem to a situation where α has its support in a small neighbourhood, U, of a boundary point. Let this boundary point be 0 and assume the tangent plane to $\partial\Omega$ at 0 is $\{\text{Im } z_n = 0\}$. Let K be a truncated open cone with vertex at 0 that contains the positive Im z_n-axis. By taking K and U small enough, we may assume that

$$\begin{aligned} p + K &\subseteq \Omega^c \quad \text{and} \\ p - K &\subseteq \Omega \quad \text{for all } p \in U \cap \partial\Omega. \end{aligned}$$

Choose a function $\varphi^+ \in C_c^\infty(K)$ such that

$$\varphi^+ \geq 0 \quad \text{and} \quad \int_{\mathbb{C}^n} \varphi^+ = 1,$$

and let

$$\varphi^-(z) = \varphi^+(-z),$$

these two functions will be used as convolutors to approximate α.

Remember that our form α is defined in Ω and has its support in U. We can extend α to a form in L^2_{loc} by letting it be identically 0 in the complement of Ω. Somewhat abusively we call this form $\chi_\Omega\alpha$, where χ_Ω is the characteristic function of Ω.

Lemma 1.5.5 *$\alpha \in Dom\ T^*$ iff*

$$\bar\partial^*_\varphi \chi_\Omega\alpha = \chi_\Omega\bar\partial^*_\varphi\alpha \quad \text{and} \quad \bar\partial^*_\varphi\alpha \in L^2$$

in the sense of distributions.

Proof. Let $u \in C_c^\infty(\mathbb{C}^n)$. Then $\alpha \in$Dom(T^*) if and only if,

$$\begin{aligned} \int \bar{\partial} u \cdot \overline{\chi_\Omega \alpha} e^{-\varphi} d\lambda &= \int_\Omega \bar{\partial} u \cdot \bar{\alpha} e^{-\varphi} =< u, T^* \alpha >= \\ &= \int_\Omega u \overline{e^\varphi \bar{\partial}^* e^{-\varphi} \alpha} e^{-\varphi} d\lambda. \end{aligned}$$

This means precisely the same as the statement in the lemma. ■

Lemma 1.5.6 *Assume $\alpha \in$ Dom T^*, and that α is supported in U. Let $\alpha^\epsilon = \chi_\Omega \alpha * \varphi_\epsilon^-$. Then $\alpha^\epsilon \in$ Dom T^*, $\alpha^\epsilon \to \alpha$ in $L^2_{(0.1)}(\Omega, \varphi)$ and $T^* \alpha^\epsilon \to T^* \alpha$ in $L^2(\Omega, \varphi)$. Moreover $\alpha^\epsilon = 0$ in Ω^c.*

Proof. By general properties of approximate identities $\alpha^\epsilon \in C^\infty$ and $\alpha^\epsilon \to X_\Omega \alpha$ in L^2. Moreover,

$$\bar{\partial}^*_\varphi \alpha^\epsilon \to \bar{\partial}^*_\varphi \alpha$$

by Lemma 1.5.5 and Proposition 1.5.2 (still in its simple form where all first order coefficients are constant). We need therefore only prove the last statement since $\alpha^\epsilon \in$ Dom T^* follows from this. But

$$\alpha^\epsilon = \int \alpha(z - \epsilon\zeta) \varphi^-(\zeta) d\lambda(\zeta)$$

so if $z \in \Omega^c$ and $\zeta \in$ supp φ^-, the integrand is zero. ■

The lemma says that it is easy to approximate α and $T^*\alpha$ if we do not care about what happens to $\bar{\partial}\alpha$. On the other hand, we can also approximate $\bar{\partial}\alpha$ if we regularize with φ^+_ϵ instead of φ^-_ϵ, but then we lose control over $T^*\alpha$. What we will do is then to decompose α into one normal and one tangential part and use φ^- for the normal part and φ^+ for the tangential one. This however requires that all partial derivatives of (coefficients of) α be in L^2 and to obtain this, we first perform a preliminary regularization "in the tangential direction". This is basically the crux of the proof.

We may assume without loss of generality that we can choose real coordinates $x_1, \dots, x_N$ ($N = 2n$) in U so that $x_N = \rho$. By the Gram-Schmidt process we can obtain $(1,0)$-forms $w_1 \dots, w_n$ that form an orthonormal basis for the $(1,0)$-forms of each point in U (possibly after shrinking U), where moreover $w_n = \partial\rho$. Then $\bar{w}_1, \dots, \bar{w}_n$ form a basis for the $(0,1)$-forms. If u is a function, we define the differential operators ∂_k and $\bar{\partial}_k$ by

$$\partial u = \sum \partial_k u w_k, \quad \bar{\partial} u = \sum \bar{\partial}_k u \bar{w}_k.$$

It is not hard to check that if we express a form

$$\alpha = \sum \alpha_k d\bar{z}_k$$

in this basis as

$$\alpha = \sum A_k \bar{w}_k$$

then

$$\bar{\partial}^*_\varphi \alpha = -\sum \partial_k A_k + \dots \text{ and} \tag{1.26}$$

$$\bar{\partial}\alpha = \sum \bar{\partial}_j A_k \bar{w}_j \wedge \bar{w}_k + \dots \tag{1.27}$$

where the dots indicate terms that contain no derivatives of the A_k:s.

Express the operators ∂_k and $\bar{\partial}_k$ in the real coordinates $x_1, \dots, x_N$ as

$$\partial_k = \sum_1^N a_{kj} \frac{\partial}{\partial x_j} \quad \bar{\partial}_k = \sum_1^N \overline{a_{kj}} \frac{\partial}{\partial x_j} \tag{1.28}$$

Then

$$a_{kN} = \partial_k x_N,$$

and since

$$w_n = \partial\rho = \partial x_N = \sum \partial_k x_N w_k$$

we see that

$$a_{nN} = 1 \quad \text{and} \quad a_{kN} = 0 \quad \text{for} \quad k < n. \tag{1.29}$$

In particular, all the derivatives with respect to x_N that occur in T^* and $\bar{\partial}$ (when expressed in the basis w_j) are constant. This means that we are in a positition to apply Proposition 1.5.2.

Regularizing the coefficients A_k with respect to the variables $x_1, \ldots, x_{N-1}$ as in the last part of Proposition 1.5.2 we obtain a sequence of forms α'^{ν} such that

$$\alpha'^{\nu} \to \alpha,\ \bar{\partial}^*_{\varphi}\alpha'^{\nu} \to \bar{\partial}'^*_{\varphi}\alpha \quad \text{and} \quad \bar{\partial}\alpha'^{\nu} \to \bar{\partial}\alpha$$

in $L^2(u, e^{-\varphi})$. We claim that moreover the α^{ν}:s still lie in Dom T^*. To see this, recall Lemma 1.3.5, which after our change of basis says that $\alpha \in$ Dom T^* iff

$$\frac{\partial}{\partial x_N}\chi_\Omega A_n = \chi_\Omega \frac{\partial A_n}{\partial x_N},$$

and note that this property evidently is unchanged by regularization in the $x_1, \ldots, x_{N-1}$ variables.

Our next claim is that all partial derivatives

$$\frac{\partial A'^{\nu}{}_k}{\partial x_j} \in L^2(U \cap \Omega).$$

This is evident if $j < N$ and follows for $j = N$ since

$$\bar{\partial}^*_{\varphi}\alpha'^{\nu} \in L^2 \quad \text{and} \quad \alpha'^{\nu} \in L^2$$

because derivatives with respect to X_N can be expressed in terms of these operators and derivatives with respect to x_j for $j < N$, using (1.26), (1.26), (1.28) and (1.29).

The conclusion of all this is that we may assume that the form we wish to approximate by smooth forms has all its partial derivatives in $L^2(U \cap \Omega)$.

We are now finally able to define the sequence α^{ν}:

Let

$$A^n u_k = A_k * \varphi^+_{\epsilon_\nu} \quad \text{if} \quad k < n$$

and

$$A^{\nu}_n = (X_\Omega A_n) * \varphi^-_{\epsilon_\nu},$$

and let

$$\alpha^{\nu} = \sum_1^n A^{\nu}_k \bar{w}_k.$$

Then

$$\|\alpha^{\nu} - \alpha\| \to 0$$

since all derivatives of components of α^{ν} converge to the corresponding expression for α. ■

1.6 Existence Theorems

We first prove

Theorem 1.6.1 *Let ϕ be a strictly plurisubharmonic function in $C^\infty(\bar{\Omega})$, where Ω is a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$. Let f be a $\bar{\partial}$-closed $(0,1)$-form in $L^2_{(0,1)}(e^{-\phi})$. Then there is a solution, u to $\bar{\partial}u = f$ which satisfies*

$$\int_\Omega |u|^2 e^{-\phi} \leq \int_\Omega \sum \phi^{j\bar{k}} f_j \bar{f}_k e^{-\phi}.$$

Proof: By Proposition 1.3.2 we need only verify the inequality

$$\int \sum \phi_{j\bar{k}} \alpha_j \bar{\alpha}_k e^{-\phi} \leq \int |\bar{\partial}^*_\phi \alpha|^2 e^{-\phi}$$

for all $\alpha \in$Dom$(T^*) \cap N$. If moreover α is smooth up to the boundary, this inequality follows immediately from Theorem 1.4.2 (remember that $\bar{\partial}\alpha = 0$ since α lies in N, and note that the boundary integral is nonnegative since Ω is pseudoconvex.)

In the general case we apply Proposition 1.5.3. Since the inequality we look for holds for each α_ν in the approximating sequence, it also holds for a general α in Dom$(T^*) \cap N$. ■

Next we will eliminate some of the smoothness assumptions in the theorem.

Lemma 1.6.2 *Let ϕ_k be a sequence of continuous functions in Ω decreasing to ϕ. Let f be a $\bar{\partial}$-closed $(0,1)$-form in Ω, and let u_k be the solution to $\bar{\partial}u = f$ which is of minimal norm in $L^2(e^{-\phi_k}) =: L^2_k$. Assume (the increasing sequence)*

$$A_k = ||u_k||_{L^2_k}$$

is bounded.

Then the sequence $\{u_k\}$ converges weakly in each L^2_m, to a function u in L^2_{loc}. The limit function u solves $\bar{\partial}u = f$, $u \in L^2(e^{-\phi})$ and

$$||u||_{L^2(e^{-\phi})} = lim A_k.$$

Furthermore, u is the solution to $\bar{\partial}u = f$ which is of minimal norm in $L^2(e^{-\phi})$.

Proof: If $k \geq m$ clearly

$$\int |u_k|^2 e^{-\phi_m} \leq \int |u_k|^2 e^{-\phi_k} = A_k^2.$$

If A_k is bounded we can therefore select a subsequence converging weakly in $L^2(e^{-\phi_m})$. By a diagonal argument we may even find a subsequence converging in $L^2(e^{-\phi_m})$ for all m. To avoid using too many indices, we still denote the subsequence u_k. Then in particular u_k converges weakly in L^2_{loc}. Call the limit function u.

Clearly $\bar{\partial}u = f$. Since weak limits decrease norms we have for any m

$$\int |u|^2 e^{-\phi_m} \leq \liminf \int |u_k|^2 e^{-\phi_m} \leq \lim A_k^2.$$

By monotone convergence

$$\int |u|^2 e^{-\phi} \leq \lim A_k^2.$$

On the other hand , if u_0 is the solution to $\bar\partial u = f$ of minimal norm in $L^2(e^{-\phi})$, then

$$\int |u_m|^2 e^{-\phi_m} \leq \int |u_0|^2 e^{-\phi_m} \leq \int |u_0|^2 e^{-\phi} \leq \int |u|^2 e^{-\phi} \leq \lim A_k^2.$$

Letting m tend to infinity we see that

$$\int |u_0|^2 e^{-\phi} = \int |u|^2 e^{-\phi} = \lim A_k^2.$$

Since the minimal solution is unique, we see that $u_0 = u$. Therefore any convergent subsequence converges to the minimal solution, so the entire sequence must converge, and the lemma is proved. ■

We will also need an analogous statement when the domain varies.

Lemma 1.6.3 *Let Ω_k be an increasing sequence of domains in $\mathbb{C}^n$ with union Ω, and let ϕ be a plurisubharmonic function in Ω. Let f be a $\bar\partial$-closed $(0,1)$-form in Ω, and let u_k be the solution to $\bar\partial u = f$ of minimal norm in $L^2(\Omega_k, e^{-\phi})$. Suppose*

$$A_k^2 = \int_{\Omega_k} |u_k|^2 e^{-\phi}$$

is a bounded sequence. Then $\{u_k\}$ converges weakly in all $L^2(\Omega_k, e^{-\phi})$, to a function, u, in $L^2_{loc}(\Omega)$. The limit function is then the $L^2(\Omega, e^{-\phi})$-minimal solution to $\bar\partial u = f$ and

$$\int_\Omega |u|^2 e^{-\phi} = \lim A_k^2.$$

Proof: As in the proof of the last lemma we can select a subsequence, still denoted u_k, that converges weakly in all $L^2(\Omega_m, e^{-\phi})$.

The limit function then lies in L^2_{loc} and solves $\bar\partial u = f$. Since,again, weak limits do not increase norms

$$\int_{\Omega_m} |u|^2 e^{-\phi} \leq \liminf \int_{\Omega_m} |u_k|^2 e^{-\phi} \leq \lim \int_{\Omega_k} |u_k|^2 e^{-\phi} = \lim A_k^2.$$

Letting m tend to infinity we get

$$\int_\Omega |u|^2 e^{-\phi} \leq \lim A_k^2.$$

On the other hand, if u_0 denotes the solution of minimal norm in $L^2(e^{-\phi})$, then

$$\int_{\Omega_m} |u_m|^2 e^{-\phi} \leq \int_{\Omega_m} |u_0|^2 e^{-\phi} \leq \int_\Omega |u_0|^2 e^{-\phi} \leq \int_\Omega |u|^2 e^{-\phi}.$$

Letting m tend to infinity again we see that

$$\int |u|^2 e^{-\phi} = \int |u_0|^2 e^{-\phi} = \lim A_k^2.$$

In particular, by the uniqueness of the minimal solution $u = u_0$, so the entire sequence is convergent and the proof is complete.

■

With the aid of these two lemmas we can prove a more general version of Theorem 1.6.1.

Theorem 1.6.4 *Let Ω be a pseudoconvex domain in $\mathbb{C}^n$, and let ϕ be plurisubharmonic in Ω. Suppose $\phi = \psi + \xi$, where ξ is an arbitrary plurisubharmonic function, and ψ is a smooth, strictly plurisubharmonic function. Then for any f, a $\bar\partial$-closed $(0,1)$-form in Ω, we can solve $\bar\partial u = f$, with u satisfying*

$$\int |u|^2 e^{-\phi} \leq \int \sum \psi^{j\bar{k}} f_j \bar{f}_k e^{-\phi},$$

provided the right hand side is finite.

Proof: We can write Ω as an increasing sequence of compactly included, smoothly bounded pseudoconvex domains Ω_k (see section 1.1). In each Ω_k we can write ξ as the limit of a decreasing sequence of smooth plurisubharmonic functions. By Lemma 1.6.1 the theorem will hold in each Ω_k, and by Lemma 1.6.2 it will also hold in Ω. ■

Assuming Ω to be bounded we may choose $\psi = |z|^2$. This gives the next Corollary.

Corollary 1.6.5 *Let Ω be a pseudoconvex domain contained in the ball with radius 1, and let ϕ be plurisubharmonic in Ω. Then, for any $\bar\partial$-closed $(0,1)$-form f we can solve $\bar\partial u = f$ with*

$$\int |u|^2 e^{-\phi} \leq e \int |f|^2 e^{-\phi},$$

provide the right hand side is finite.

1.7 The method of three weights

The technically most complicated part of the proof of the existence theorems in the previous section was the proof of the approximation lemma, Proposition 1.5.3. The main difficulty there comes from the regularization of a form near the boundary, where we need to respect the boundary conditions implicit in the condition $\alpha \in \mathrm{Dom}(T^*)$. There is one case in which this difficulty does not appear, namely when there is no boundary, i e when $\Omega = \mathbb{C}^n$. In that case it is not hard to see that compactly supported forms are dense in the domain of T^*, and regularization is achived by a trivial convolution with an approximate identity. With more work, the same situation can be arranged in general domains, by choosing weight functions that explode near the boundary. This is the approach taken in [2], and in this section we shall give a brief indication of how it works.

Let as before ϕ be a weight function which is smooth inside Ω, and let in addition ψ be another smooth function in Ω, which will be specified later. We shall use the following three weighted L^2-spaces, consisting of functions, $(0,1)$-forms and $(0,2)$-forms respectively.

$$L^2(e^{-\phi}) = \{u \in L^2_{loc}; \int |u|^2 e^{-\phi} < \infty\} =: H_1,$$

$$L^2_{(0,1)}(e^{-\phi-\psi}) = \{f; f(0,1)-\text{form}, \int |f|^2 e^{-\phi-\psi} < \infty\} =: H_2,$$

and

$$L^2_{(0,2)}(e^{-\phi-2\psi}) = \{g; g(0,2)-\text{form}, \int |g|^2 e^{-\phi-2\psi} < \infty\} =: H_3.$$

As before we get a densily defined operator, T, from H_1 to H_2 by letting $Tu = \bar\partial u$, for any u such that $\bar\partial u$ in the sense of distributions lies in H_2. We then have the following analog of Proposition 1.3.2

Proposition 1.7.1 *Let $\mu = (\mu_{j\bar{k}})$ be a function defined in Ω whose values are positive definite hermitean $n \times n$ matrices. Assume that μ is uniformly bounded and uniformly strictly positive definite on Ω.Suppose that for any α in Dom(T^*) such that $\bar{\partial}\alpha = 0$ it holds*

$$\int \sum \mu_{j\bar{k}} \alpha_j \bar{\alpha}_k e^{-\phi-\psi} \leq \int |T^*\alpha|^2 e^{-\phi}.$$

Then, for any $f \in H_2$ satisfying $\bar{\partial} f = 0$, there is a solution, u to $\bar{\partial} u = f$ satisfying

$$\int |u|^2 e^{-\phi} \leq \int \sum \mu^{j\bar{k}} f_j \bar{f}_k e^{-\phi-\psi}.$$

If μ is a constant multiple of the identity matrix, the converse to this also holds.

Since the proof follows exactly the lines of the proof of Proposition 1.3.2, we omit it.

Next, let Ω_k be a sequence of relatively compact subdomains of Ω, with union equal to Ω. Choose also a sequence of functions χ_k with compact support in Ω such that $\chi_k = 1$ on Ω_k. Assume now that ψ tends to infinity at the boundary, so rapidly that

$$|d\chi_k|^2 \leq e^{\psi}, \tag{1.30}$$

for all k.

Proposition 1.7.2 *Assume $\alpha \in$ Dom(T^*) satisfies $\bar{\partial}\alpha \in H_3$. Then there is a sequence α^k, whose coefficients are in $C_c^\infty(\Omega)$, such that*

$$\|\alpha^k - \alpha\|_{H_1} + \|T^*(\alpha^k - \alpha)\|_{H_2} + \|\bar{\partial}(\alpha^k - \alpha)\|_{H_3}$$

tends to zero.

Proof: First we make a preliminary definition of α^k as

$$\alpha^k = \chi_k \alpha.$$

Then clearly α^k tends to α in H_1. Moreover $\bar{\partial}\alpha^k = \chi_k \bar{\partial}\alpha + (\bar{\partial}\chi)\alpha$. By condition 1.30 and dominated convergence, the second term here tends to zero in H_3, so it also follows that $\bar{\partial}\alpha^k$ tends to $\bar{\partial}\alpha$ in H_3. Finally, by testing the definition of T^* on a function in H_1 which is smooth with compact support, one sees that

$$T^*\alpha = \bar{\partial}^*_\phi (e^{-\psi}\alpha),$$

so

$$T^*\alpha^k = \chi_k T^*\alpha - \alpha \cdot \partial\xi,$$

and from this it easily follows that $T^*\alpha^k$ tends to $T^*\alpha$ in H_1. This way we have managed to approximate α with a form with compact support, and the proof is then completed by taking a convolution with a smooth approximation to the identity.

■

By Proposition 1.7.2 it therefore suffices to verify the hypothesis in proposition 1.7.1 under the additional assumption that α is smooth and has compact support. We can then use the fundamental identity, Theorem 1.4.2. If we apply that identity to $e^{-\psi}\alpha$ we obtain

$$\int \sum \phi_{j\bar{k}} \alpha_j \bar{\alpha}_k e^{-\phi-2\psi} \leq \int |T^*\alpha|^2 e^{-\phi} + \int |\bar{\partial}(e^{-\psi}\alpha)|^2 e^{-\phi} \leq$$

$$\leq \int |T^*\alpha|^2 e^{-\phi} + 2\int |\bar{\partial}\alpha|^2 e^{-\phi-2\psi} + 2\int |d\psi|^2 |a|^2 e^{-\phi-2\psi}.$$

Altogether this gives the next lemma.

Lemma 1.7.3 *Assume ψ satisfies 1.30 and that*

$$(\phi_{j\bar{k}})e^{-\psi} \geq (\mu_{j\bar{k}}) + 2|d\psi|^2 e^{-\psi}(\delta_{j\bar{k}}), \tag{1.31}$$

where $\mu_{j\bar{k}}$ is uniformly bounded and positive definite in Ω. Then, for any $f \in H_2$ such that $\bar{\partial} f = 0$ there is a solution u to $\bar{\partial} u = f$ satisfying

$$\int |u|^2 e^{-\phi} \leq \int \sum \mu^{j\bar{k}} f_j \bar{f}_k e^{-\phi-\psi}.$$

It now only remains to get rid of the factor $e^{-\psi}$ in the estimate of the Lemma. Surprisingly, this is quite easy. First note that the condition 1.30 is satisfied with $\psi = 0$ in Ω_1. Since we may of course throw away any finite number of indices k in the beginning and then renumber, we see that we may actually choose $\psi = 0$ on any relatively compact subdomain, Ω', given in advance. Next, let ϕ_0 be any strictly plurisubharmonic function in Ω, which is smooth up to the boundary, and let ξ be a smooth, strictly plurisubharmonic exhaustion function in Ω. Choose $\Omega' = \{\xi < C\}$ and replace ξ by $\xi_C =: \max(\xi, C)$. Apply the lemma with $\phi = \phi_0 + k(\xi_c)$, where k is a convex function which equals 0 for $\xi < C$. Note

$$(k(\xi)_{j\bar{k}}) \geq k'(\xi_{j\bar{k}}).$$

From here we see that 1.31 will be satisfied if we only choose k with k' sufficiently large. Hence we obtain from the lemma a solution to $\bar{\partial} u = f$ satisfying

$$\int_{\Omega'} |u|^2 e^{-\phi_0} \leq \int_{\Omega} \sum \phi^{j\bar{k}} f_j \bar{f}_k e^{-\phi} \leq \int_{\Omega} \sum \phi_0^{j\bar{k}} f_j \bar{f}_k e^{-\phi_0}.$$

Letting C tend to infinity we now obtain Theorem 1.6.1 from Lemma 1.6.2, and the rest of the existence theorems follow as in the previous section.

1.8 A more refined estimate

In Theorem 1.6.1 we have estimated the solution to $\bar{\partial} u = f$ in terms of the right hand side measured in the norm

$$\sum \phi^{j\bar{k}} f_j \bar{f}_k.$$

In many situations, if the weight function ϕ is sufficiently plurisubharmonic this means that we gain quite a lot, as compared to an estimate in terms of $|f|^2$. This gain, however, is independent of the domain, and it turns out that for special domains one can in many cases do better. We shall first give two simple examples of when this situation occurs.

First, let us consider the one variable case, and choose Δ, the unit disk, for our domain. In that case one would expect the estimate

$$\int |u|^2 \leq C \int (1 - |z|^2)^2 |f|^2 \tag{1.32}$$

to hold, since one should roughly gain one unit when solving the equation $\bar{\partial} u = f$. Comparing to the estimate in Theorem 1.2.3 this means that we would like to choose ϕ so that

$$\Delta\phi = (1 - |z|^2)^{-2}.$$

There is however no bounded function ϕ satisfying this, so 1.32 can not be proved this way.

As a second example of a similar problem remember that in Corollary 1.6.5, when the domain was contained in a ball with radius 1, we chose the weight function $\psi = |z|^2$, and got a uniform constant in the estimate for all such domains. Applying the same argument to an arbitrary bounded domain we get the constant e^{R^2} if the domain is contained in a ball of radius R. It is clear that this is not optimal, since a simple scaling argument shows that the right constant is of the order R^2.

In both of these examples we were trying to choose a uniformly bounded weight function, with Hessian $\phi_{j\bar{k}}$ as large as possible. The main point of the results in this section is that it is actually enough to produce a weight function which satisfies a good bound *on the gradient*, and has a large Hessian. We shall consider weight functions ψ that satisfy the condition

$$|\alpha \cdot \partial\psi|^2 \leq \sum \psi_{j\bar{k}}\alpha_j\bar{\alpha}_k, \tag{1.33}$$

uniformly in our domain Ω, for any $\alpha \in \mathbb{C}^n$. Notice that this is a more liberal condition than requiering that ψ be uniformly bounded, since if e g $-1 < \phi < 0$ we can put $\psi = e^{\phi}$, and obtain a function which satisfies 1.33, and has a Hessian larger than that of ϕ.

We shall now state and prove a theorem, in essence due to Donelly and Fefferman [4], that in particular solves the difficulties we encountered above.

Theorem 1.8.1 *Let Ω be a pseudoconvex domain in $\mathbb{C}^n$, and let ϕ be plurisubharmonic in Ω. Let ψ be smooth and strictly plurisubharmonic in Ω and suppose ψ satisfies the condition 1.33 for all α. Then, for any $\bar{\partial}$-closed $(0,1)$-form, f, we can solve $\bar{\partial}u = f$ with*

$$\int |u|^2 e^{-\phi} \leq 4\int \sum \psi^{j\bar{k}} f_j \bar{f}_k e^{-\phi}.$$

In particular, if Ω is bounded we can choose $\psi = R^{-2}|z-c|^2$ where $B(c,R)$ is the smallest ball containingΩ, and this way we get the right dependence of the constant in the estimates in terms of the diameter. In the second example above, we can take $\psi = -\log(1-|z|^2)$. and this way we see that 1.32 holds.

Proof of Theorem 1.8.1: By shrinking the domain slightly, and then passing to a limit like we did in section 1.6 we may assume that ψ and ϕ are smooth up to the boundary, and that $\Omega = \{\rho < 0\}$ is a smoothly bounded domain. From Theorem 1.4.2, with ϕ replaced by $\phi + \psi$ we see that if α is smooth and satisfies the boundary condition $\alpha \cdot \partial\rho = 0$, then

$$\int \sum \psi_{j\bar{k}}\alpha_j\bar{\alpha}_k e^{-\psi-\phi} \leq \int |\bar{\partial}^*_{\psi+\phi}\alpha|^2 e^{-\psi-\phi} + \int |\bar{\partial}\alpha|^2 e^{-\psi-\phi}. \tag{1.34}$$

Note that

$$\bar{\partial}^*_{\psi+\phi}\alpha = \bar{\partial}^*_{\psi/2+\phi}\alpha + \alpha \cdot \partial\psi/2.$$

Hence

$$|\bar{\partial}^*_{\psi+\phi}\alpha|^2 \leq 2|\bar{\partial}^*_{\psi/2+\phi}\alpha|^2 + |\alpha \cdot \partial\psi|^2/2.$$

The condition 1.33 on ψ now implies that the second term on the right hand side can be controlled by

$$\frac{1}{2}\sum \psi_{j\bar{k}}\alpha_j\bar{\alpha}_k.$$

Using this in 1.34 we obtain

$$\int \sum \psi_{j\bar{k}}\alpha_j\bar{\alpha}_k e^{-\psi-\phi} \leq 4\int |\bar{\partial}^*_{\psi/2+\phi}\alpha|^2 e^{-\psi-\phi} + 2\int |\bar{\partial}\alpha|^2 e^{-\psi-\phi}.$$

By the approximation lemma it follows that

$$\int \sum \psi_{j\bar{k}}\alpha_j\bar{\alpha}_k e^{-\psi-\phi} \leq 4\int |\bar{\partial}^*_{\psi/2+\phi}\alpha|^2 e^{-\psi-\phi},$$

for any $\alpha \in \mathrm{Dom}(T^*) \cap N$, where now T is regarded as an operator

$$T : L^2(e^{-\phi-\psi/2}) \to L^2_{(0,1)}(e^{-\phi-\psi/2}).$$

Now we invoke Proposition 1.3.4 with $w = e^{-\psi/2}$ and $\mu_{j\bar{k}} = \psi_{j\bar{k}} e^{-\psi/2}$ and ϕ replaced by $\phi + \psi/2$. This completes the proof. ■

The original theorem of Donelly and Fefferman deals with forms of arbitrary bidegree (p, q) and involves estimates with respect to a more general Kähler metric. It will be given in Chapter 3 (Theorem 3.9.6). Theorem 1.8.1 correspends to the case $(p, q) = (n, 1)$ of that theorem, but we have rearranged the proof to avoid at this point the use of generalizations of Theorem 1.4.2 to general metrics. However we remark already here that the condition 1.33 can be expressed equivalently as saying that the differential $\partial\psi$ has norm not exceeding one when measured in the metric $(\psi_{j\bar{k}})$.

We end this section by one more application of the same idea.

Theorem 1.8.2 *Let Ω be a pseudoconvex domain in $\mathbb{C}^n$, and let ϕ be plurisubharmonic in Ω. Let ψ be smooth and strictly plurisubharmonic in Ω and suppose ψ satisfies the condition 1.33. Let $\delta > 0$. Then, for any $\bar{\partial}$-closed $(0,1)$-form, f, we can solve $\bar{\partial} u = f$ with*

$$\int |u|^2 e^{-\phi+(1-\delta)\psi} \leq C_\delta \int \sum \psi^{j\bar{k}} f_j \bar{f}_k e^{-\phi+(1-\delta)\psi}.$$

Proof: We follow the proof of the previous theorem (which of course corresponds to the case $\delta = 1$). By Theorem 1.4.2 we have

$$\int \sum \psi_{j\bar{k}} \alpha_j \bar{\alpha}_k e^{-\psi-\phi} \leq \int |\bar{\partial}^*_{\psi+\phi}\alpha|^2 e^{-\psi-\phi} + \int |\bar{\partial}\alpha|^2 e^{-\psi-\phi}, \tag{1.35}$$

for any $\alpha \in D_{(0,1)}$ satisfying the boundary condition. Changing only slightly the preceeding proof we write

$$\bar{\partial}^*_{\psi+\phi}\alpha = \bar{\partial}^*_{\delta\psi/2+\phi}\alpha + (1 - \delta/2)\alpha \cdot \partial\psi.$$

Therefore

$$|\bar{\partial}^*_{\psi+\phi}\alpha|^2 \leq (1 + 1/\epsilon)|\bar{\partial}^*_{\delta\psi/2+\phi}\alpha|^2 + (1 + \epsilon)(1 - \delta/2)^2 |\alpha \cdot \partial\psi|^2.$$

If ψ satisfies 1.33, we can estimate the second term on the right by

$$(1 - \delta/2) \sum \psi_{j\bar{k}} \alpha_j \bar{\alpha}_k,$$

if we choose ϵ small enough. Using this in 1.35 we find

$$\int \sum \psi_{j\bar{k}} \alpha_j \bar{\alpha}_k e^{-\psi-\phi} \leq C_\delta \left(\int |\bar{\partial}^*_{\delta\psi/2+\phi}\alpha|^2 e^{-\psi-\phi} + \int |\bar{\partial}\alpha|^2 e^{-\psi-\phi} \right).$$

Again, the approximation lemma implies that

$$\int \sum \psi_{j\bar{k}} \alpha_j \bar{\alpha}_k e^{-\psi-\phi} \leq C_\delta \left(\int |\bar{\partial}^*_{\delta\psi/2+\phi}\alpha|^2 e^{-\psi-\phi} \right)$$

holds for all $\alpha \in \mathrm{Dom}(T^*) \cap N$ where now T is regarded as an operator from

$$T : L^2(e^{-\phi-\delta\psi/2}) \to L^2_{(0,1)}(e^{-\phi-\delta\psi/2}).$$

Applying Proposition 1.3.4, with ϕ replaced by $\phi + \delta\psi/2$, $w = e^{-(1-\delta/2)\psi}$ and $\mu_{j\bar{k}} = \psi_{j\bar{k}} e^{-(1-\delta/2)\psi}$, the theorem follows. ■

The point of this theorem is that, under the conditions stated, it allows for weight functions which have the opposite sign to the usual one. Applying it e g to the case of the disk in one variable, with $\psi = \log(1-|z|^2)$, we obtain, for positive δ

$$\int |u|^2 e^{-\phi}/(1-|z|^2)^{1-\delta} \leq C_\delta \int (1-|z|^2)^{1+\delta} |f|^2 e^{-\phi}.$$

This is clearly false for $\delta = 0$ so we see that Theorem 1.8.2 is quite sharp.

As a final illustration of Theorem 1.8.1 we can let the domain $\Omega = B_n$ be the unit ball in $\mathbb{C}^n$, with $n > 1$, and choose $\psi = -\log(1-|z|^2)$, as in the disk case. A direct computation then shows that 1.33 holds. The conclusion of Theorem 1.8.1 then is that the (weighted) L^2-norm of u can be estimated by the integral

$$\int |f|^2_B e^{-\phi},$$

where $|f|^2_B$ stands for the norm of f measured in the *Bergman* metric. It is interesting to note that this fact can *not* be deduced from the standard duality formulation, Proposition 1.3.2. Indeed, this would require the condition of Proposition 1.3.2 to be satisfied with $\mu_{j\bar{k}} = \psi_{j\bar{k}}$, which is easily seen to force that $\alpha = 0$ on the boundary.

Chapter 2

The $\bar{\partial}$-Neumann problem

We shall now consider a somewhat different functional analytic set-up which can also be used to treat the $\bar{\partial}$-equation. The content of the method is to reduce the $\bar{\partial}$-system, which is overdetermined, to a system of equations with equal numbers of unknowns and equations. For motivation we first look at a finite dimensional analog.

Let

$$E \stackrel{A}{\longrightarrow} F$$

be a linear map between finite dimensional spaces. A will later correspond to the $\bar{\partial}$-operator, but let us first assume that A is surjective. We suppose E and F are equipped with scalar products, and look for a solution e_0 to

$$Ae = f$$

which has minimal norm in E. This means that $e_0 \perp N(A)$, where $N(A)$ is the kernel of A. Now A has an adjoint

$$A^* : F \to E$$

and since our spaces are of finite dimension, it holds that

$$N(A)^\perp = R(A^*),$$

where R means the image space of an operator. Thus e_0 must have the form

$$e_0 = A^* h, \quad h \in F,$$

so we must solve

$$AA^* h = f.$$

But since A is surjective, so is AA^*, and therefore AA^* is invertible since it is a map from F to itself. In conclusion the e_0 we are looking for is given by

$$e_0 = A^*(AA^*)^{-1} f.$$

Now we leave the assumption that A be surjective but instead assume given a third space G and a map $B : F \to G$, such that the sequence

$$E \stackrel{A}{\longrightarrow} F \stackrel{B}{\longrightarrow} G$$

is exact. (This means that $R(A) = N(B)$). We can then decompose F

$$F = R(A) \oplus R(A)^\perp = R(A) \oplus N(B)^\perp = R(A) \oplus R(B^*).$$

Define the map

$$E \oplus G \overset{A \oplus B^*}{\longrightarrow} F$$

by $A \oplus B^*(e+g) = Ae + B^*g$. Then $A \oplus B^*$ is surjective so our previous considerations apply. Thus the solution of minimal norm to

$$A \oplus B^*(x) = f$$

is

$$x_0 = (A \oplus B^*)^*((A \oplus B^*)(A \oplus B^*)^*)^{-1} f.$$

But clearly

$$(A \oplus B^*)^* = A^* \oplus B$$

and

$$(A \oplus B^*)(A^* \oplus B) = AA^* + B^*B.$$

Hence

$$x_0 = (A^* \oplus B)[AA^* + B^*B]^{-1} f =: (A^* \oplus B)h.$$

Let us now consider in particular f such that $f \in R(A)$. Write the equation $f = (A \oplus B^*)x_0$ as

$$f - Ax_0 = B^*x_0.$$

Here the left hand side lies in $R(A)$ and the right hand side is orthogonal to $R(A)$, since $BA = 0$. Hence both sides vanish, so $f = Ax_0$. Recalling $x_0 = (A^* \oplus B)h$, we find

$$f = AA^*h.$$

In conclusion, we have showed that if the equation

$$Ae = f$$

is solvable, then the solution of minimal norm is

$$e_0 = A^*h = A^*(AA^* + B^*B)^{-1} f$$

(this is of course easy to verify directly).

We shall now imitate this method to treat the $\bar\partial$-operator. Our three Hilbert spaces are

$$L^2(\Omega, \varphi),\ L^2_{(0,1)}(\Omega, \varphi),\ L^2_{(0,2)}(\Omega, \varphi)$$

and we have a sequence of operators

$$L^2 \overset{\bar\partial =: T}{\longrightarrow} L^2_{(0,1)} \overset{\bar\partial =: S}{\longrightarrow} L^2_{(0,2)}$$

which (at least) we hope is exact. Following the finite-dimensional analogy, we set up the

$\bar\partial$-Neumann problem:

Suppose $f \in L^2_{(0,1)}(\Omega, \varphi)$. Solve

$$(TT^* + S^*S)h = f$$

with $h \in L^2_{(0,1)}$. To have the operator in the left hand side defined, we require that

$$h \in \text{Dom } (S) \cap \text{Dom } (T^*)$$

and

$$Sh \in \text{Dom }(S^*),\ T^*h \in \text{Dom }(T).$$

Before we discuss the solvability of this problem, we shall analyze what it means concretely. First, we assume $\varphi = 0$ and compute the operators T^* and S^*. From Chapter 1 we already know that if $h = \sum h_j d\bar{z}_j$ then

$$T^*h = -\sum \frac{\partial h_j}{\partial z_j} = \bar{\partial}^* h,$$

provided $h \in \text{Dom } T^*$. Moreover

$$Sh = \bar{\partial} h = \sum \frac{\partial h_j}{\partial \bar{z}_k} d\bar{z}_k \wedge d\bar{z}_j = \frac{1}{2} \sum \left(\frac{\partial h_j}{\partial \bar{z}_j} - \frac{\partial h_k}{\partial \bar{z}_j}\right) d\bar{z}_k \wedge d\bar{z}_j.$$

If $g \in L^2_{(0,1)}$, we write

$$g = \sum g_{kj} d\bar{z}_k \wedge d\bar{z}_j \quad g_{kj} = -g_{jk},$$

and we define the scalar product in $L^2_{(0,2)}$ by

$$< g, g >= \int \sum_{kj} |g_{kj}|^2.$$

The adjoint of S is defined by

$$< S\alpha, g >=< \alpha, S^* g > \quad \forall \alpha \in \text{Dom }(S).$$

Since smooth forms with compact support are dense in Dom (S), we can also define S^* by the same relation for all test forms α. Then

$$\begin{aligned} < S\alpha, g > &= \int \sum \left(\frac{\partial \alpha_j}{\partial \bar{z}_k} - \frac{\partial \alpha_k}{\partial \bar{z}_j}\right) \overline{g_{kj}} = \\ &= -\int \sum \alpha_j \frac{\overline{\partial g_{kj}}}{\partial z_k} - \sum \alpha_k \frac{\overline{\partial g_{kj}}}{\partial z_j} = 2 \int \sum \alpha_k \frac{\overline{\partial g_{kj}}}{\partial z_j}. \end{aligned}$$

Hence

$$S^* g = 2 \sum \frac{\partial g_{kj}}{\partial z_j} d\bar{z}_k \quad \text{if} \quad g \in \text{Dom }(S^*).$$

In particular,

$$S^* S h = \sum \frac{\partial}{\partial z_j} \left(\frac{\partial h_j}{\partial \bar{z}_k} - \frac{\partial h_k}{\partial \bar{z}_j}\right) d\bar{z}_k.$$

Since

$$TT^* h = -\bar{\partial} \sum \frac{\partial h_j}{\partial z_j} = -\sum \frac{\partial^2 h_j}{\partial \bar{z}_k \partial z_j} d\bar{z}_k,$$

we obtain

$$(TT^* + S^* S) h = -\sum \frac{\partial^2 h_k}{\partial z_j \partial \bar{z}_j} d\bar{z}_k.$$

Thus, the $\bar{\partial}$-Neumann problem amounts to solving the system of equations

$$-\Delta h_k = f_k \qquad h = 1, \ldots, n$$

with a form h such that $h \in \text{Dom }(T^*)$, $Sh \in \text{Dom }(S^*)$ and $h \in \text{Dom }(S)$, $T^*h \in \text{Dom }(T)$. Of these condition the last two mean only that h and T^*h should be sufficiently differentiable, but the first two contain assumptions on the boundary values of h and Sh, and this is what makes the problem difficult.

Remark. In the general case when $\varphi \equiv 0$, let us denote the adjoints by T^*_φ and S^*_φ. Using the obvious relations

$$T^*_\varphi = e^\varphi T^* e^{-\varphi},\ S^*_\varphi = e^\varphi S^* e^{-\varphi},$$

one can show that

$$(TT^*_\varphi + S^*_\varphi S)h = -\sum \Delta h_k d\bar{z}_k + \sum \frac{\partial h_k}{\partial \bar{z}_j}\frac{\partial \varphi}{\partial z_j} d\bar{z}_k + \sum h_j \varphi_{j\bar{k}} d\bar{z}_k.$$

2.1 Existence of solutions to the $\bar{\partial}$-Neumann problem

To prove existence we first give our problem a dual formulation. Let

$$E = C^\infty_{(0,1)} \cap \text{Dom } T^*.$$

On $L^2_{(0,1)} \cap \text{Dom } T^* \cap \text{Dom } S$ we define a bilinear from

$$Q(\alpha, \beta) =< T^*\alpha, T^*\beta > + < S\alpha, S\beta >$$

where the scalar products are taken in $L^2(\Omega, \varphi)$ and $L^2_{(0,1)}(\Omega, \varphi)$ respectively. Suppose now that we have a solution to the $\bar{\partial}$-Neumann problem

$$\begin{aligned}
&(TT^* + S^*S)h = f \\
&h \in \text{Dom } (S) \cap \text{Dom } (T^*) \\
&Sh \in \text{Dom } (S^*),\ T^*h \in \text{Dom } T.
\end{aligned}$$

If $\alpha \in E$, we get

$$Q(h, \alpha) =:< T^*h, T^*\alpha > + < Sh, S\alpha >=< f, \alpha > \tag{2.1}$$

Denote by $\bar{E}^Q$ the completion of E with respect to the (pseudo)norm Q. By this we mean that $h \in E^Q$ if $h \in L^2_{(0,1)}$ and there is a sequence $h_\nu \in E$ such that $h_\nu \to h$ in $L^2_{(0,1)}$ and h_ν is a Cauchy sequence with respect to Q. Then, our dual formulation is:

Proposition 2.1.1 *Suppose $h \in \bar{E}^Q$ and that (2.1) holds for all $\alpha \in E$. Then h solves the $\bar{\partial}$-Neumann problem with right hand side f.*

In the proof we shall use the following lemma.

Lemma 2.1.2 *Suppose $h \in$ Dom $T^* \cap$ Dom S and that (2.1) holds for all $\alpha \in E$. then*

$$| < Sh, S\alpha > | \leq \|f\| \|\alpha\|$$

and

$$| < T^*h, T^*\alpha > | \leq \|f\| \|\alpha\|.$$

Proof. The assumption means that the sum of our two scalar products satisfies an estimate of the type we claim, but we want to prove that each one of them also does. Therefore we decompose $\alpha = \alpha^1 + \alpha^2$ where $S\alpha^1 = 0$ and $\alpha^2 \perp N(S)$. Then $\alpha^2 \perp R(T)$ so $T^*\alpha^2 = 0$ whence

$$\alpha^1 \in \text{Dom } (T^*) \quad \text{and} \quad T^*\alpha^1 = T^*\alpha.$$

This gives

$$< T^*h, T^*\alpha >=< T^*h, T^*\alpha^1 >= Q(h, \alpha^1)$$

and

$$< Sh, S\alpha >=< Sh, S\alpha^2 >= Q(h, \alpha^2).$$

But if (2.1) holds for all $\alpha \in E$, it actually also holds for $\alpha \in$ Dom $T^* \cap$Dom S by the approximation lemma. Hence

$$|Q(h, \alpha^1)| = | < f, \alpha^1 > | \leq \|f\|\|\alpha^1\| \leq \|f\|\|\alpha\|$$

and

$$|Q(h, \alpha^2)| = | < f, \alpha^2 > | \leq \|f\|\|\alpha\|.$$

■

Proof of Proposition 2.1.1. First note that if $h \in \bar{E}^Q$, then $h \in$ Dom $T^* \cap$ Dom S, since T^* and S are closed operators. Lemma 2.1.2 implies that

$$| < T^*h, T^*\alpha > | \leq C\|\alpha\|$$

for in particular all smooth forms with compact support. From this it follows that $T^*h \in$ Dom T (and that $\|TT^*h\| \leq C$). What remains to prove is that $Sh \in$ Dom S^*, which means that

$$| < Sh, S\alpha > | \leq C\|\alpha\| \quad \text{for} \quad \alpha \in \text{Dom } S. \tag{2.2}$$

We know that (2.2) holds if $\alpha \in E$, and by the approximation lemma it suffices to prove (2.2) for $\alpha \in C^\infty_{(0,1)}(\bar{\Omega})$. If α has compact support, (2.2) follows since α then lies in E. Hence we may assume that α has support near $\partial\Omega$. Write

$$\alpha = \alpha^0 + a\bar{\partial}\rho$$

where $< \alpha^0, \bar{\partial}\rho >\equiv 0$. Then $\alpha^0 \in E$ so

$$| < Sh, S\alpha^0 > | \leq C\|\alpha^0\| \leq C\|\alpha\|$$

so we need only control

$$< Sh, S(a\bar{\partial}\rho) > .$$

Take a sequence of smooth functions $\chi_\epsilon(t)\epsilon \to 0$ such that

$$\begin{aligned} \chi_\epsilon(t) = 1 \qquad & t \leq -\epsilon \\ \chi_\epsilon(t) = 0 \qquad & t \geq -\epsilon/2. \end{aligned}$$

We get with $\chi_\epsilon = \chi_\epsilon(\rho)$

$$\begin{aligned} < Sh, S(a\bar{\partial}\rho > \quad &= \quad \lim_{\epsilon \to 0} < Sh, \chi_\epsilon S(a\bar{\partial}\rho) >= \\ &= \quad \lim_{\epsilon \to 0}(< Sh, S\chi_\epsilon a\bar{\partial}\rho > - < Sh, a\bar{\partial}\chi_\epsilon \wedge \bar{\partial}\rho >) \end{aligned}$$

But $\bar{\partial}\chi_\epsilon \wedge \bar{\partial}\rho = 0$ and $\chi_\epsilon a\bar{\partial}\rho \in E$ for fixed ϵ. Thus

$$| < Sh, S(a\bar{\partial}\rho) > | \leq C\|a\bar{\partial}\rho\| \leq C\|\alpha\|,$$

so we see that indeed $Sh \in$ Dom (S^*). But then

$$Q(h, \alpha) =< (T^*T + SS^*)h, \alpha >$$

so (2.1) implies that

$$(T^*T + SS^*)h = f.$$

Thus, h satisfies all three criteria for a solution to the $\bar{\partial}$-Neumann problem and Proposition 2.1.3 is proved. ■

It is now easy to give a criterium for the solvability of the $\bar{\partial}$-Neumann problem.

Proposition 2.1.3 *Suppose there is a constant $\lambda_0 > 0$ such that*

$$\lambda_0 \|\alpha\|^2 \leq Q(\alpha, \alpha) \tag{2.3}$$

for all α in E. Then for any $f \in L^2_{(0,1)}(\Omega, \varphi)$ there is a unique $h \in L^1_{(0,1)}$ which solves the $\bar{\partial}$-Neumann problem with right hand side f. Moreover,

$$\lambda_0 \|h\|^2 \leq Q(h, h) \leq \lambda_0^{-1} \|f\|^2$$

Proof. Define an antilinear functional on E by

$$L\alpha =< f, \alpha > .$$

Then

$$|L\alpha|^2 \leq \|f\|^2 \|\alpha\|^2 \leq \lambda_0^{-1} Q(\alpha, \alpha) \|f\|^2,$$

so L is continuous for the Q-norm. Hence there is an element h in the completion of E with respect to Q, such that

$$< f, \alpha >= Q(h, \alpha) \quad \text{for all} \quad \alpha \in E.$$

But (2.3) implies that the Q-completion of E is precisely what we have called $\bar{E}^Q$, so the existence follows from Proposition 2.1.1. Uniqueness follows since

$$< (TT^* + S^*S)h, h >= Q(h, h).$$

■

Corollary 2.1.4 1) *Suppose Ω is pseudoconvex with boundary in C^2 and that $\varphi \in C^2(\bar{\Omega})$ is strictly plurisubharmonic. Then for $f \in L^2_{(0,1)}(\Omega, \varphi)$ there is a form h which solves the $\bar{\partial}$-Neumann problem with right hand side f and is such that*

$$\int \sum \varphi_{j\bar{k}} h_j \bar{h}_k e^{-\varphi} \leq \int \sum \varphi^{j\bar{k}} f_j \bar{f}_k e^{-\varphi}$$

(here $(\varphi^{j\bar{k}}) = (\varphi_{j\bar{k}})^{-1}$).

2) *Suppose Ω is pseudoconvex with boundary in C^2. Then there is a constant $C = C(\Omega)$ such that for any $f \in L^2_{(0,1)}(\Omega, 0)$ the $\bar{\partial}$-Neumann problem with right hand side f is solvable and the solution satisfies*

$$\int_\Omega |h|^2 \leq C \int_\Omega |f|^2.$$

Proof.

(1) By the basic identity, we have if $\alpha \in C^1(\bar{\Omega}) \cap \mathrm{Dom}\, T^*$

$$\begin{aligned} |< f, \alpha >|^2 &\leq \int \sum \varphi^{j\bar{k}} f_j \bar{f}_k e^{-\varphi} \int \varphi_{j\bar{k}} \alpha_j \bar{\alpha}_k e^{-\varphi} \leq \\ &\leq \int \sum \varphi^{j\bar{k}} f_j \bar{f}_k e^{-\varphi} Q(\alpha, \alpha). \end{aligned}$$

By the proof of Proposition 3 there is a solution h such that

$$Q(h, h) \leq \int \sum \varphi^{j\bar{k}} f_j \bar{f}_k e^{-\varphi}.$$

But the basic identity also says that

$$\int \sum \varphi_{j\bar{k}} h_j \bar{h}_k e^{-\varphi} \leq Q(h,h)$$

since $h \in \text{Dom } T^*$.

(2) Let $\varphi = t|z|^2$. Then the basic identity says that

$$t \int |\alpha|^2 e^{-t|z|^2} \leq \int (|\bar{\partial}^*_\varphi \alpha| + |\bar{\partial}\alpha|^2) e^{-t|z|^2}.$$

But

$$\begin{aligned} \bar{\partial}^*_\varphi \alpha &= \bar{\partial}^*_0 \alpha + t \sum \alpha_j \bar{z}_j, \\ |\bar{\partial}^*_\varphi \alpha|^2 &\leq 2|\bar{\partial}^*_0 \alpha|^2 + 2t^2 |\alpha \cdot \bar{z}|^2 \leq 2|\bar{\partial}^*_0 \alpha|^2 + 2t^2 |z|^2 |\alpha|^2. \end{aligned}$$

If t is sufficiently small, $2t^2|z|^2 \leq t/2$ in Ω, so we get

$$t/2 \int |\alpha|^2 \leq C \int |\bar{\partial}^*_0 \alpha|^2 + |\bar{\partial}\alpha|^2.$$

Hence (2.2) follows from Proposition 2.1.3. ■

Finally we return to the $\bar{\partial}$-equation.

Proposition 2.1.5 *Let $f \in L^2_{(0,1)}(\Omega, \varphi)$ be such that $Sf = 0$, and let h be the solution to the $\bar{\partial}$-Neumann problem with right hand side f. Then*

$$u = T^* h$$

is the minimal solution to

$$\bar{\partial} u = f$$

in $L^2_{(0,1)}(\Omega, \varphi)$.

Proof.

$$S^* S h = f - TT^* h.$$

Here the left hand side is orthogonal to $N(S)$, and the right hand side lies in $N(S)$. Hence both terms are zero, i.e.,

$$\bar{\partial} u = f.$$

Since $u \in R(T^*)$, u must be minimal. ■

2.2 Regularity of solutions to the $\bar{\partial}$-Neumann problem

In the previous section we have showed the existence of solutions to the $\bar{\partial}$-Neumann problem in the weak sense. It is of course natural to ask whether we also have classical solutions if the right hand side is smooth. As far as interior smoothness is concerned, it is not hard to prove that this is the case. Since the leading term of the operator $(TT^* + S^*S)$ is just the Laplacian on each component of our form, standard elliptic theory shows that our solution h is roughly two units smoother than the right hand side.

The problem of boundary smoothness is however much more complicated, and is actually still unsolved in the general case. The best results so far are based on a general theorem of Kohn and Nirenberg [7], which specialized to our situation says: (Throughout this section we shall let our weight function ϕ be identically 0, although the same arguments with minor modifications work as well with an arbitrary weight which is smooth up to the boundary.)

Theorem 2.2.1 *Suppose that Ω is pseudoconvex with smooth boundary. Assume that the unit ball defined by Q*

$$K = \{\alpha \in E; Q(\alpha, \alpha) \leq 1\}$$

is relatively compact in $L^2_{(0,1)}$. Then the solution to the $\bar{\partial}$-Neumann problem is smooth up to the boundary, provided the right hand side is smooth up to the boundary.

We shall not try to prove this theorem but rather give some easy corollaries. Let us first of all note however, what the conclusion means. Since h is smooth up to the boundary, the condition that $h \in \text{Dom } T^*$ simply means

$$(i) \qquad \sum h_j \frac{\partial \rho}{\partial z_j} = 0 \quad \text{on} \quad \partial\Omega.$$

In a similar way one sees that the condition $Sh \in \text{Dom } S^*$ means that

$$(ii) \qquad \sum_k \left(\frac{\partial h_j}{\partial \bar{z}_k} - \frac{\partial h_k}{\partial \bar{z}_j}\right)\frac{\partial \rho}{\partial z_k} = 0 \quad \text{on} \quad \partial\Omega \quad \forall j.$$

Thus, we have a solution to a certain boundary value problem, where the boundary conditions are of mixed Dirichlet and Neumann type.

For the rest of this section we will assume that Ω is pseudoconvex with smooth boundary and that $\varphi \in C^\infty(\bar{\Omega})$. Let, as before, ρ be a function such that $\Omega = \{\rho < 0\}$ and $d\rho \neq 0$ on $\partial\Omega$.

Proposition 2.2.2 *Assume Ω is strictly pseudoconvex. Then there is a constant C such that*

$$\int_\Omega |\alpha|^2 + \int_\Omega (-\rho)|\nabla\alpha|^2 \leq CQ(\alpha, \alpha)$$

for all $\alpha \in E$.

Proof. Since the basic identity (Theorem 1.2.2) already gives a good estimate for the derivatives $\frac{\partial \alpha_j}{\partial \bar{z}_k}$ we shall first consider the integral

$$I = \int \sum \left|\frac{\partial \alpha_j}{\partial z_k}\right|^2 (-\rho).$$

But

$$\begin{aligned}
\int \left|\frac{\partial \alpha_j}{\partial z_k}\right|^2 (-\rho) &= \int \frac{\partial \rho}{\partial z_k}\alpha_j \overline{\frac{\partial \alpha_j}{\partial z_k}} - \int (-\rho)\alpha_j \overline{\frac{\partial^2 \alpha_j}{\partial z_k \partial \bar{z}_k}} = \\
&= \int \left|\frac{\partial \rho}{\partial z_k}\right|^2 |\alpha_j|^2 \frac{dS}{|d\rho|} - \int \frac{\partial^2 \rho}{\partial z_k \partial \bar{z}_k}|\alpha_j|^2 \\
&\quad - \int \frac{\partial \rho}{\partial z_k}\frac{\partial \alpha_j}{\partial \bar{z}_k}\bar{\alpha}_j + \int (-\rho)\left|\frac{\partial \alpha_j}{\partial \bar{z}_k}\right|^2 - \int \frac{\partial \rho}{\partial \bar{z}_k}\alpha_j \overline{\frac{\partial \alpha_j}{\partial \bar{z}_k}}.
\end{aligned}$$

Rearranging we get

$$\begin{aligned}
&\int \Delta\rho|\alpha|^2 + \int \sum \left|\frac{\partial \alpha_j}{\partial z_k}\right|^2 (-\rho) \leq \\
&C\{\int_\partial |\alpha|^2 dS + (\int |\alpha|^2)^{1/2}(\int \sum \left|\frac{\partial \alpha_j}{\partial \bar{z}_k}\right|^2)^{1/2} + \int (-\rho)\sum \left|\frac{\partial \alpha_j}{\partial \bar{z}_k}\right|^2\}.
\end{aligned} \tag{2.4}$$

With no loss of generality we may assume that $\Delta\rho \geq 1$. then (2.4) implies that

$$\int |\alpha|^2 + \int (-\rho) \sum \left|\frac{\partial \alpha_j}{\partial z_k}\right|^2 \leq C\{\int_\partial |\alpha|^2 dS + \int \sum \left|\frac{\partial \alpha_j}{\partial \bar{z}_k}\right|^2\}.$$

By the basic identity the right hand side is dominated by a constant times $Q(\alpha, \alpha)$, so the proof is complete. ■

To prove the regularity of solutions to the $\bar{\partial}$-Neumann problem, it is now sufficient to prove that the norm in the left hand side of Proposition 2.2.2 defines a precompact unit ball. We shall prove the following stronger statement.

Proposition 2.2.3 *Let $\varphi(t)$ be a positive continuous function on $(0, \infty)$. Suppose*

$$\int_0 \frac{tdt}{\varphi(t)} < \infty. \tag{2.5}$$

Let

$$|||u|||^2 = \int |u|^2 + \int \varphi(d) |\nabla u|^2$$

where d is the distance to the boundary. Then the set

$$\{u \in C^\infty(\bar{\Omega}); |||u|||^2 \leq 1\}$$

is relatively compact in L^2.

Proof. It is easy to see that if $\chi \in C^\infty(\bar{\Omega})$, then we can estimate $|||\chi u|||$ with $|||u|||$. We may therefore consider only functions that have support near a boundary point. Since moreover our norm is not changed much if we transform by a change of coordinates, we may assume that Ω is the ball.

Suppose now that u_n is a sequence of functions such that $|||u_n||| \leq 1$. By the Rellich lemma there is, for each $\Omega' \subseteq\subseteq \Omega$, a subsequence that converges in $L^2(\Omega')$. Taking a diagonal sequence, we can assume that u_n is convergent in $L^2(\Omega')$ for any $\Omega' \subseteq\subseteq \Omega$. We claim that then actually u_n converges in $L^2(\Omega)$. To prove this, it is enough to prove that for any $\epsilon > 0$ there is a $\delta > 0$ such that if $|||u|||^2 \leq 1$ then

$$\int_{1-\delta<|x|<1} |u|^2 \leq \epsilon.$$

By rotational symmetry this follows if we can prove that

$$\int_0^\delta (v(t))^2 dt \leq \epsilon$$

when $\int_0^1 (v'(t))^2 \phi(t) dt + \int_0^1 (v(t))^2 dt \leq 1$. Take $a \in (0, 1)$. Then

$$\begin{aligned} |v(t)| &\leq |v(a)| + \int_t^a |v'| dx \leq \\ &\leq |v(a)| + (\int_0^1 |v'|^2 \phi dx)^{2/1} (\int_t^a \frac{dx}{\phi})^{1/2}. \end{aligned}$$

Hence

$$|v(t)|^2 \leq 2|v(a)|^2 + 2\int_t^1 \frac{dx}{\phi(x)}.$$

Integrating over $a \in (1/2, 1)$ we get

$$|v(t)|^2 \leq 2 + 2\Phi(t)$$

where

$$\Phi(t) = \int_t^1 \frac{dx}{\varphi(x)}.$$

Hence $\int_0^\delta |v(t)|^2 dt \leq 2\delta + 2\int_0^\delta \Phi(t)dt$.

But our hypothesis means precisely that $\int_0^1 \Phi(t)dt < \infty$, so the proof is complete. ∎

Corollary 2.2.4 *Assume Ω is strictly pseudoconvex. Then we have smoothness up to the boundary for solutions to the $\bar\partial$-Neumann problem.*

It is clear from the proof that the condition of strict pseudoconvexity can be relaxed a lot. The crucial part of the argument was that we managed to dominate

$$\int (-\rho)|\nabla\alpha|^2$$

by the energy form $Q(\alpha, \alpha)$. Actually Proposition 2.2.3 shows that it would have been enough to prove

$$I_\epsilon =: \int (-\rho)^{2-\epsilon}|\nabla\alpha|^2 \leq CQ(\alpha, \alpha), \tag{2.6}$$

for any positive ϵ. We shall close this chapter by showing that for any bounded pseudoconvex domain, the Q-norm dominates I_0, and that for positive ϵ Q dominates I_ϵ for a class of domains that is considerably more general than the strictly pseudoconvex ones. The proof uses an idea of Catlin [3], and we refer to that article for optimal results in this genre.

The next proposition follows from an argument similar to the one used in the proof of Proposition 2.2.2.

Proposition 2.2.5 *Let Ω be a smoothly bounded domain given by $\Omega = \{\rho < 0\}$, where ρ is a smooth defining function satisfying $d\rho \neq 0$ on $\partial\Omega$. Let $0 \leq \epsilon < 1$. Then*

$$\int (-\rho)^{2-\epsilon}|\nabla\alpha|^2 \leq C\left(\int (-\rho)^{2-\epsilon}\sum|\frac{\partial\alpha_j}{\partial\bar z_k}|^2 + \int (-\rho)^{-\epsilon}|\alpha|^2\right).$$

Note that the fundamental identity, Theorem 1.4.2, gives a bound of the first term in the right hand side in terms of $Q(\alpha, \alpha)$. It follows that in order to prove 2.6 it suffices to prove that Q dominates

$$\int (-\rho)^{-\epsilon}|\alpha|^2.$$

When $\epsilon = 0$ such an inequality follows from the proof of Corollary 2.1.4. For positive ϵ we can use the following proposition.

Proposition 2.2.6 *Suppose there exists a function v which is strictly plurisubharmonic in Ω with $(v_{j\bar k}) \geq (\delta_{j\bar k})$, which moreover satisfies an inequality*

$$0 < -v \leq C(-\rho)^\epsilon,$$

for some positive ϵ. Then the inequality

$$\int (-\rho)^{-\epsilon/2}|\alpha|^2 \leq CQ(\alpha, \alpha)$$

holds for all $\alpha \in Dom(T) \cap C^\infty(\bar\Omega)$.

Proof: Let $\psi = -(-v)^{1/2} + A$, where A is chosen so large that $\psi \geq 1$. Then ψ is bounded, and satisfies

$$(\psi_{j\bar{k}}) \geq 1/2(-\rho)^{-\epsilon/2}.$$

Replacing if necessary ψ by $c\psi^2$ we may assume that

$$|\alpha \cdot \partial\psi|^2 \leq 1/3 \sum \psi_{j\bar{k}}\alpha_j\bar{\alpha}_k$$

for all α. We now apply the fundamental identity, Theorem 1.4.2, with ϕ replaced by ψ. Discarding some positive terms we find

$$\int \sum \psi_{j\bar{k}}\alpha_j\bar{\alpha}_k e^{-\psi} \leq \int |\bar{\partial}^*_\psi \alpha|^2 e^{-\psi} + \int |\bar{\partial}\alpha|^2 e^{-\psi}.$$

Using

$$|\bar{\partial}^*_\psi \alpha|^2 = |\bar{\partial}^*_0 \alpha + \alpha \cdot \partial\psi|^2 \leq 2|\bar{\partial}_0 \alpha|^2 + 2|\alpha \cdot \partial\psi|^2,$$

and keeping in mind that ψ is bounded, we obtain

$$\int \sum \psi_{j\bar{k}}\alpha_j\bar{\alpha}_k \leq CQ(\alpha, \alpha).$$

This completes the proof. ■

Note the kinship of this proof with the argument used in the proof of Theorem 1.8.1

We collect the result of this discussion in the following Corollary.

Corollary 2.2.7 *Assume Ω is a smoothly bounded domain that satisfies the hypothesis of Proposition 2.2.6. Then we have smoothness up to the boundary for solutions to the $\bar{\partial}$-Neumann problem.*

Chapter 3

L^2-theory on complex manifolds

Now we shall generalize the results of the first chapter to the setting of complex manifolds. The first step is to develop a coordinate-free formalism for the concepts that we have already used. This requires quite a lot of preparations, but once it is done, results like "the basic identity", will follow almost immediately (and in a much more general form).

We suppose the reader has some familiarity with the basic theory of real manifolds.

3.1 Real and complex structures

First we define a complex manifold as a manifold where the local coordinate systems can be chosen holomorphic. More precisely:

Definition: A complex manifold M is a Hausdorff space which can be covered by local coordinate patches in the following way.

i) $M = \cup U_j$, U_j open.

ii) for each j there is a homeomorphism
$$z^{(j)} : U_j \to U_j' \subset \mathbb{C}^n$$
where U_j' is open in $\mathbb{C}^n$.

iii) the functions
$$z^{(i)} \circ {z^{(j)}}^{-1}$$
are holomorphic where they are defined.

Fixing a system of local coordinates, we usually just write $z = (z_1, \ldots, z_n)$. Clearly if $z_k = x_k + iy_k$ the functions
$$(x_1, y_1, \ldots, x_n, y_n)$$
will form a real coordinate system. Our first concern is how one can recover the complex structure from the system of real coordinates.

Recall that the real tangent space of M at a point $p \in M$ is defined as the set of real derivations on functions defined near p. In other words
$$v \in T_p(M)$$

if v is a linear map,

$$v : \{f; f \text{ real-valued function defined near } p\} \to \mathbb{R}$$

satisfying

$$v(fg) = g(p)v(f) + f(p)v(g).$$

The derivations $\frac{\partial}{\partial x_j}$ and $\frac{\partial}{\partial y_j}$ form a basis for $T_p(M)$. Our next objective is to show that M:s structure as *a complex* manifold, makes each T_p into a *complex* vector space, in such a way that a map between complex manifolds is holomorphic if an only if its differential is complex linear. Let us first discuss complex structure is general.

Suppose E is a given finite dimensional real vector space. How can we make E into a vector space over the complex numbers? Clearly, what we need is a definition of what iv is if v is a vector in E (and $i = \sqrt{-1}$). This definition must be such that the map

$$v \to J(v) =: iv$$

is $\mathbb{R}$-linear. Moreover we must demand that $J^2 = -id$. It is easy to see that if J is a map satisfying these two conditions then the rule

$$(a + ib)v =: av + bJ(v)$$

makes E into a complex vector space. Therefore we call such a J a *complex structure* on E.

It is sometimes useful to describe the complex structure in terms of the complexification of $E, E^{\mathbb{C}}$. Formally, $E^{\mathbb{C}}$ is defined as

$$E^{\mathbb{C}} = E \otimes_{\mathbb{R}} \mathbb{C},$$

which means that

$$E^{\mathbb{C}} = \{v + iw; v, w \in E\}.$$

(We could also define $E^{\mathbb{C}}$ as the set of $\mathbb{R}$-linear maps from E^* to $\mathbb{C}$, where E^* is the dual to E.)

Any vector in $E^{\mathbb{C}}$ can be written

$$v = e + if$$

where $e, f \in E$, and any $\mathbb{R}$-linear map between real vector spaces can be extended to a $\mathbb{C}$-linear map between the complexifications, simply by putting

$$Tv = Te + iTf.$$

In particular, given a complex structure J on E, we may extend to a $\mathbb{C}$-linear map

$$J : E^{\mathbb{C}} \to E^{\mathbb{C}}.$$

Clearly, it still holds that $J^2 = -id$. This implies that we have a decomposition as a direct sum

$$E^{\mathbb{C}} = E_{1,0} \oplus E_{0,1}$$

where

$$Jv = iv \qquad \text{if} \quad v \in E_{1,0}$$

and

$$Jv = -iv \qquad \text{if} \quad v \in E_{0,1}.$$

Explicitly the decomposition is given by

$$v = \frac{v - iJv}{2} + \frac{v + iJv}{2}. \tag{3.1}$$

Let us denote by $\pi_{1,0}$ and $\pi_{0,1}$ the projections on $E_{1,0}$ and $E_{0,1}$ respectively.

Lemma 3.1.1 *$\pi_{1,0}$ is a $\mathbb{R}$-linear isomorphism between E and $E_{(1,0)}$. If we let J define a structure as complex vector space on E then $\pi_{1,0}$ is also $\mathbb{C}$-linear.*

Proof. Clearly,

$$\pi_{1,0} = \frac{1 - iJ}{2}$$

is a linear map. If $\pi_{1,0}v = 0$ then

$$v = \pi_{1,0}v + \overline{\pi_{1,0}v} = 0,$$

so $\pi_{1,0}$ is injective. Moreover

$$J\pi_{1,0} = \frac{J + i}{2} = i\frac{1 - iJ}{2} = i\pi_{1,0}$$

so $R(\pi_{1,0}) \subseteq E_{1,0}$. Since both spaces have the same dimension over $\mathbb{R}$, $\pi_{1,0}$ is an isomorphism. Finally,

$$\pi_{1,0}J = J\pi_{1,0} = i\pi_{1,0},$$

so $\pi_{1,0}$ is $\mathbb{C}$-linear. ■

Summing up, we have seen that a complex structure J on E gives us a splitting of E:s complexification

$$\mathbb{E}^{\mathbb{C}} = E_{1,0} \oplus E_{0,1}$$

such that

(i) $E_{1,0} = \overline{E_{0,1}}$

and

(ii) $Jv = iv \qquad \text{if} \quad v \in E_{1,0}$.

Conversely, if we have splitting of $E^{\mathbb{C}}$, satisfying (i), we may define J on $E^{\mathbb{C}}$ by

$$J(v_{1,0} + v_{0,1}) = iv_{1,0} - iv_{0,1}.$$

Then J commutes with conjugation, so $J : E \to E$. Since clearly $J^2 = -id$, we get back our complex structure. Hence we can regard a complex structure either as a map $J : E \to E$ satisfying $J^2 = -id$, or as a decomposition of $E^{\mathbb{C}}$ into a subspace plus its conjugate.

Let us now return to our original situation where $E = T_p(M) = T(M)$ (we drop the index p in the sequel to avoid too many subscripts). Given holomorphic coordinates

$$z_j = x_j + iy_j$$

we get a basis for T

$$\frac{\partial}{\partial x_1}\frac{\partial}{\partial y_1}\cdots\frac{\partial}{\partial x_n}\frac{\partial}{\partial y_n}.$$

A vector

$$v = \sum \alpha_j \frac{\partial}{\partial x_j} + \sum \beta_j \frac{\partial}{\partial y_j}$$

has the coordinates

$$(\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n)$$

which intuitively correspond to the complex numbers

$$(\alpha_1 + i\beta_1, \ldots, \alpha_n + i\beta_n).$$

It is therefore natural to define Jv as the vector corresponding to

$$(i(\alpha_1 + i\beta_1), \ldots, i(\alpha_n + i\beta_n))$$

i.e., the vector whose coordinates are

$$(-\beta_1, \alpha_1, \ldots, -\beta_n, \alpha_n).$$

This means that

$$J(\frac{\partial}{\partial x_j}) = \frac{\partial}{\partial y_j}$$

and

$$J(\frac{\partial}{\partial y_j}) = -\frac{\partial}{\partial x_j}.$$

The following lemma says that J does not depend on the choice of coordinates but only on the complex structure on M.

Lemma 3.1.2 *Let J' be defined in the same way as J, but using a coordinate system $\zeta = (\zeta_1, \ldots, \zeta_n)$ instead. Then $J = J'$ if and only if the change of coordinates $z \circ \zeta^{-1}$ is holomorphic.*

Proof. Let $\zeta_j = \xi_j + i\eta_j$. Then by the chain rule $\frac{\partial}{\partial \xi_i} = \sum \frac{\partial x_k}{\partial \xi_j} \frac{\partial}{\partial x_k} + \sum \frac{\partial y_k}{\partial \xi_j} \frac{\partial}{\partial y_k}$ and $\frac{\partial}{\partial \eta_j} = \sum \frac{\partial x_k}{\partial \eta_j} \frac{\partial}{\partial x_k} + \sum \frac{\partial y_k}{\partial \eta_j} \frac{\partial}{\partial y_k}$. Thus

$$J(\frac{\partial}{\partial \xi_j}) = \sum \frac{\partial x_k}{\partial \xi_j} \frac{\partial}{\partial y_k} - \sum \frac{\partial y_k}{\partial \xi_j} \frac{\partial}{\partial x_k}$$

and

$$J(\frac{\partial}{\partial \eta_j}) = \sum \frac{\partial x_k}{\partial \eta_j} \frac{\partial}{\partial y_k} - \sum \frac{\partial y_k}{\partial \eta_j} \frac{\partial}{\partial x_k}.$$

On the other hand

$$J'(\frac{\partial}{\partial \xi_j}) = \frac{\partial}{\partial \eta_j}$$

and

$$J'(\frac{\partial}{\partial \eta_j}) = -\frac{\partial}{\partial \xi_j}.$$

Thus $J = J'$ if and only if

$$\frac{\partial x_k}{\partial \xi_j} = \frac{\partial y_k}{\partial \eta_j}$$

and

$$\frac{\partial x_k}{\partial \eta_j} = -\frac{\partial y_k}{\partial \xi_j}.$$

By the Cauchy-Riemann equations this means that $z \circ \zeta^{-1}$ is holomorphic. ■

If M and N are two complex manifolds and $f : M \to N$ is a map, we say that f is holomorphic if the functions

$$\zeta \circ f \circ z^{-1}$$

are holomorphic, whenever ζ and z are complex coordinates on N and M respectively. Precisely as in the proof of Lemma 3.1.2, we see that f is holomorphic if and only if the differential df is $\mathbb{C}$-linear as a map between tangent spaces.

Note that our complex structure J on the tangent spaces T_p, induces a complex structure (still denoted J) on the cotangent spaces T_p^*, by

$$J\omega(v) = \omega(Jv).$$

We can now apply our previous discussion of complex structures on real vector spaces to T and T^*. We then get decompositions

$$T^{\mathbb{C}} = T_{1,0} \oplus T_{0,1} \tag{3.2}$$

$$T^{*\mathbb{C}} = T_{1,0}^* \oplus T_{0,1}^*. \tag{3.3}$$

As mentioned above, we have a natural representation of $T^{*\mathbb{C}}$ as the space of $\mathbb{R}$-linear maps from T to $\mathbb{C}$. Using this interpretation of $T^{*\mathbb{C}}$, we see that the condition that ω belong to $T_{1,0}^*$

$$J\omega = i\omega, \quad \text{i.e.} \quad \omega(Jv) = i\omega(v)$$

menas just that ω is $\mathbb{C}$-linear for the complex structure J on T. More generally, (3.3) decomposes a $\mathbb{R}$-linear map into one $\mathbb{C}$-linear and one $\mathbb{C}$-antilinear part. In terms of local coordinates $z = (z_j)$, $z_j = x_j + iy_j$ we have that

$$\begin{aligned} J(dx_j) &= -dy_j \\ J(dy_j) &= dx_j, \end{aligned}$$

so $dz_1, \ldots, dz_n$ span $T_{1,0}^*$ and $d\bar{z}_1, \ldots, d\bar{z}_n$ span $T_{0,1}^*$. If $f : M \to \mathbb{C}$ is a differentiable complex valued function, then clearly df is an element in $T^{*\mathbb{C}}$. We then define

$$\partial f = \pi_{1,0}(df), \quad \bar{\partial} f = \pi_{(0,1)}(df),$$

so that

$$df = \partial f + \bar{\partial} f$$

is the decomposition of df into $\mathbb{C}$-linear and $\mathbb{C}$-antilinear parts. In particular, f is holomorphic if and only if df is $\mathbb{C}$-linear, i.e., if and only if $\bar{\partial} f = 0$. In terms of our local coordinates

$$df = \sum \frac{\partial f}{\partial z_j} dz_j + \sum \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j \tag{3.4}$$

can be taken as definition of the operators $\frac{\partial}{\partial z_j}$ and $\frac{\partial}{\partial \bar{z}_j}$. Writing $dz_j = dx_j + idy_j$, $d\bar{z}_j = dx_j - idy_j$ and identifying coeffcients in (3.4) we see that

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)$$

and

$$\frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right).$$

Similarily we see that $\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}$ span $T_{1,0}$ and $\frac{\partial}{\partial \bar{z}_j}, \ldots, \frac{\partial}{\partial \bar{z}_n}$ span $T_{0,1}$.

We say that a differential form is of degree one is of bidegree $(0,1)$ if at each point it lies in the $T_{1,0}^*$ part of the decomposition (3.3). Bidegree $(0,1)$ is defined analogously, and thus any 1-from can be uniquely splitted into a $(1,0)$ and a $(0,1)$ part. We also say that a k-form is of bidegree (p,q) if it can be written as a (sum of) product(s) of p $(1,0)$-forms and $q(0,1)$-forms. Since $\omega_1 \ldots, \omega_n, \bar{\omega}_1, \ldots, \bar{\omega}_n$ form a basis for T^* if $\omega_1 \ldots, \omega_n$ form a basis for $T_{1,0}^*$ any k-form, ω can be written uniquely

$$\omega = \sum_{p+q=k} \omega_{p,q}$$

where $\omega_{p,q}$ is of bidegree (p,q).

The preceding discussion can be carried through even if M is not a complex manifold, as soon as we have a complex structure on each $T_p(M)$ which varies smoothly with p. (By this we mean that the matrix for J with respect to a smooth local basis for $T(M)$ is smooth, or equivalently that locally there is a smooth basis for $T_{1,0}$.) Such a structure is called an *almost complex* structure on M. An almost complex structure is called *integrable* if it is induced by a structure on M as a complex manifold. If this is the case, then we can choose

$$\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}$$

as a local basis for $T_{1,0}$. From this it follows that if Z and W are vector fields of bidegree $(1,0)$, then their commutator [Z,W] is also of bidegree $(1,0)$. A famous theorem of Newlander and Nirenberg asserts that the converse of this is also true: *an almost complex structure is integrable if and only if the space of vector fields of type* $(1,0)$ *is closed under the formation of Liebrackets.* Sine a vectorfield is of type $(1,0)$ if and only if it is annihilated by any $(0,1)$-form, and since any 1-form ω satisfies

$$d\omega(Z,W) = Z(\omega(W)) - W(\omega(Z)) + \omega([Z,W]),$$

this condition is equivalent to saying that if ω is $(0,1)$ then $d\omega$ has no $(2,0)$ component.

We end this section with a discussion of scalar products in the real and the complex sense. Let us return to our real vector space E with a complex structure J. It is clear that if $(,)$ is a complex scalar product on E, then

$$<,>=: Re(,) \tag{3.5}$$

is a real scalar product. Which products arise in this way? Clearly a necessary condition is that $< Jv, Jw >=< v, w >$ for all $v, w \in E$, i.e., $<,>$ is J-invariant. Note also that if (3.5) holds, then

$$< v, Jw >= Re - i(v,w) = \text{Im } (v,w).$$

We can therefore try to define

$$(v,w) =< v, w > +i < v, Jw > . \tag{3.6}$$

If now $<,>$ is J-invariant, then

$$< Jv, w >=< J^2 v, Jw >= - < v, Jw >$$

so (3.6) implies that

$$(v,w) = \overline{(w,v)}.$$

Moreover

$$(Jv,w) =< Jv, w > +i < Jv, Jw >= - < v, Jw > +i < v, w >= i(v,w),$$

so $(,)$ is a complex scalar product. In other words we have a one-to-one correspondence between complex scalar products and J-invariant real scalar products. This can also be seen in the following way using the complexification of E. Given a J-invariant real scalar product on E, we can extend $<,>$ to a complex symmetric bilinear form on $E^{\mathbb{C}}$ in a unique way. If $v, w \in E_{1,0}$, then

$$< v, w >=< Jv, Jw >= - < v, w >$$

so

$$< v, w >= 0.$$

Therefore the form on $E^{\mathbb{C}}$

$$(v,w) =:< v, \bar{w} >$$

is sesquilinear and $E_{1,0}$ is orthogonal to $E_{0,1}$ with respect to this form. Moreover, it holds that

$$(\bar{v}, \bar{w}) =< \bar{v}, w >= \overline{(v,w)}.$$

If now v is a real vector, i.e., $v \in E$, then $v = \pi_{1,0}v + \overline{\pi_{1,0}v}$ so

$$< v, v >= (\pi_{1,0}v + \overline{\pi_{1,0}v},\ \pi_{1,0}v + \overline{\pi_{1,0}v}) = 2Re(\pi_{1,0}v, \pi_{1,0}v).$$

Thus the $\mathbb{C}$-linear isomorphism $v \to \pi_{1,0}v$ makes $(,)$ into a complex scalar product on E which satisfies

$$< v, w >= 2Re(\pi_{1,0}v, \pi_{1,0}w).$$

The conclusion of all this is that we may define a (complex) metric on a complex manifold either as a J-invariant Riemannian metric on M, or as a smoothly varying Hermitean form on $T_{(1,0)}$ (which is everywhere positive definite).

3.2 Connections on the tangent bundle

To start with, we consider M with only its real structure. A *connection* is a rule which allows us to differentiate a vectorfield along another field. Take two vector fields.

$$X = \sum X_j \frac{\partial}{\partial x_j},\ Y = \sum Y_j \frac{\partial}{\partial x_j} \tag{3.7}$$

and let us try to define $''X(Y)''$ – the derivative of Y in the direction X. If we demand that differentiation satisfy the product rule, we get

$$“X(Y)'' = \sum X(Y_j)\frac{\partial}{\partial x_j} + \sum Y_j X\big(\frac{\partial}{\partial x_j}\big).$$

The problem is that it is not clear what $X\big(\frac{\partial}{\partial x_j}\big)$ should be. We could try to put it equal to zero, but then the definition will depend on which coordinates we have chosen, and it will in general be impossible to get a global definition this way. A connection is an arbitrary (but consistent) definition of $X\big(\frac{\partial}{\partial x_j}\big)$.

Definition. Let $\chi(M)$ be the space of vector fields on M. A connection ∇ is a bilinear map

$$\nabla : \chi(M) \times \chi(M) \to \chi(M),$$

written

$$\nabla(X, Y) = \nabla_X Y,$$

which satisfies

i)
$$\nabla_{fX} Y = f \nabla_X Y$$

and

ii)
$$\nabla_X fY = f\nabla_X Y + X(f)Y.$$

It follows from the definition that ∇ is a *local* operator, i.e., its value in a point depends only on X, and Y in any neighbourhood of that point. Consequently $\nabla_X Y$ is well defined for vector fields that are only locally defined. If X and Y are given by (3.7) we find by ii) and i)

$$\begin{aligned}\nabla_X Y &= \sum X(Y_j)\frac{\partial}{\partial x_j} + \sum Y_j \nabla_X \frac{\partial}{\partial x_j} \\ &= \sum X(Y_j)\frac{\partial}{\partial x_j} + \sum Y_j X_k \nabla_{\frac{\partial}{\partial x_k}} \frac{\partial}{\partial x_j}.\end{aligned}$$

Hence the connection is determined by $\nabla_{\frac{\partial}{\partial x_k}} \frac{\partial}{\partial x_j}$. Say,

$$\nabla_{\frac{\partial}{\partial x_k}} \frac{\partial}{\partial x_j} = \sum_l \Gamma^l_{kj} \frac{\partial}{\partial x_l}.$$

The Γ^l_{kj}:s are called the connection coefficients or Christoffel symbols of ∇.

It is evident that we can find many different connections, but we shall now show that each Riemannian metric on M will give us a unique associated connection. Let $X, Y \to < X, Y >$ be a Riemannian metric on M. We say that ∇ is compatible with $<,>$ if the product rule

$$X < Y, Z > = < \nabla_X Y, Z > + < Y, \nabla_X Z > \tag{3.8}$$

holds. This condition does not in itself determine ∇ since we may e.g. add a linear map in Y which is antisymmetric w.r.t. $<,>$. Therefore we introduce one more restriction on ∇.

Definition. ∇ is symmetric if

$$\nabla_X Y - \nabla_Y X = [X, Y]. \tag{3.9}$$

If we take $X = \frac{\partial}{\partial x_k}$, $Y = \frac{\partial}{\partial x_j}$, we see that (3.9) implies

$$\Gamma^l_{kj} = \Gamma^l_{jk} \quad \text{for all} \quad l. \tag{3.10}$$

Conversely one sees directly that (3.10) implies (3.9). We now have

Theorem 3.2.1 *Given a Riemannian metric $<,>$ there is precisely one symmetric connection which is compatible with $<,>$.*

Proof. It is enough to prove this in a coordinate neighbourhood since the unicity statement implies that our definitions will agree on overlaps. If ∇ is compatible with $<,>$, we get from (3.8) with $X = \frac{\partial}{\partial x_k}$, $Y = \frac{\partial}{\partial x_i}$, $Z = \frac{\partial}{\partial x_j}$ that

$$\frac{\partial}{\partial x_k} g_{ij} = \Gamma^j_{ki} + \Gamma^i_{kj} \tag{3.11}$$

in a given point p if we have chosen coordinates so that

$$g_{ij} =: < \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} > = \delta_{ij}$$

in p (this is always possible by a linear change of coordinates). Now assume that ∇ and ∇' are two symmetric connections compatible with $<,>$, and let Δ^l_{jk} be the difference of the Christoffel symbols. Then (3.10) and (3.11) give

(a) $$\Delta^j_{ki} = \Delta^j_{ik} \quad \text{and}$$

(b) $$\Delta^j_{ki} + \Delta^i_{kj} = 0.$$

If we permute indices in the second equation, we get

$$\Delta^k_{ij} + \Delta^j_{ik} = 0.$$

If we use $\Delta^j_{ik} = \Delta^j_{ki}$, this gives us together with (b) that

$$\Delta^k_{ij} = \Delta^i_{kj}.$$

But this (together with (a)) says that Δ is symmetric in *any* pair of indices so (b) gives that $\Delta = 0$. Hence the coefficients are uniquely determined in p (and therefore in any point).

To show existence note that the space of coefficients Γ, satisfying

$$\Gamma^l_{jk} = \Gamma^l_{kj}$$

has dimension $\binom{n}{2}n$. Moreover, the space of possible left hand sides in (3.11)

$$\frac{\partial}{\partial x_k} g_{ij}|_p$$

also has dimension $\binom{n}{2}n$ since we have symmetry in the indices i and j. Therefore (3.11) is a quadratic system of equations, so the unicity implies existence of solutions. ■

Notice that we have also shown that $\Gamma = 0$ in any point where $dg_{ij} = 0$.

Finally, we shall consider the interaction between the connection and the complex structure. Recall from the previous paragraph how our metric $<,>$ induces a complex scalar product on $T^{\mathbb{C}}$.

Assuming $<,>$ is J-invariant, we extended $<,>$ to a $\mathbb{C}$-bilinear form on $T^{\mathbb{C}}$. Then we define

$$(z, w) =< z, \bar{w} >,$$

and this way $(,)$ became a complex metric on $T^{\mathbb{C}}$. Now we also extend the definition of ∇ with $\mathbb{C}$-linearity:

$$\begin{aligned} \nabla_{X+iY} &= \nabla_X + i\nabla_Y \\ \nabla(X + iY) &= \nabla X + i\nabla Y. \end{aligned}$$

If ∇ is compatible with the metric, one sees directly that

$$V(Z, W) = (\nabla_V Z, W) + (Z, \nabla_{\bar{V}} W).$$

In general, we have no reason to believe that $\nabla_V Z$ will be a $(1, 0)$ vector field if Z is a $(1, 0)$ vector field, so that ∇ will in general not operate on $T^{\mathbb{C}}_{1,0}$. We shall see later however that this will be the case if our metric satisfies one more condition, known as the Kähler condition.

3.3 Vector bundles

Let M be a complex manifold. A complex vector bundle over M is, loosely speaking, a family of complex vector space indexed by the points in M, which depend on the point in a smooth way.

Definition. Let E be a manifold and let $\pi : E \to M$ be a surjective map. We say that (E, π, M) is a complex vector bundle of rank r over M if

1. $\forall_p \in M \quad E_p =: \pi^{-1}(p)$ is a complex vector space of rank r.

2. For all $p \in M$ there is a neighbourhood U of p and a diffeomorphism.

$$\varphi_U : \pi^{-1}(U) \to U \times \mathbb{C}^r$$

such that

$$\varphi_U(\xi) = (\pi(\xi), \tilde{\varphi}_U(\xi))$$

where $\tilde{\varphi}_u$ is $\mathbb{C}$-linear on each $\pi^{-1}(p)$.

If U and V are intersecting neighbourhoods, we can define

$$G_{VU} = \varphi_V \circ \varphi_U^{-1} : (U \cap V) \times \mathbb{C}^r \to (U \cap V) \times \mathbb{C}^r.$$

Note that

$$G_{VU}(p, v) = (p, g_{VU}(p)v)$$

where g_{VU} is a smooth function on $U \cap V$ whose values and complex $r \times r$-matrices. The g_{VU}:s are called the transition functions of the bundle and they clearly satisfy

(i) $g_{U_1 U_2} = g_{U_2 U_1}^{-1}$

(ii) $g_{U_1 U_2} g_{U_2 U_3} g_{U_3 U_1} = id$.

Conversely one can prove that given a locally finite covering of M by open sets, and a collection of transition functions, one for each pair of sets in the covering, that satisfy i) and ii), there is always a complex vector bundle over M associated to this collection of transition functions.

In the sequel we shall denote the bundle simply by E, when π and M are understood. A (local) section to E is a map

$$\xi : U \subseteq M \to E \quad \text{such that} \quad \pi \circ \xi = id_U.$$

A set of r local sections $e_1, \ldots, e_r$ such that $\{e_j(p)\}$ is linearly independent at each p, is called a frame for E. Given a local frame an arbitrary section ξ can be written uniquely

$$\xi = \sum_1^r \xi_\nu e_\nu$$

where ξ_ν are complex valued functions.

We also say that our bundle E is holomorphic if E is a complex manifold and the local trivializations can be chosen holomorphic. Observe that this is just the same as saying that the transition functions can be chosen holomorphic. It is also equivalent to saying that E is a complex manifold and that we have a local frame of holomorphic sections near each point. Clearly, an arbitrary section ξ is holomorphic if and only if its "coordinates" ξ_ν are holomorphic, provided the frame is holomorphic.

A *hermitian metric* on E is a complex scalar product $<,>_p$ on each E_p with the property that

$$< \xi, \eta >$$

is a smooth function if ξ and η are smooth sections to E. Given a smooth local frame $<,>$ is clearly represented by an hermitean matrix $(h_{\nu\mu})$ whose entries are smooth functions.

Example. If M is a complex manifold,

$$T_{1,0}(M) = \cup_{p \in M} T^{\mathbb{C}}_{p(1,0)}$$

has a natural structure as holomorphic vector bundle over M. Given local coordinates $z_1, \ldots, z_n$, the fields $\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}$ constitute a local holomorphic frame for E.

We can now mimick the definition of connection on the tangent bundle to define connections on general bundles. Let $\Gamma(E)$ be the space of global sections to E, and similarly, let $\Gamma(T^{\mathbb{C}}(M))$ be the space of global complex vector fiels on M.

Definition. A connection on E is a bilinear map

$$\nabla : \Gamma(T^{\mathbb{C}}(M)) \times \Gamma(E) \to \Gamma(E)$$

that satisfies

1. $\nabla_V f\xi = V(f)\xi + f\nabla_V \xi$

2. $\nabla_{fV}\xi = f\nabla_V \xi$.

We say that ∇ is compatible with the metric $<,>$ if

3. $v < \xi, \eta > = < \nabla_V \xi, \eta > + < \xi, \nabla_{\bar{V}} \eta >$.

∇ is said to be holomorphic if

4. $\nabla_V \xi = 0$ if V is of the type $(0,1)$ and ξ is holomorphic.

Earlier we have seen that on a Riemannian manifold there is exactly one connection that is compatible with the metric and also is symmetric. The analogous statement for holomorphic vector bundles is:

Theorem 3.3.1 *Let E be a holomorphic vector bundle with an hermitian metric. Then there is precisely one connection on E that is both holomorphic and compatible with the metric.*

Proof. It is enough to prove this over a local trivialization since the local connections then must agree on overlaps. Say $e_1, \ldots, e_r$ is a local holomorphic frame, and that

$$h_{\nu\mu} = < e_\nu, e_\mu > .$$

If ∇ meets both our conditions, then

$$\frac{\partial h_{\nu\mu}}{\partial z_m} = < \nabla_{\frac{\partial}{\partial z_m}} e_\nu, e_\mu >$$

since e_μ is holomorphic. Let

$$\nabla_{\frac{\partial}{\partial z_m}} e_\nu = \sum \Gamma^\lambda_{m\nu} e_\lambda .$$

Then we get

$$\frac{\partial h_{\nu\mu}}{\partial z_m} = \sum \Gamma^\lambda_{m\nu} h_{\lambda\mu} .$$

Solving for $\Gamma^\lambda_{m\nu}$, we get

$$\Gamma^\lambda_{m\nu} = \sum_\mu \left(\frac{\partial h_{\nu\mu}}{\partial z_m}\right) h^{\mu\lambda},$$

so ∇e_ν is uniequely determined since

$$\nabla_{\frac{\partial}{\partial \bar{z}_m}} e_\nu = 0.$$

Conversely, this same formula evidently defines a connection which is both holomorphic and compatible with the mertric. ■

3.4 Kähler manifolds

Let us now consider a complex manifold M with an hermitian metric (g_{jk}), which we shall think of as a complex scalar product on $T_{1,0}$. Locally we can find an orthonormal basis for the space of $(1,0)$-forms, say

$$w_1, \ldots, w_n .$$

We then put

$$\Omega = i \sum_1^n w_k \wedge \bar{w}_k ,$$

so that Ω is a $(1,1)$-form. If $Z = \sum Z_j \frac{\partial}{\partial z_j}$ is a local vector field, then

$$\Omega(Z, \bar{Z}) = i|Z|^2.$$

Thus Ω is independent of our choice of orthonormal basis and has the form

$$\Omega = i \sum g_{ik} dz_j \wedge d\bar{z}_k$$

in our standard basis.

Definition. The metric (g_{ik}) is a Kähler metric if $d\Omega = 0$.

Given a point $p \in M$ we can always change our local coordinates by a complex-linear transformation so as to achieve $(g_{jk}) = (\delta_{jk})$ in p. If moreover it holds that $dg_{jk} = 0$ in p, we say that the coordinates are *normal* in that point. In that case we find that

$$d\Omega = \sum dg_{jk} \wedge dz_j \wedge d\bar{z}_k = 0$$

in p. Since the left hand side does not depend on our choice of coordinates, we see that if we can find normal (holomorphic) coordinates at each point the metric must be Kähler. Conversely we have:

Proposition 3.4.1 *If Ω defines a Kähler metric, there are for each point p in M local holomorphic coordinates near p that are normal in p.*

Proof. Assume $z_1, \ldots, z_n$ are holomorphic coordinates near p such that $z(p) = 0$ and $g_{jk} = \delta_{jk}$ at p. Let

$$z_j = \zeta_j + \sum A^j_{st} \zeta_s \zeta_t, \; A^j_{st} = A^j_{ts}$$

be a quadratic change of coordinates. Then

$$\begin{aligned} \sum g_{jk} dz_j \wedge d\bar{z}_k &= \sum g_{jk} d\zeta_j \wedge d\bar{\zeta}_k + 2 \sum g_{jk} A^j_{st} \zeta_t d\zeta_s \wedge d\bar{\zeta}_k + \\ &\quad + 2 \sum g_{jk} \bar{A}^k_{st} \bar{\zeta}_t d\zeta_j \wedge d\bar{\zeta}_s + O(|\zeta|^2). \end{aligned}$$

If $\tilde{g}_{j\bar{k}}$ denotes the components of the metric in the ζ-coordinates, we must have

$$\sum g_{jk} dz_j \wedge d\bar{z}_k = \sum \tilde{g}_{jk} d\zeta_j \wedge d\bar{\zeta}_k$$

so that

$$\tilde{g}_{jk} = g_{jk} + 2 \sum g_{rk} A^r_{jt} \zeta_t + 2 \sum g_{jr} \bar{A}^r_{kt} \bar{\zeta}_t + O(|\zeta|)^2).$$

Therefore, at p,

$$\frac{\partial \tilde{g}_{jk}}{\partial \zeta_m} = \frac{\partial g_{jk}}{\partial \zeta_m} + 2A^k_{jm} \quad \text{and} \quad \frac{\partial \tilde{g}_{jr}}{\partial \bar{\zeta}_m} = \frac{\partial g_{jk}}{\partial \bar{\zeta}_m} + 2\bar{A}^j_{km}$$

since $g_{jk} = \delta_{jk}$ at p. Moreover $\frac{\partial}{\partial \zeta_m} = \frac{\partial}{\partial z_m}$ and $\frac{\partial}{\partial \bar{\zeta}_m} = \frac{\partial}{\partial \bar{z}_m}$ at p so our new coordinates are normal if

$$A^k_{jm} = -\frac{1}{2} \frac{\partial g_{jk}}{\partial z_m}(p) \quad \text{and} \quad \bar{A}^j_{km} = -\frac{1}{2} \frac{\partial g_{jk}}{\partial \bar{z}_m}(p).$$

Since $g_{jk} = \bar{g}_{kj}$, these two equations are equivalent. If the metric is Kähler, we can define A_{jm} by these equations since the Kähler condition means precisely that A^k_{jm} then is symmetric in j and m. ■

Let us now consider the bundle of vectors of type $T_{1,0}$ over M as a holomorphic vector bundle. We then have two ways of defining a connection on $T_{1,0}$:

i) By the previous section there is a unique connection on $T_{1,0}$ that is both compatible with the metric and holomorphic.

ii) By the theory of Riemannian manifolds there is the Levi-Civita connection on $T_{1,0}$.

Actually, the second connection does not necessarily map a $(1,0)$ field to a field of the same type, so it is not, properly speaking, a connection on $T_{1,0}$ in general. We shall next prove that if the metric is Kähler, then the Levi-Civita connection does preserve type of the vector field, and moreover the two "canonical" connections are equal.

Proposition 3.4.2 *Let (M, g) be a Kähler manifold. Let ∇ be the canonical connection on $T_{1,0}$ that is both holomorphic and compatible with the metric. Let D be the Levi-Civita connection induced by the Riemannian structure. Then D maps $(1,0)$ fields to $(1,0)$ fields and $D = \nabla$ on such fields.*

Proof. It is enough to prove this in a fixed but arbitrary point, p. Let z_j be normal coordinates at p. Then

$$D_X(\sum Z_j \frac{\partial}{\partial z_j}) = \sum X(Z_j)\frac{\partial}{\partial z_j}$$

at p, by the comment after the proof of the existence of the Levi-Civita connection. In particular $D_X(Z)$ is a $(1,0)$-field if Z is a $(1,0)$-field. Moreover

$$D_X(Z) = 0$$

if Z is holomorphic and X is of bidegree $(0,1)$. Hence D is a holomorphic connection. Since moreover D is compatible with the metric, $D = \nabla$. ■

Our next objective is to generalize the fundamental identity from Chapter 1 to forms on a Kähler manifold with values in a holomoprhic bundle. This requires some algebraic perparations which we will take care of in the next section.

3.5 The Kähler identities

Let M be a Kähler manifold and denote the associated scalar product on $T^{\mathbb{C}}$ by $(,)$. In local coordinates

$$\begin{aligned}
(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}) &= g_{jk} \\
(\frac{\partial}{\partial \bar{z}_j}, \frac{\partial}{\partial \bar{z}_k}) &= \overline{g_{jk}} \\
(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_k}) &= 0.
\end{aligned}$$

The dual space of $T^{\mathbb{C}}$, $T^{*\mathbb{C}}$ is the space of complex one-forms and we can let the scalar product define an anti-linear isomorphism between $T^{\mathbb{C}}$ and $T^{*\mathbb{C}}$ by setting, for a vector field Z, Z^*, be the form satisfying

$$Z^*(W) = (W, Z)$$

for all $W \in T^{\mathbb{C}}$. Then we can define the scalar product also on $T^{*\mathbb{C}}$ by

$$(Z^*, W^*) = (W, Z).$$

This implies $|Z^*|^2 = |Z|^2$ and that

$$|Z^*| = \sup_{|W^*|\leq 1} |(Z^*, W^*)| = \sup_{|W|\leq 1} |Z^*(W)|$$

so the norm of a form with respect to the scalar product coincides with the norm as a linear functional.

We shall now extend the definition of the scalar product to forms of arbitrary degree. If v and w are two decomposable p-forms, i.e.,

$$\begin{aligned} v &= v_1 \wedge \dots v_p \\ w &= w_1 \wedge \dots w_p \end{aligned}$$

we let

$$(v, w) = \det((v_i, w_j)),$$

and then we extend the definition to arbitrary forms by linearity. The usual rules for determinants imply that this definition is independent of the representation of a p-form as a sum of decomposable ones. Note also that if

$$v_1, \dots, v_{2n}$$

is an orthonormal basis for $(T^*)^{\mathbb{C}}$, then

$$v_I = v_{i_1} \wedge \dots v_{i_p}$$

where $I = (i_1, \dots, i_p)$ runs over all increasing multiindices, is an orthonormal basis for $\wedge^p(T^{*\mathbb{C}})$. We also see that forms of different bidegrees are orthogonal.

We shall also take the opportunity to review the definition of the $*$-operator. First we consider M with only its real structure, i.e., as a N-dimensional real manifold, with a Riemannian metric. We also suppose M is oriented so that we have a globally defined volume form, ω_M, of degree N. For v, a k-form and w a $(N-k)$-form we can define a pairing $[v, w]$ by

$$v \wedge w = [v, w]\omega_N.$$

This is a non-singular pairing which gives an isomorphism between the dual of the space of k-forms and the space of $N-k$-forms. On the other hand, the space of k-forms is also dual to itself by the scalar product. Hence there are a linear operators

$$* : \wedge^k \to \wedge^{N-k}$$

defined by the property

$$[v, *w] =< v, w >,$$

i.e.,

$$v \wedge *w =< v, w > \omega_M.$$

If $e_1, \dots, e_N$ is an orthonormal basis for the one-forms, which is oriented so that $\omega_M = e_1 \wedge \dots e_N$, then clearly

$$*e_1 \wedge \dots e_k = e_{k+1} \wedge \dots e_N.$$

This could also have been taken as the definition but one would then need to prove that it is independent of the choice of basis. We leave it to the reader to verify that if v is a k-form then

$$* * v = (-1)^{k(N-k)} v.$$

If we consider complex-valued forms, we extend $*$ by complex-linearity. We then have

$$v \wedge *\bar{w} = (v, w)\omega_M.$$

With the aid of the scalar product we can now define the important operation of *interior multiplication.* Let θ be a r-form and r-form and v, a p-form with $p \geq r$. Then we define

$$\theta \lrcorner v$$

by the relation

$$(\theta \lrcorner v, w) = (v, \bar{\theta} \wedge w)$$

for all $(p-r)$-forms w. Observe that we have chosen the conjugate sign so that the operation becomes complex linear in θ:

$$(a\theta_1 + b\theta_2) \lrcorner v = a(\theta_1 \lrcorner v) + b(\theta_2 \lrcorner v).$$

Proposition 3.5.1 *Let θ be a 1-form and let $v = v_1 \wedge \dots v_p$. Then*

$$\theta \lrcorner v = (\theta \lrcorner v_1) v_2 \wedge \dots v_1 - (\theta \lrcorner v_2) v_1 \wedge \hat{v}_2 \wedge \dots v_p + \dots$$

(Note also that $\theta \lrcorner v_j = (\theta, \bar{v}_j)$).

Proof. We may assume that the v_j:s are the first p elements of an orthonormal basis, $v_1, \dots, v_{2n}$. Then $\bar{v}_j$ is also an orthonormal basis, so we may assume that $\theta = \bar{v}_k$ for some $k = 1, \dots, 2n$. What we must prove is then that

$$\theta \lrcorner v = 0 \qquad \text{if } k > p$$

and that

$$\theta \lrcorner v = (-1)^{k-1} v_1 \wedge \dots \hat{v}_k \wedge v_p \quad \text{if} \quad k \leq p.$$

The first claim follows from

$$(\theta \lrcorner v, w) = (v, v_k \wedge w) = 0 \qquad \text{if } k > p$$

and the second one follows from

$$((-1)^{k-1} v_1 \wedge \dots \hat{v}_k \wedge \dots v_p, w) = (v_1 \wedge \dots v_p, v_k \wedge w)$$

which is seen by expanding w in the same basis. ■

Let

$$\Omega = i \sum g_{jk} dz_j \wedge d\bar{z}_k$$

be the fundamental form of the Kähler metric. We now let L be the operator

$$w \to Lw = \Omega \wedge w$$

which sends p-forms to $(p+2)$-forms. Let us also use the notation

$$\theta^* w = \theta \lrcorner w.$$

Lemma 3.5.2

$$[L, \theta^*] w =: (L\theta^* - \theta^* L) w = i\theta \wedge w$$

if θ is a $(1,0)$-form.

Proof. Recall that Ω can be written

$$\Omega = i \sum w_k \wedge \overline{w_k}$$

if $\{w_k\}$ is an orthonormal basis for the $(1,0)$-forms. If $\theta = \sum \theta_j w_j$ we get from the previous proposition

$$\theta \lrcorner \Omega = -i\theta.$$

If w is an arbitarary form, the same proposition gives

$$\theta \lrcorner (\Omega \wedge w) = (\theta \lrcorner \Omega) \wedge w + \Omega \wedge (\theta \lrcorner w).$$

This gives the lemma. ■

We shall also have use for the operator Λ that is adjoint to L, defined by

$$(\Lambda v, w) = (v, Lw)$$

if v is a p-form and w is a $(p-2)$-form. ($\Lambda v = 0$ if v is of degree 0 or 1).

Lemma 3.5.3 *If θ is a $(1,0)$-form,*

$$\theta^* = i[\Lambda, \theta] =: i(\Lambda(\theta\wedge) - \theta \wedge \Lambda).$$

Proof. This is the dual of the previous lemma:

$$\begin{aligned}
(\theta^* v, w) &= (v, \bar\theta \wedge w) = (\theta \wedge \bar w, \bar v) = \\
&= i((\theta^* L - L\theta^*)\bar w, \bar v) = \\
&= i(\bar w, [\Lambda, \bar\theta]\bar v) = \\
&= i([\Lambda, \theta]v, w).
\end{aligned}$$

(Note that $\bar\Lambda = \Lambda$ since $\bar\Omega = \Omega$). ■

Our next aim is to find a useful formula for the adjoint of the $\bar\partial$-operator. Recall that in Section 3.1 we have defined $\bar\partial f$ when f is a function. If z is a local holomorphic coordinate system and f is a (p,q)-form,

$$f = \sum f_{IJ} dz_I \wedge d\bar z_J$$

($I = (i_1 \dots i_p)$, $J = (j_1, \dots, j_q)$, $dz_I = dz_1 \wedge \dots dz_{i_p}$) we let

$$\bar\partial f = \sum \bar\partial f_{IJ} \wedge dz_I \wedge d\bar z_J.$$

We then define the formal adjoint operator ϑ by

$$\int (\bar\partial v, w) = \int (v, \vartheta w)$$

for all smooth forms v with compact support. Here the integrals are taken with respect to the volume element on M, defined by

$$dV = \Omega^n / n!,$$

but we do not write that out in general.

First we shall give a formula for ϑ in terms of the connection D on M. The connection is originally defined on vector fields but can also be defined on 1-forms by the product rule:

$$(D_X v)(Z) + v(D_X(Z)) = X(v(Z))$$

if v is a 1-form and X, Z are vector fields. Note that D_X preserves the type ($(1,0)$ or $(0,1)$) of the form v, since we know that this is the case for vector fields. Next we define D_X on p-forms again by a product rule:

$$D_X(v \wedge w) = D_X v \wedge w + (-1)^{\deg v} v \wedge D_X w$$

if v and w are forms of arbitrary degree.

Lemma 3.5.4 *Let $Z_1, \ldots, Z_n$ be a basis for the vector fields of type $(1,0)$ and let $w_1, \ldots, w_n$ be the dual basis for the $(1,0)$-forms (so that $w_j(Z_k) = \delta_{jk}$). Then*

$$\bar{\partial} = \sum \overline{w_j} \wedge D_{\bar{Z}_j}.$$

Proof. Note first that the right hand side is independent of the choice of basis. Fix a point $p \in M$ and choose Z_j so that

$$Z_j = \frac{\partial}{\partial z_j},\ w_j = dz_j$$

in the point p, and assume also that z_j are normal coordinates at p. Then

$$D_{\frac{\partial}{\partial z_k}} dz_j = D_{\frac{\partial}{\partial \bar{z}_j}} d\bar{z}_j = D_{\frac{\partial}{\partial \bar{z}_k}} dz_j = 0$$

in p. Hence, if

$$\begin{aligned} v &= g dz_I \wedge d\bar{z}_J \\ \sum \overline{w_j} \wedge D_{\bar{Z}_j} v &= \sum \frac{\partial g}{\partial \bar{z}_j} d\bar{z}_j \wedge dz_I \wedge d\bar{z}_J = \bar{\partial} v \end{aligned}$$

at p. ■

Proposition 3.5.5 *Let again Z_j be a basis for the (1,0) vector field and let w_j be the dual basis for (1,0)-forms. Then*

$$\vartheta = -\sum w_j \lrcorner D_{Z_j}.$$

Proof. Observe again that the expression in the right hand side is independent of choice of basis. Therefore we may assume that

$$Z_j = \frac{\partial}{\partial z_j},\ w_j = dz_j$$

where z_j are normal coordinates at p — a given point. The operator ϑ is defined by

$$\int (\bar{\partial} v, w) = \int (v, \vartheta w)$$

for all smooth v:s with compact support. Let us first assume that the metric is flat near p, i.e., Z_j is orthogonal in a whole neighbourhood of p. Choosing v with support in that neighbourhood we get if

$$\begin{aligned} v &= g dz_I \wedge d\bar{z}_J,\ w = h dz_K \wedge d\bar{z}_L \\ \int (\bar{\partial} v, w) &= \int (\sum \frac{\partial g}{\partial \bar{z}_J} d\bar{z}_j \wedge dz_I \wedge d\bar{z}_J,\ h dz_K \wedge d\bar{z}_L) \\ &= \sum \int \frac{\partial g}{\partial \bar{z}_j} \bar{h} (dz_I \wedge d\bar{z}_J, dz_j \lrcorner (dz_K \wedge d\bar{z}_L)) \\ &= -\sum \int g \frac{\overline{\partial h}}{\partial z_j} (dz_I \wedge d\bar{z}_J, dz_j \lrcorner (dz_K \wedge d\bar{z}_L)) \\ &= \int (v, -\sum dz_j \lrcorner D_{\frac{\partial}{\partial z_j}} w) \end{aligned}$$

where the last equality follows since $D dz_k \wedge d\bar{z}_L = 0$. This proves the proposition if the metric is flat. In the general case we can argue in precisely the same way and we get the same result apart from possibly some extra terms containing first derivatives of the metric. All of those terms will vanish in the point p since we have normal coordinates so the proposition holds in general. ■

The last part of the argument may well be formulated as a general principle. *If a formula is independent of choice of coordinates, depends only on derivatives of the metric to order 0 and 1, and holds in the flat case, then it holds on any Kähler manifold.*

It will be useful to rewrite the formula for ϑ using the Λ operator.

Proposition 3.5.6

$$\vartheta = i[\partial, \Lambda].$$

Proof. It is enough to compute ϑv, where

$$v = g dz_I \wedge d\bar{z}_J,$$

in a given point where our coordinates are normal. According to the previous proposition

$$\vartheta v = -\sum \frac{\partial g}{\partial z_j} dz_j \lrcorner (dz_I \wedge d\bar{z}_J) = -\theta \lrcorner dz_I \wedge d\bar{z}_J$$

if

$$\theta = \partial g.$$

Hence Lemma 3.5.3 gives

$$\vartheta v = i[\theta, \Lambda] dz_I \wedge d\bar{z}_J.$$

On the other hand

$$\partial v = \theta \wedge dz_I \wedge d\bar{z}_J$$

and

$$\partial \Lambda v = \theta \wedge \Lambda dz_I \wedge d\bar{z}_J.$$

(The last statement follows since $\partial \Lambda dz_I \wedge d\bar{z}_J = 0$, which is true since it holds in the flat case and our coordinates are normal.) Hence

$$i[\partial, \Lambda]v = i[\theta, \Lambda]v = \vartheta v$$

and the proposition is proved. ∎

Taking conjugates we get, since $\bar{\Lambda} = \Lambda$:

Proposition 3.5.7 *The adjoint of $\partial, \bar{\vartheta}$ satisfies*

$$\bar{\vartheta} = -i[\bar{\partial}, \Lambda].$$

Recall now that in Riemannian geometry one defines the Laplace operator by

$$\Delta = d^* d + d d^*$$

where d^* is the adjoint of the exterior differentiation operator d under the scalar product. Using the operators ∂ and $\bar{\partial}$, we can then define in the same way

$$\Box = \partial^* \partial + \partial \partial^*$$

and

$$\bar{\Box} = \bar{\partial}^* \bar{\partial} + \bar{\partial} \bar{\partial}^*$$

where we have written $\bar{\partial}^*$ and ∂^* instead of ϑ and $\bar{\vartheta}$. Then both $\Box$ and $\bar{\Box}$ send (p, q)-forms to (p, q)-forms. We then have the following generalization of the elementary formula

$$\frac{\partial^2}{\partial z \partial \bar{z}} = \frac{1}{4} \Delta :$$

Theorem 3.5.8 *On a Käher manifold*

$$\Box = \bar{\Box} = \frac{1}{2}\Delta.$$

Remark. The factor $\frac{1}{2}$ instead of $\frac{1}{4}$ depends on our having chosen the metric so that dz_j is of norm 1.

Proof. By Proposition 3.5.6

$$\begin{aligned} \bar{\Box} &= \bar{\partial}^*\bar{\partial} + \bar{\partial}\bar{\partial}^* = i([\partial,\Lambda]\bar{\partial} + \bar{\partial}[\partial,\Lambda])) \\ &= i(\partial\Lambda\bar{\partial} - \Lambda\partial\bar{\partial} + \bar{\partial}\partial\Lambda - \bar{\partial}\Lambda\partial). \end{aligned}$$

Hence $\bar{\Box}$ is a real operator so $\bar{\Box} = \Box$. Propositions 3.5.6 and 3.5.7 also give that

$$\partial^*\bar{\partial} + \bar{\partial}\partial^* = i([\Lambda,\bar{\partial}]\bar{\partial} + \bar{\partial}[\Lambda,\bar{\partial}]) = 0$$

and

$$\bar{\partial}^*\partial + \partial\bar{\partial}^* = 0.$$

Writing $d = \partial + \bar{\partial}$, $d^* = \partial^* + \bar{\partial}^*$ we therefore get

$$\Delta = \Box + \bar{\Box}$$

since the "mixed" terms vanish. ∎

Proposition 3.5.9 *On a Kähler manifold*

$$[\Box, L] = 0 = [\Box, \Lambda].$$

Proof: Since $\Box^* = \Box$ it suffices to prove the second equality. But, since $[\partial^*, \Lambda] = 0$,

$$[\Lambda, \Box] = [\Lambda, \bar{\partial}]\bar{\partial}^* + \bar{\partial}^*[\Lambda, \bar{\partial}] = -i(\partial^*\bar{\partial} + \bar{\partial}\partial^*) = 0,$$

by the proof of the previous Proposition. ∎

In the last section we shall also have use for the next proposition.

Proposition 3.5.10 *Let M be a Riemannian manifold. Then*

$$[\Delta, *] = 0.$$

If M is a Kähler manifold it also holds

$$[\Box, *] = 0.$$

Proof: We need of course only prove the first statement. First we need to verify that if v is a k-form then

$$d^*v = (-1)^{(k+1)N-1} * d * v,$$

if N is the (real) dimension of M. To see this, let u be a $k-1$-form with compact support (if $k = 0$, the formula is trivial). By definition

$$\int (du, v) = \int (u, d^*v).$$

On the other hand, by the definition of the $*$-operator

$$\int (du, v) = \int du \wedge *v = \int d(u \wedge *v) + (-1)^k \int u \wedge d * v,$$

and the first term on the right hand side vanishes if u has compact support, by Stokes theorem. Since $** = (-1)^{p(N-p)}$ on p-forms, it follows that $** (d * v) = (-1)^{(N-k+1)(k-1)}(d * v) = (-1)^{N(k+1)+k+1}(d * v)$, and from this the formula for d^* follows.

That

$$*\Delta = *(d^*d + dd^*) = (d^*d + dd^*)* = \Delta*$$

now follows from a direct verification, using again that $** = (-1)^{p(N-p)}$ on p-forms . ■

3.6 The Lefschetz isomorphism

In this section we shall prove some further identities for forms in a point. Apart from the first two propositions they will be used only at one place in the sequel, namely in the proof of the so-called "hard Lefschetz Theorem". Since we are dealing with forms in a fixed point, we assume throughout that $dz_1, \ldots, dz_n$ is orthonormal, i.e., our Kähler form is

$$\Omega = i \sum dz_j \wedge d\bar{z}_j.$$

Definition. A form α is primitive if

$$\Lambda\alpha = 0.$$

Let us use the following notation:

$$\begin{aligned} dV_i &= dz_i \wedge d\bar{z}_i,\ dV_I = dV_{i_1} \wedge \ldots dV_{i_p}, \\ \alpha_{I,J,K} &= dV_I \wedge dz_J \wedge d\bar{z}_K \end{aligned}$$

where I, J, K are disjoint multiindices.

Let us also write $I + i = \{i, I\}$ and $I - i =$ the multiindex I with i removed regardless of place.

Proposition 3.6.1

$$\Lambda\alpha_{I,J,K} = -i \sum_{j \in I} \alpha_{I-j,J,K}$$

Proof. If $\beta = \alpha_{L,M,N}$,

$$L\beta = \Omega \wedge \beta = i \sum_{j \notin L,M,N} \alpha_{L+j,M,N}.$$

Λ is defined by

$$< \Lambda\alpha, \beta > = < \alpha, L\beta > .$$

The scalar product in the right hand side is $\neq 0$ if and only if there is a $j \notin L \cup M \cup N$ such that $L + j = I$, $M = J$, $K = N$. In this case it equals $-i$. This proves the proposition. ■

Proposition 3.6.2 *Suppose α is a form of degree k. Then*

$$[\Lambda, L]\alpha = (n - k)\alpha.$$

Proof. We may assume $\alpha = \alpha_{I,J,K}$. Then

$$L\alpha = i \sum_{j \notin I \cup J \cup K} \alpha_{I+j,J,K}$$

so

$$\Lambda L\alpha = \sum_{\substack{k \in I \\ j \notin I \cup J \cup K}} \alpha_{I+j-k,J,K} + (n - (|I| + |J| + |K|))\alpha$$

(the last term comes from when $j = k$). Moreover

$$L\Lambda\alpha = \sum_{\substack{j \in I \\ k \notin I \cup J,K}} \alpha_{I-j+k,J,K} + |I|\alpha.$$

Combining we get

$$[\Lambda, L]\alpha = (n - (2|I| + |J| + |K|))\alpha = (n - k)\alpha.$$

■

Proposition 3.6.3 *Suppose α is a k-form. Then*

$$[\Lambda, L^p]\alpha = C_{k,p} L^{p-1}\alpha$$

where $C_{k,p} = p(n - k + 1 - p)$.

Proof. Note that $L^{p-1}\alpha$ is of degree $k + 2p - 2$. Hence

$$\begin{aligned} \Lambda L^p \alpha &= \Lambda L L^{p-1}\alpha = [\Lambda, L]L^{p-1}\alpha + L\Lambda L^{p-1}\alpha = \\ &= (n - (k + 2p - 2))L^{p-1}\alpha + C_{k,p-1}L^{p-1}\alpha + L^p\Lambda\alpha \end{aligned}$$

where we have used the previous proposition and an induction hypotehesis. Hence

$$[\Lambda, L^p]\alpha = C_{k,p} L^{p-1}\alpha$$

where

$$C_{k,p} = C_{k,p-1} + (n - k - 2p + 2).$$

This difference equation implies that $C_{k,p}$ is a second degree polynomial in p with leading coefficient $-p^2$,

$$C_{k,p} = -p^2 + xp + C_{k,0}.$$

Since $C_{k,0} = 0$ and $C_{k,1} = n - k$, we get $x = n - k + 1$ so

$$C_{k,p} = p(n - k + 1 - p).$$

■

Theorem 3.6.4 *Let α be a k-form. Then α can be written*

$$\alpha = v_0 + Lv_1 + \dots L^s v_s$$

with v_j primitive $(k - 2j)$-forms. Moreover, the decomposition is unique in the sense that $\alpha = 0$ implies $L^j v_j = 0$, $j = 0, \dots s$.

Proof. We can always write

$$\alpha = v_0 + \alpha^1$$

where α^1 is orthogonal to the kernel of Λ, and v_0 is primitive. Since our space of forms in a point is of finite dimension $\alpha^1 = L\alpha_1$ for some α_1 of degree $(k-2)$. Repeating the argument with α replaced by α_1, etc., we get the existence of a decomposition. To prove unicity, it is enough to prove that the terms are pairwise orthogonal. Say $k < j$ and v_k, v_j are primitive. Then

$$(L^k v_k, L^j v_j) = (L^{k-1} v_k, \Lambda L^j v_j) = (L^{k-1} v_k, L^{j-1} v_j) C_{j,g_j}$$

where $g_j =$ degree (v_j). Here we have used our commutator formula and $\Lambda v_j = 0$. Continuing k times we find that the terms are indeed orthogonal. ∎

A natural question that arises is if $L^j v_j = 0$ implies $v_j = 0$? The answer is given by

Proposition 3.6.5 *Suppose v is a primitive k-form. Then*

a) $k \leq n \quad$ if $\quad v \neq 0$.
b) $L^{n-k} v = 0 \Rightarrow v = 0$.
c) $L^{n-k+1} v = 0$.

Proof. Suppose v is a primitive k-form and $L^s v = 0$. Then $[\Lambda, L^s] v = 0$ so

$$s(n-k+1-s) L^{s-1} v = 0.$$

If now $s \leq n-k$, we get $L^{s-1} v = 0$. Iterating we find $v = 0$, so b) is proved. On the other hand, $L^s v = 0$ always if s is sufficiently big. Then it follows again that even $L^{s-1} v = 0$ if $s > n-k+1$. Iterating we find $L^{n-k+1} v = 0$ if $n-k+1 \geq 0$, so c) also follows. If again $n-k+1 \leq 0$, the process stops at the stage $v = 0$, so we also obtain a). ∎

We can then improve the formulation of Theorem 3.6.4.

Theorem 3.6.6 *Any k-form α can be written*

$$\alpha = \sum_{j \geq k-n} L^j v_j$$

with v_j primitive. If such a sum vanishes,then $v_j = 0$ for all j.

Proof. We know

$$\alpha = \sum_0^\infty L^j v_j$$

and deg $v_j = k - 2j =: d_j$. The previous proposition implies that

$$L^j v_j = 0,$$

if $j > n - d_j = n - k + 2j$, i.e., if $j < k - n$. Thus

$$\alpha = \sum_{j \geq k-n} L^j v_j.$$

If in this sum $L^j v_j$ vanishes, then $v_j = 0$ since $j \leq n - d_j$. ∎

Proposition 3.6.7 *Let α be a k-form with $k \leq n$. Assume $L^{n-k+s}\alpha = 0$. Then the primitive decomposition*

$$\alpha = \sum_{j \geq 0} L^j v_j$$

conists only of s terms (i.e., $L^j v_j = 0$, $j \geq s$).

Proof. We have already seen that $L^{n-k+s}\alpha = 0$ implies $L^{n-k+s+j}v_j = 0$, $j = 0, 1, 2, \ldots$. This in turn implies $v_j = 0$ if

$$n - k + s + j \leq n - \deg\,(v_j) = n - (k - 2j),$$

i.e., if $s \leq j$. ■

Corollary 3.6.8 *Let α be a k-form with $k \leq n$. Then α is primitive if and only if $L^{n-k+1}\alpha = 0$.*

Proof. The "if"-direction follows from the last proposition, and the other is Proposition 3.6.5. c). ■

The main result of this section is also an immediate consequence:

Theorem 3.6.9 *Let $k \leq n$. Then the map*

$$L^{n-k} : E^k \to E^{2n-k}.$$

(E^j is the space of j-forms) is an isomorphism.

Proof. Injectivity is the case $s = 0$ of Proposition 3.6.7. Surjectivity follows from 3.6.6

$$E^{2n-k} \ni \alpha = \sum_{j \geq n-k} L^j v_j$$

(or by comparing dimensions). ■

Our next goal is to compare the isomorphism L^{n-k} with the isomorphism defined by the $*$-opertor (see §5). We start with the case v primitive, and then want to compute $*v$.

Note that a form of the type

$$\alpha dz_I \wedge d\bar{z}_K \qquad I \cap K = \phi$$

is always primitive by Proposition 3.6.1. The same evidently holds for forms that can be written in this form after a unitary change of coordinates. Our next Lemma says that this gives us spanning set for all primitive forms.

Lemma 3.6.10 *Let v be a primitive (p, q)-form. Then v can be written as a sum of terms of the type*

$$* \qquad a_1 \wedge \ldots a_p \wedge \bar{b}_1 \wedge \ldots \bar{b}_q,$$

where a_i, b_j are $(1, 0)$-forms such that

$$(a_i, b_j) = 0 \qquad \forall i, j.$$

Proof. It is enough to show that if v is primitive and orthogonal to all forms of type $*$ then $v = 0$. Suppose that e.g. $p \geq q$. Take an arbitrary $(1,0)$-form a. Then

$$\bar{a} \lrcorner v$$

satisfies the same hypothesis as v on $a^\perp$, i.e., $a \lrcorner v$ is orthogonal to all forms of type

$$a_2 \wedge \dots a_p \wedge \bar{b}_1 \wedge \dots \bar{b}_q$$

where a_i and b_j are pairwise orthogonal and orthogonal to a. Moreover, $\Lambda \bar{a} \lrcorner v = \bar{a} \lrcorner \Lambda v = 0$ so $\bar{a} \lrcorner v$ is primitive. We can then assume by induction that $\bar{a} \lrcorner v = 0$ on $a^\perp$, i.e.,

$$\bar{a} \lrcorner v \perp a_2 \wedge \dots \wedge a_p \wedge \bar{b}_1 \wedge \dots \bar{b}_q$$

if $a_2, \dots a_p$, $b_1, \dots b_q \perp a$. In other words,

$$v \perp a \wedge a_2 \wedge a_p \wedge \bar{b}_1 \wedge \dots \bar{b}_q.$$

Of course this last relation holds even if the a_j:s are not orthogonal to a since the component of a_j that is parallel to a gives a zero contribution to the wedge product. On the other hand, there is nothing special about the first factor in $a \wedge a_2 \wedge \dots a_p$ so actually

$$v \perp a_1 \wedge a_2 \wedge \dots a_p \wedge \bar{b}_1 \wedge \dots \bar{b}_q$$

as soon as there is some linear combination of $a_1, \dots, a_p$ that is orthogonal to all the b_j:s. Otherwise $p \leq q$, so $p = q$ and v can be written

$$v = \sum \lambda_I dV_I$$

(just expand $v = \sum \lambda_{IJK} \alpha_{IJK}$). Now take $j \neq k$ and a multiindex J that does not contain j or k. Since v is orthogonal to

$$(dz_j - dz_k) \wedge (d\bar{z}_j + d\bar{z}_k) \wedge dV_J,$$

we see that

$$\lambda_{j \cup J} = \lambda_{k \cup J}.$$

This means that all the λ_I:s are equal, so

$$v = c\Omega^p$$

where Ω is the Kähler form. But then v can be primtive only if $v = 0$. ■

It is now easy to compare the $*$-operator and the operator L^{n-k} on k-forms. First assume that v is a primitive k-form. By the previous Lemma it is enough to treat the case $v = dz_I \wedge d\bar{z}_J$, $I \cup J = \emptyset$. Then it is easily seen that

$$L^{n-k} v = a_{n,p,q} * v.$$

If $v = L^j w$ where w is a primitive k-form, we use the relation

$$*L = \Lambda *$$

and find that

$$*L^j w = \Lambda^j * w = A_{n,p,q,j} L^{n-k-j_w}.$$

If we then use the orthogonal decomposition

$$\alpha = \sum L^j v_j$$

with v_j primitive, we see that $*$ and L^{n-k} are related by a multiplicative constant at each level, and this constant depends on n, j and the bidegree.

3.7 Vector bundles over Kähler manifolds

If we for a moment recall the proof of the $\bar{\partial}$-estimates over open sets in $\mathbb{C}^n$ from Chapter 1, we see that a crucial role was played by the weight factor $e^{-\varphi}$. The counterpart of this in our present setting is a choice of hermitian metric on a complex vector bundle over our complex manifold M. The weight factor $e^{-\varphi}$ from Chapter 1, can then be interpreted as a metric on the trivial line bundle over $\mathbb{C}^n$. In this section we assume that M is a Kähler manifold with a fixed Kähler metric.

Let (E, M, π) be a complex vector bundle over M, endowed with an hermitian metric g. Let ∇ be the canonical conection on E which is both holomorphic and compatible with the metric. We shall now regard ∇ from a slightly different point of view, and as a preparation we first need to consider differential forms on M with values in E. An ordinary complex k-form on M is, for each $p \in M$, an alternating form v_p on

$$T_p^{\mathbb{C}} \times \ldots T_p^{\mathbb{C}} \qquad (k \text{ times })$$

such that the function of p

$$v(Z_1, \ldots, Z_n)$$

is smooth if $Z_1, \ldots, Z_n$ are smooth vector fields. A k-form with values in E is then, for each $p \in M$, a map

$$\xi_p : T_p^{\mathbb{C}} \times \ldots T_p^{\mathbb{C}} \to E_p,$$

which is linear in each argument, alternating and smoth. This means that if $Z_1, \ldots, Z_p$ are smooth vector fields, then

$$\xi(Z_1, \ldots, Z_n)$$

is a smooth section to E. In particular, a 0-form with values in E is just a section to E, and in general if $e_1, \ldots, e_r$ is a local frame of sections to E, a k-form with values in E can be written locally

$$\xi = \sum_1^n \xi_\nu e_\nu$$

where ξ_ν are complex valued k-forms. Sometimes we write

$$\xi = \sum_1^r \xi_\nu \otimes e_\nu,$$

where the tensor product, $\xi = v \otimes s$, of a form and a section is defined in the obvious way

$$\xi(Z_1, \ldots, Z_k) = v(Z_1, \ldots, Z_k)s.$$

Of course we can consider E-valued forms that are only locally defined. The space of E-valued k-forms over $U \subseteq M$ is written

$$C_k^\infty(U, E),$$

and the decomposition of scalar forms in bidegrees induces a decomposition

$$C_k^\infty(U, E) = \sum_{p+q=k} C_{p,q}^\infty(U, E)$$

in the obvious way. Now observe the important fact that the $\bar{\partial}$ operator can be defined in a natural way on $C_{p,q}^\infty(U, E)$. Simply let

$$\bar{\partial}\xi = \sum \bar{\partial}\xi_\nu \otimes e_\nu$$

if e_ν is a holomorphic frame. It is immediately clear that this definition is independent of which holomorphic frame we have chosen. On the other hand, the operator d has no canonical definition on E-valued forms, and we shall now see that what corresponds to d for E-valued forms is our connection ∇.

We have defined (see §3) ∇ as a bilinear map

$$\nabla : \Gamma(T^{\mathbb{C}}(M)) \times \Gamma(E) \to (E)$$

with certain additional properties. Equivalently, we can consider ∇ as a map

$$\nabla : \Gamma(E) \to C_1^\infty(E)$$

from sections to formvalued sections, where if ξ is a scalar-valued section, $\nabla\xi$ is defined by

$$\nabla\xi(Z) = \nabla_Z\xi.$$

The defining properties of a connection mean precisely that $\nabla\xi$ is an E-valued 1-form, and moreover,

$$\nabla f\xi = df \otimes \xi + f\nabla\xi$$

if f is a function. Remember that ∇ is said to be holomorphic if $\nabla_Z\xi = 0$ whenever ξ is a holomorphic section, and Z is of bidegree $(0,1)$. Decompose

$$\nabla = \nabla' + \nabla''$$

where ∇'_ξ is the $(1,0)$ component and ∇''_ξ is the $(0,1)$ component of the 1-form $\nabla\xi$. If Z is of bidegree $(0,1)$ and $\xi = \sum \xi_\nu \otimes e_\nu$,

$$\nabla_Z\xi = \nabla''_Z\xi$$

and

$$\nabla''\xi = \sum \bar\partial\xi_\nu \otimes e_\nu + \sum \xi_\nu \nabla'' e_\nu.$$

Hence ∇ is holomorphic if and only if

$$\nabla'' = \bar\partial.$$

We have previously defined the connection coefficients $\Gamma^\mu_{m\nu}$ by

$$\nabla_{\partial_m} e_\nu = \sum \Gamma^\mu_{m\nu} e_\mu$$

(where $\partial_m = \frac{\partial}{\partial z_m}$) if $\{e_\nu\}$ is a frame field. Equivalently,

$$\nabla' e_\nu = \sum \Gamma^\mu_{m\nu} dz_m \otimes e_\mu.$$

We can then define

$$\theta^\mu_\nu = \sum \Gamma^\mu_{m\nu} dz_m,$$

and get

$$\nabla'\xi = \sum \partial\xi_\nu \otimes e_\nu + \sum \xi_\nu \theta^\mu_\nu \otimes e_\mu$$

or in short-hand

$$\nabla' = \partial + \theta. \tag{3.12}$$

So, for each frame, we get a matrix of $(1,0)$-forms θ. Now, let g be a hermitian metric on E so that

$$g_{\nu\mu} =< e_\nu, e_\mu >,$$

and let ∇ be the canonical connection for this metric. Then we know that

$$\Gamma^\mu_{m\nu} = \sum_\lambda g^{\lambda\mu} \frac{\partial}{\partial z_m} g_{\nu\lambda}$$

(see the proof of Theorem 3.3.1), so

$$\theta_\nu^\mu = \sum_\lambda g^{\lambda\mu} \partial g_{\nu\lambda}$$

or

$$\theta = h^{-1}\partial h \quad \text{if} \quad h = g^t.$$

In particular, we see that if E is a line bundle (i.e., $r = 1$) and

$$g = g_{11} = e^{-\varphi},$$

then

$$\theta = e^{\varphi}\partial e^{-\varphi} = -\partial\varphi. \tag{3.13}$$

Next, we can combine the scalar products that we have on forms and on sections to E to get a scalar product on E-valued forms. Concretely, if

$$\xi = \sum \xi_\nu \otimes e_\nu \quad \text{and} \quad \eta = \sum \eta_\nu \otimes e_\nu,$$

then

$$< \xi, \eta >=: \sum < \xi_\nu, \eta_\mu >< e_\nu, e_\mu >= \sum g_{\nu\mu} < \xi_\nu, \eta_\mu > .$$

This definition implies that

$$< \xi \otimes s, \eta \otimes t >=< \xi, \eta >< s, t >$$

if ξ and η are forms, and s and t are scalar sections. Therefore, the definition is independent of choice of frame.

We can let the connection ∇ act on E-valued forms by demanding

$$\nabla v \wedge \xi = dv \wedge \xi + (-1)^m v \wedge \nabla\xi \tag{3.14}$$

if v is a scalar m-form, and ξ is an E-valued form. If, with respect to a frame $\xi = \sum \xi_\nu \otimes e_\nu$, then (3.14) implies

$$\begin{aligned} \nabla\xi &= \sum d\xi_\nu \otimes e_\nu + (-1)^m \sum \xi_\nu \wedge \theta_\nu^\mu \otimes e_\mu \\ &= \sum d\xi_\nu \otimes e_\nu + \sum \theta_\nu^\mu \wedge \xi_\nu \otimes e_\mu = (d+\theta)\xi. \end{aligned} \tag{3.15}$$

Conversely if (3.15) holds for some choice of frame, then (3.14) holds, so we really get a good definition.

The last new concept that we need is the curvature of the connection. Consider the operator

$$\xi \to \nabla^2\xi$$

that sends (E-valued) k-forms to $(k+2)$-forms. If f is a function

$$\begin{aligned} \nabla^2(f\xi) &= \nabla(df \wedge \xi + f\nabla\xi) = \\ &= d^2 f \wedge \xi - df \wedge \nabla\xi + df \wedge \xi + f\nabla^2\xi = f\nabla^2\xi. \end{aligned}$$

This means that ∇^2 (contrary to ∇) is $\mathbb{C}^\infty$-linear.

With respect to a frame $\nabla^2\xi = \sum \xi_\nu \otimes \nabla^2 e_\nu = \sum \xi_\nu \wedge \Theta_\nu^\mu \otimes e_\mu$. The operator

$$\nabla^2 =: \Theta$$

is called the curvature of connection and is represented with respect to a frame by the matrix of 2-forms

$$(\Theta^\mu_\nu).$$

We can compute Θ by

$$\begin{aligned}\nabla^2\xi &= (d+\theta)(d\xi+\theta\wedge\xi) = \\ &= d^2\xi + d\theta\wedge\xi - \theta\wedge d\xi + \theta\wedge d\xi + \theta\wedge\theta\wedge\xi = (d\theta+\theta\wedge\theta)\xi.\end{aligned}$$

In other words,

$$\Theta^\mu_\nu = d\theta^\mu_\nu + \sum_\lambda \theta^\mu_\lambda \wedge \theta^\lambda_\nu$$

Proposition 3.7.1 *Let ∇ be the canonical connection with respect to some metric. Then*

$$\partial\theta + \theta\wedge\theta = 0 \quad \textit{and} \quad \Theta = \bar{\partial}\theta.$$

Proof. We know that

$$\theta = h^{-1}\partial h.$$

Hence

$$\partial\theta = -h^{-1}\partial h h^{-1}\partial h = -\theta\wedge\theta,$$

which proves the first equation. The second one follows since

$$\Theta = d\theta + \theta\wedge\theta = \partial\theta + \theta\wedge\theta + \bar{\partial}\theta.$$

■

Example. If E is a trivial line bundle and $g = (g_{11})$, where

$$g_{11} = e^{-\varphi},$$

we know that ((3.13)) $\theta = -\partial\varphi$. Hence

$$\Theta = -\bar{\partial}\partial\varphi = \partial\bar{\partial}\varphi.$$

If ξ and η are two E-valued forms of which at least one has compact support, we can define

$$<\xi,\eta>_M = \int_M <\xi,\eta>,$$

where, as usual, the integral is taken with respect to the volume element

$$dV = \Omega^n/n!,$$

where Ω is the Käher form of M. As before, we define the adjoint operators to ∇' and ∇'' by

$$<\nabla'\xi,\eta>_M = <\xi,(\nabla')^*\eta>_M$$

and

$$<\nabla''\xi,\eta>_M = <\xi,(\nabla'')^*\eta>_M\ .$$

Suppose Ω defines a Kähler metric on M, and define the Λ-operator on E-valued forms by

$$\Lambda\sum\xi_\nu\otimes e_\nu = \sum(\Lambda\xi_\nu)\otimes e_\nu$$

(see §5 for the definition of Λ).

Proposition 3.7.2

$$\begin{aligned}(\nabla')^* &= -i[\bar\partial, \Lambda] \quad \textit{and} \\ (\nabla'')^* &= i[\nabla', \Lambda].\end{aligned}$$

Proof. Take a holomorphic frame $\{e_\nu\}$ and let

$$\begin{aligned}g_{\nu\mu} &= < e_\nu, e_\mu > \\ h_{\nu\mu} &= g_{\mu\nu}\end{aligned}$$

as before. For the moment we use the notation

$$\{\xi, \eta\} = \sum < \xi_\nu, \eta_\nu >$$

if $\xi = \sum \xi_\nu \otimes e_\nu$, $\eta = \sum \eta_\nu \otimes e_\nu$, and let h operate on ξ by

$$h\xi = \sum h_{\nu\mu}\xi_\mu \otimes e_\nu.$$

We know that

$$\nabla'\xi = \partial\xi + (h^{-1}\partial h)\xi = h^{-1}\partial(h\xi)$$

and

$$< \xi, \eta > = \{h\xi, \eta\} = \{\xi, g\eta\}.$$

Hence

$$< \nabla'\xi, \eta >_M = \int_M \{\partial(h\xi), \eta\} = \int_M \{h\xi, \partial^*\eta\} = < \xi, \partial^*\eta >_M,$$

where $\partial^*\eta = \sum \partial^*\eta_\nu \otimes e_\nu$. Since by Proposition 3.5.7 $\partial^* = -i[\bar\partial, \Lambda]$, we have proved the first statement. On the other hand,

$$< \nabla''\xi, \eta >_M = \int \{h\bar\partial\xi, \eta\} = \int \{\bar\partial\xi, g\eta\} = < \xi, g^{-1}\bar\partial^*(g\eta) >_M,$$

where $\bar\partial^*\eta = \sum \bar\partial^*\eta_\nu \otimes e_\nu$. Thus

$$(\nabla'')^*\eta = g^{-1}\bar\partial^*(g\eta) = ig^{-1}[\partial, \Lambda]g\eta = i[g^{-1}\partial g, \Lambda]\eta = i[\nabla', \Lambda]$$

by Proposition 3.5.6 since g and g^{-1} commute with Λ. ■

Now we are finally ready to prove the fundamental identity that generalizes Theorem 1.4.2 and is the key to the vanishing theorems that we will prove in the next section. In analogy with §5 we can use the operators ∇', ∇'' to define two Laplace-operators for forms with values in a vector bundle.

$$\Box' = \nabla'(\nabla')^* + (\nabla')^*\nabla' \tag{3.16}$$

and

$$\Box'' = \nabla''(\nabla'')^* + (\nabla'')^*\nabla''. \tag{3.17}$$

When E is the trivial line bundle with trivial metric $\Box' = \Box$ and $\Box'' = \bar\Box$, so it follows from §5 that $\Box' = \Box''$. In general, the difference of the two operators depends on the curvature of the connection on E.

Theorem 3.7.3

$$\Box'' = \Box' + i[\Theta, \Lambda].$$

Proof. As before, we choose a holomorphic frame $e_1, \ldots, e_r$ so that

$$\nabla = d + \theta$$

with respect to this frame. By the previous proposition

$$(\nabla')^* = -i[\bar{\partial}, \Lambda] = \partial^*$$

where $\partial^*\xi$ just means

$$\partial^* \sum \xi_\nu \otimes e_\nu = \sum (\partial^* \xi_\nu) \otimes e_\nu.$$

In the same way,

$$(\nabla'')^* = \bar{\partial}^* + i[\theta, \Lambda].$$

Therefore,

$$\Box' = \Box + \theta\partial^* + \partial^*\theta$$

and

$$\Box'' = \bar{\Box} + i[\theta, \Lambda]\bar{\partial} + i\bar{\partial}[\theta, \Lambda],$$

where $\Box$ and $\bar{\Box}$ are the Laplacians from §5 acting on each component ξ_ν. Since our metric on M is Kähler $\Box = \bar{\Box}$, so

$$\begin{aligned} \Box'' - \Box' &= i([\theta, \Lambda]\bar{\partial} + \bar{\partial}[\theta, \Lambda]) - \theta\partial^* - \partial^*\theta = \\ &= i([\theta, \Lambda]\bar{\partial} + \bar{\partial}[\theta, \Lambda] + \theta[\bar{\partial}, \Lambda] + [\bar{\partial}, \Lambda]\theta) = i[\bar{\partial}\theta + \theta\bar{\partial}, \Lambda]. \end{aligned}$$

Now, the expression $\bar{\partial}\theta + \theta\bar{\partial}$ stands for the operator

$$\xi \to \bar{\partial}(\theta \wedge \xi) + \theta \wedge \bar{\partial}\xi$$

which equals

$$\xi \to (\bar{\partial}\theta) \wedge \xi = \Theta \wedge \xi.$$

Hence

$$\Box'' - \Box' = i[\Theta, \Lambda]$$

and we are done. ∎

3.8 Vanishing theorems

To explain the title of this paragraph, we first review the definition of the Dolbeault cohomology groups. First, let

$$Z^{p,q}(M, E) = \{\xi \in C^\infty_{p,q}(M, E); \bar{\partial}\xi = 0\}$$

and

$$B^{p,q}(M, E) = \{\bar{\partial}\eta; C^\infty_{p,q-1}(M, E)\}$$

($B^{p,-1} =: 0$). Thus $Z^{p,q}$ is the space of smooth $\bar{\partial}$-closed (p,q)-forms, and $B^{p,q}$ is the space of $\bar{\partial}$-exact (p,q)-forms. Note that $Z^{p,0}$ is the space of holomorphic p-forms, i.e., forms of bidegree $(p, 0)$ that have all their coefficients holomorphic. Clearly $B^{p,q} \subseteq Z^{p,q}$.

Definition 3.8.1

$$H^{p,q}(M, E) = Z^{p,q}(M, E)/B^{p,q}(M, E)$$

is the (p, q):th Dolbeault cohomology group with coefficients in E.

This, $H^{p,q}$ measures to what extent the equation $\bar\partial\eta = \xi$ is solvable, and theorems to the effect that this equation is always solvable are called vanishing theorems since they mean that $H^{p,q} = 0$.

The formalism of the preceding section makes it easy to give the analog of Theorem 1.4.2.

Theorem 3.8.2 *Let E be a hermitian vector bundle over the Kähler manifold M, and let ξ be a (p,q)-form with values in E and compact support. Then*

$$\int_M |\bar\partial\xi|^2 + \int_M |(\nabla'')^*\xi|^2 = \int_M |\nabla'\xi|^2 + \int_M |(\nabla')^*\xi|^2 + \int_M i < [\Theta,\Lambda]\xi, \xi > .$$

Proof. This follows directly from Theorem 3.7.3 since

$$\int < \Box''\xi, \xi > = \int |\bar\partial\xi|^2 + \int |(\nabla'')^*\xi|^2$$

and

$$\int < \Box'\xi, \xi > = \int |\nabla'\xi|^2 + \int |(\nabla')^*\xi|^2.$$

∎

To get existence theorems for the $\bar\partial$-equation, we need to analyze the expression

$$i[\Theta,\Lambda]\xi.$$

We shall do this in some detail when E is a line bundle, but to warm up, we first study the case when $M = \mathbb{C}^n$, and E is the trivial line bundle with metric $e^{-\varphi}$. This means that we consider the scalar product

$$< \xi, \eta > = (\xi,\eta)e^{-\varphi},$$

where $(,)$ denotes the standard scalar product for forms on $\mathbb{C}^n$ defined by

$$(dz_j, dz_k) = \delta_{jk}.$$

For reasons that will become apparent we choose for ξ a $(n,1)$-form

$$\xi = \sum \xi_j d\bar z_j \wedge dz, \quad dz = dz_1 \wedge \dots dz_n.$$

Then we have

$$\Lambda d\bar z_j \wedge dz = i(-1)^{j-1} dz_1 \wedge \dots \widehat{dz_j} \wedge \dots dz_n = i\widehat{dz_j}$$

(see §6), and

$$\Theta = \sum \varphi_{j\bar k} dz_j \wedge d\bar z_k.$$

Since $\Theta \wedge \xi = 0$ for bidegree reasons,

$$i[\Theta,\Lambda]\xi = -\Theta \wedge \sum \xi_j \widehat{dz_j} = \sum \varphi_{j\bar k}\xi_j d\bar z_k \wedge dz.$$

Hence

$$i < [\Theta,\Lambda]\xi, \xi > = \sum \varphi_{j\bar k}\xi_j \bar\xi_k e^{-\varphi},$$

and the theorem says that

$$\int_M \sum \varphi_{j\bar k}\xi_j\bar\xi_k e^{-\varphi} \le \int_M |\bar\partial\xi|^2 e^{-\varphi} + \int_M |\bar\partial^*_\varphi\xi|^2 e^{-\varphi}$$

(where, for a moment, we let $|\cdot|$ denote the non-weighted norm.) This is precisely the inequality that we used in the proof of the existence theorem in Chapter 1.

Now, let E be a general hermitean line bundle over M. Given a local frame e, the curvature operator is represented by a $(1,1)$-form so that

$$\Theta(\xi \otimes e) =: \nabla^2(\xi \otimes e) = (\Theta \wedge \xi) \otimes e$$

(for simplicity, we use the same notation for the operator and the form). Notice that the form Θ is actually independent of choice of frame. Choose a (local) basis for the space of $(1,0)$-forms, $w_1, \ldots, w_n$. Then

$$\Theta = \sum c_{j\bar{k}} w_j \wedge \bar{w}_k.$$

We say that Θ is (semi)-positive if $(c_{j\bar{k}})$ is (semi)-positively definite, and define negativity in the same way. This is of course independent of choice of basis. At a given point, we can choose a basis that is orthonormal for the Kähler metric on M and moreover diagonalizes Θ.

$$\Theta = \sum \lambda_j w_j \wedge \bar{w}_j.$$

Let us first assume that Θ is positive (everywhere). Observe that Θ is always a closed form. (This can be seen in many ways: with respect to a local frame the metric can be written

$$g = (g_{11}),\ g_{11} = e^{-\varphi}$$

and

$$\Theta = \partial\bar{\partial}\varphi.$$

Hence $d\Theta = 0$. Or, in general $\Theta = \bar{\partial}\theta$ where $\partial\theta + \theta\wedge\theta = 0$. Hence $\bar{\partial}\Theta = 0$ and $\partial\Theta = +\bar{\partial}(\theta\wedge\theta) = 0$.)

Therefore, we can give M a new Kähler metric whose Kähler form is

$$\Omega' = i\Theta.$$

Then

$$i[\Theta, \Lambda]\xi = [L, \Lambda]\xi = (k-n)\xi$$

if ξ is a k-form by Proposition 3.6.2. With this new Kähler metric Theorem 3.8.2 implies

$$(k-n)\int |\xi|^2 \leq \int |\bar{\partial}\xi|^2 + \int |(\nabla'')^*\xi|^2. \tag{3.18}$$

If, on the other hand, Θ is negative, we choose for our new Kähler metric

$$\Omega' = -i\Theta$$

and get

$$(n-k)\int |\xi|^2 \leq \int |\bar{\partial}\xi|^2 + \int |(\nabla'')^*\xi|^2. \tag{3.19}$$

Lemma 3.8.3 *Suppose that E is a hermitian vector bundle over a complex, compact manifold M. Suppose that for all E-valued (p,q)-forms ξ it holds*

$$c\int |\xi|^2 \leq \int |(\nabla'')^*\xi|^2 + \int |\bar{\partial}\xi|^2 \tag{3.20}$$

for some fixed $c > 0$. Then $H^{p,q}(M,E) = 0$.

Proof. This is basically a repetition of Proposition 1.3.2, but, for no particular reason we shall formulate the duality argument differently this time. Let f be a E-valued (p,q)-form.

Then

$$|\int < f, \xi > |^2 \leq \frac{1}{c}\int |f|^2 \{\int |(\nabla'')^*\xi|^2 + \int |\bar{\partial}\xi|^2\}$$

for all (p,q)-forms ξ. By elementary Hilbert space theory (cf. proof of Propostion 1.2.1.) it follows that there are a $(p,q-1)$-form u and a $(p,q+1)$-form v such that

$$\int <f,\xi>=\int <u,(\nabla'')^*\xi>+\int <v,\bar\partial\xi>$$

for all ξ and

$$\int |u|^2+|v|^2\leq \frac{1}{c}\int |f|^2.$$

This means that

$$f=\bar\partial u+(\nabla'')^*v$$

(in the weak sense). If now $\bar\partial f=0$, we have that

$$f-\bar\partial u=(\nabla'')^*v$$

is both $\bar\partial$-closed and orthogonal to closed forms. Hence

$$\bar\partial u=f$$

and we are done, except for the question of regularity. We shall discuss this question only briefly. Notice that we have proved that any f, E-valued form in L^2, can be written

$$f=\bar\partial u+(\nabla'')^*v.$$

Alternately, we could notice that (3.20) implies

$$c\int |\xi|^2\leq\int <\Box''\xi,\xi>$$

so

$$c^2\int |\xi|^2\leq\int |\Box''\xi|^2.$$

This implies that we can actually solve the equation

$$f=\Box''g=(\bar\partial(\nabla'')^*+(\nabla'')^*\bar\partial)g$$

by the same Hilbert space argument as before. In other words, if $\bar\partial f=0$, we can choose our solution u to $\bar\partial u=f$ of the form

$$u=(\nabla'')^*g\quad\text{where}\quad(\nabla'')^*\bar\partial g=0.$$

If moreover f is smooth, it follows relatively easily that g is smooth since $\Box''$ is an elliptic operator. Therefore, this special choice of u will also be smooth, and we are done. ∎

It is now easy to prove the Kodaira-Nakano vanishing theorem. First, a

Definition. A complex line bundle E over M is called positive if E can be given a hermitian metric with positive curvature form. E is negative if E has a metric with negative curvature.

Theorem 3.8.4 *Let E be a line bundle over a compact complex manifold of dimension n. Then $H^{p,q}(M,E)=0$ if*

a) *E is positive and $p+q>n$,*
or b) *E is negative and $p+q<n$.*

Proof. This follows immediately from Lemma 3.8.2 and the comments before it. ■

We shall end this section by giving a generalization of Theorem 3.8.3, known as Girbau's Vanishing Theorem.

Definition. A curvature form Θ on a line bundle is called k-positive if it is semipositive and at each point at most k eigenvalues are equal to zero. A line bundle is k-positive if it has a metric with k-positive curvature.

Lemma 3.8.5 *Let $\omega_1, \ldots, \omega_n$ be an orthonormal basis for the space of $(1,0)$-forms, and let*

$$\Theta = \sum \lambda_j \omega_j \wedge \bar{\omega}_j.$$

Let $\omega_I = \omega_{i_1} \wedge \ldots \omega_{i_p}$ if $I = (i_1, \ldots, i_p)$. Then

$$[i\Theta, \Lambda]\omega_I \wedge \bar{\omega}_J = \lambda_{IJ}\omega_I \wedge \bar{\omega}_J$$

where

$$\lambda_{IJ} = \sum_{i \in I} \lambda_i + \sum_{i \in J} \lambda_i - \sum_1^n \lambda_j.$$

Proof. Let

$$V_K = \omega_{i_1} \wedge \bar{\omega}_{i_1}, \wedge \ldots \omega_{i_r} \wedge \bar{\omega}_{i_r} \quad \text{if} \quad K = (i_1, \ldots, i_r),$$

and write

$$\omega_I \wedge \bar{\omega}_J = V_K \wedge \omega_L \wedge \bar{\omega}_M$$

where K, L, M are pairwise disjoint. As in the proof of Proposition 3.6.2, one verifies that

$$|i\Theta, \Lambda]V_K \wedge \omega_L \wedge \bar{\omega}_M = (\sum_{j \in K} \lambda_j - \sum_{j \notin K \cup L \cup M} \lambda_j) V_K \wedge \omega_L \wedge \bar{\omega}_M.$$

This means that $\omega_I \wedge \bar{\omega}_J$ is an eigenvector for the operator $[i\Theta, \Lambda]$ with eigenvalue equal to the sum of all λ_j's in $I \cap J$ minus the sum of all λ_j's outside $I \cup J$. This can also be written

$$\sum_{j \in I} \lambda_j + \sum_{j \in J} \lambda_j - \sum_1^n \lambda_j$$

and the proof is complete. ■

Theorem 3.8.6 *Let E be a k-positive line bundle over a compact Kähler manifold M. Then*

$$H^{p,q}(M, E) = 0 \quad \textit{for} \quad p + q > n + k.$$

Proof. Choose a metric with k-positive curvature form Θ. We would like to give M the Kähler metric with fundamental form $i\Theta$ as before, but that is no longer possible since Θ is not positively definite. Let Ω be the fundamental form of some arbitrary Kähler metric and consider the metrics defined by

$$\Omega' = i\Theta + \epsilon\Omega.$$

For each $\epsilon > 0$ this defines a new Kähler metric and if $\lambda_1 \leq \lambda_2 \leq \ldots \lambda_n$ are the eigenvalues of Θ with respect to Ω, then

$$\lambda'_j = \frac{\lambda_j}{\epsilon + \lambda_j}$$

are the eigenvalues of Θ with respect to Ω'. (If

$$\Theta = \sum \lambda_j \omega_j \wedge \bar{\omega}_j,$$

where ω_j are orthonormal w.r.t. Ω, then

$$\omega_j' = \omega_j(\lambda_j + \epsilon)^{1/2}$$

are orthonormal w.r.t. Ω', and

$$\Theta = \sum \lambda_j' \omega_j' \wedge \bar{\omega}_j'.$$

As ϵ tends to $0, \lambda_j'$ tends to 1 or 0, depending on whether $\lambda_j > 0$ or $\lambda_j = 0$. By Lemma 3.8.4

$$i[\Theta, \Omega]\xi = \sum \lambda_{I\bar{J}} \xi_{IJ} \omega_I \wedge \bar{\omega}_J$$

if $\xi = \sum \xi_{IJ} \omega_I \wedge \bar{\omega}_J$, where

$$\lambda_{I\bar{J}} = \sum_{j \in I} \lambda_j + \sum_{j \in J} \lambda_j - \sum_1^n \lambda_j.$$

Replacing λ_j by λ_j' and λ_{IJ} by $\lambda_{I,J}'$, we see that

$$\lim_{\epsilon \to 0} \lambda_{I\bar{J}}' \geq (p-l)^+ + (q-l)^+ - (n-l)$$

where l is the number of eigenvalues that vanish. Since

$$p + q > n + k \geq n + l$$

this limit is always ≥ 1. Hence, for ϵ sufficiently small

$$< i[\Theta, \Lambda]\xi, \xi > \geq \frac{1}{2}|\xi|^2$$

and the proof is completed just as in the Kodaira-Nakano case. ■

3.9 Vanishing theorems on complete manifolds

In the previous section we have shown the principal analogs of Theorem 1.6.2 for compact manifolds. It should be noted that one aspect of the proofs actually is much easier in the compact case. Namely, as soon as we have the inequality of Lemma 3.8.3, we get existence theorems for the $\bar{\partial}$-operator. This is no longer the case if our manifold is non-compact. The argument in the proof of Lemma 3.8.3 still gives that we can solve

$$f = \bar{\partial} u + (\nabla'')^* v$$

but this no longer implies $f = \bar{\partial} u$ since $(\nabla'')^* v$ need not be orthogonal to $\bar{\partial}$-closed forms if M is open.

We shall now see that if our manifold is *complete*, there is a way to circumvent this difficulty, which in particular will give us a new approach to the theorems in Chapter 1.

Definition. A Riemannian manifold M is called complete if there is a smooth function

$$\varphi : M \to \mathbb{R}$$

such that

i) $\varphi^{-1}[-\infty, c]$ is compact for each c

and

ii) $|d\varphi|$ is uniformly bounded with respect to the riemannian metric.

A complete Kähler manifold is then a Kähler manifold which is complete as a Riemannian manifold. Recall that a complex manifold is a Stein manifold if there is a strictly plurisubharmonic function ψ on M such that $\psi^{-1}(-\infty, c]$ is relatively compact for all c.

Lemma 3.9.1 *Any Stein manifold has a complete Kähler metric.*

Proof. Let ψ be a strictly plurisubharmonic function such that $\psi^{-1}(-\infty, c]$ is relatively compact for all c. We shall find a Kähler metric such that $d\psi$ is bounded. If Ω is the Kähler form, this means precisely that

$$c\Omega - i\partial\psi \wedge \bar{\partial}\psi$$

is non-negative for some constant c. Let k be some convex increasing function on $\mathbb{R}$. Then

$$\partial\bar{\partial}k \circ \psi = k'\partial\bar{\partial}\psi + k''\partial\psi \wedge \bar{\partial}\psi,$$

so it is enough to take $\Omega = i\partial\bar{\partial}(k \circ \psi)$ where k is strictly increasing and $k''(t) \geq 1$ for $t \geq 0$. ■

Lemma 3.9.2 *Assume M is complete. Then there is a sequence $\{\chi_\nu\}$ of smooth functions with compact support such that χ_ν increases to 1 everywhere and $d\chi_\nu$ is uniformly bounded.*

Proof. Let g_ν be a sequence of smooth functions on $\mathbb{R}$, such that $g_\nu(x) = 1$ for $x \leq \nu$, $g_\nu(x) = 0 \quad x > \nu + 1$ and $|g'_\nu| \leq 2$. Take $\chi_\nu = g_\nu \circ \varphi$. ■

Lemma 3.9.3 *Let M be a complete Kähler manifold and let E be a hermitian vector bundle over M. Suppose that f is a $\bar{\partial}$-closed E-valued (p,q) form in L^2, and that for any E-valued (p,q)-form with compact support ξ it holds*

$$|\int < f, \xi > |^2 \leq C \int |\bar{\partial}\xi|^2 + |(\nabla)''^*\xi|^2. \tag{3.21}$$

Then, there is a solution u to $\bar{\partial}u = f$ with

$$\int |u|^2 \leq 2C.$$

Proof. We shall again repeat the arguments from Chapter 1. Take an E-valued test-form ξ and decompose $\xi = \xi^1 + \xi^2$, where ξ^1 is $\bar{\partial}$-closed and ξ^2 is orthogonal to the $\bar{\partial}$-closed forms. In particular ξ^2 is orthogonal to all forms of type $\bar{\partial}\eta$, so $(\nabla'')^*\xi^2 = 0$, whence

$$(\nabla'')^*\xi^1 = (\nabla'')^*\xi.$$

Moreover $\Box''\xi^1 = \bar{\partial}(\nabla'')^*\xi^1 = \bar{\partial}(\nabla'')^*\xi$ is smooth, so ξ^1 is smooth. Let χ_ν be the sequence from Lemma 3.9.2. Then $\chi_\nu\xi$ is a test-form, so

$$\begin{aligned} |\int < f, \xi > |^2 &= |\int < f, \xi^1 > |^2 = \lim |\int < f, \chi_\nu\xi^1 > |^2 \leq \\ &\leq C \int \chi_\nu^2 |\bar{\partial}\xi^1|^2 + \chi_\nu^2 |(\nabla'')^*\xi^1|^2 + B \int |\xi^1|^2 |d\chi_\nu|^2. \end{aligned}$$

Since $d\chi_\nu$ tends to 0 pointwise and boundedly it follows that

$$|\int < f, \xi > |^2 \leq 2C \int |(\nabla'')^* \xi^1|^2 = 2C \int |(\nabla'')^* \xi|^2.$$

By the usual Hilbert space argument there is a u such that

$$\int < f, \xi >= \int < u, (\nabla'')^* \xi > \quad \text{for all} \quad \xi$$

and $\int |u|^2 \leq 2C$. Then $\bar{\partial} u = f$ and we are done. ■

Recall now the basic identity, Theorem 3.8.1. Let us assume that E is actually a line bundle and consider the curvature term

$$< [i\Theta, \Lambda]\xi, \xi >$$

for (n, q)-forms ξ. Assume Θ is positive. Then

$$| < f, \xi > |^2 \leq \|f\|^2_{\Omega,\Theta} < i[\Theta, \Lambda]\xi, \xi > \tag{3.22}$$

where $\|f\|_{\Omega,\Theta}$ is simply the supremum of $| < f, \xi > |$ over all ξ such that the curvature form is bounded by 1. If, for example, the curvature form dominates $c|\xi|^2_\Omega$, then $\|f\|_{\Omega,\Theta} \leq \frac{1}{c}|f|^2_\Omega$. For the next theorem we need to study how $\|f\|_{\Omega,\Theta}$ depends on the metric Ω.

Choose an orthonormal basis $\omega_1, \ldots, \omega_n$ for the $(1, 0)$-forms. Write

$$\begin{aligned} \Theta &= \sum \Theta_{jk} \omega_j \wedge \bar{\omega}_k, \ \xi = \sum \xi_J \omega \wedge \bar{\omega}_J, \\ f &= \sum f_J \omega \wedge \bar{\omega}_J \end{aligned}$$

where $\omega = \omega_1 \wedge \ldots \omega_n$ and $\omega_J = \wedge_{i \in J} \omega_i$. Then $< f, \xi >= \sum f_J \xi_J$, and a computation like in the beginning of Section 6 shows that

$$< i[\Theta, \Lambda]\xi, \xi >= \sum_{|I|=q-1} \sum \Theta_{jk} \xi_{I \cup \{j\}} \bar{\xi}_{I \cup \{k\}}. \tag{3.23}$$

Now let Ω' be another Kähler form with $\Omega' \geq \Omega$. We can assume $\{\omega_j\}$ is chosen so that Ω' is also diagonal

$$\Omega' = i \sum \gamma_j \omega_j \wedge \bar{\omega}_j$$

where we must have $\gamma_j \geq 1$. Let $\omega'_j = \sqrt{\gamma_j} \omega_j$, so that ω'_j is orthonormal for Ω. Let $\gamma = \gamma_1 \ldots \gamma_n$ and $\gamma_J = \gamma_{j_1} \ldots \gamma_{j_q}$ if $J = (j_1, \ldots, j_q)$. Then

$$\xi = \sum \xi'_J \omega' \wedge \bar{\omega}'_j, \text{ and } \Theta = \sum \Theta'_{jk} \omega'_j \wedge \bar{\omega}'_k,$$

where $\xi'_J = \xi_J / \sqrt{\gamma \gamma_J}$ and $\Theta'_{jk} = \Theta_{jk} / \sqrt{\gamma_j \gamma_k}$. Therefore we get if we consider the Ω' metric

$$\begin{aligned} < i[\Theta, \Lambda']\xi, \xi >' &= \sum_{|I|=q-1} \sum \frac{\Theta_{ik}}{\sqrt{\gamma_i \gamma_k}} \xi_{I \cup \{i\}} \bar{\xi}_{I \cup \{k\}} = \\ &= \sum \frac{1}{\gamma_I} \sum \Theta_{ik} \frac{\xi'_{I \cup \{i\}} \xi'_{I \cup \{k\}}}{\sqrt{\gamma_i \gamma_k}} = \gamma \sum \Theta_{ik} \xi''_{I \cup \{i\}} \bar{\xi}''_{I \cup \{k\}} \end{aligned}$$

where we define $\xi''_J = \xi'_J / \sqrt{\gamma \cdot \gamma_J}$. Since $\|f\|_{\Omega,\Theta}$ is the supremum of $\sum f'_J \xi'_J$ over all ξ's such that the curvature form with respect to Ω' is bounded by 1, we get

$$\|f\|^2_{\Omega' \Theta} \leq \frac{1}{\gamma} \|f\|^2_{\Omega,\Theta}.$$

If we let dV_Ω denote the volume element $\pm i^n\omega\wedge\bar\omega$ we have $dV_{\Omega'} = \gamma dV_\Omega$ so, finally we have

$$\|f\|^2_{\Omega'\Theta} dV_{\Omega'} \leq \|f\|_{\Omega,\Theta} dV_\Omega. \tag{3.24}$$

Note also that for any (n,q)-form

$$|\xi|^2_{\Omega'} dV_{\Omega'} = \sum |\xi_J|^2/\gamma\gamma_J dV_{\Omega'} \leq \sum |\xi_J|^2 dV_\Omega = |\xi|^2_\Omega dV_\Omega. \tag{3.25}$$

We are now all set for the principal result of this section.

Theorem 3.9.4 *Let M be a complex manifold which has a complete Kähler metric, and let Ω be some Kähler metric on M (complete or not). Let E be a hermitian line bundle over M with semi-positive curvature form Θ. Let f be a $\bar\partial$-closed (n,q)-form on M in L^2, with values in E. Then we can solve $\bar\partial u = f$ with*

$$\int |u|^2_\Omega \leq 2\int \|f\|^2_{\Omega,\Theta}.$$

Proof. If Ω itself is complete, this follows directly from the previous lemma since

$$|<f,\xi>|^2 \leq \|f\|^2_{\Omega,\Theta} <[i\Theta,\Lambda]\xi,\xi>$$

and

$$\int <[i\Theta,\Lambda]\xi,\xi> \leq \int |\bar\partial\xi|^2 + |(\nabla'')^*\xi|^2$$

by Theorem 3.8.2.

In general, let ω be a complete metric and let

$$\Omega_k = \Omega + \frac{1}{k}\omega.$$

Then Ω_k is complete for any k since $|d\varphi|^2_{\Omega k} \leq k|d\varphi|_\omega$. Hence we get, for each k, a solution u_k to $\bar\partial u_k = f$ with

$$\int |u_k|^2_{\Omega_k} dV_{\Omega_k} \leq 2\int \|f\|^2_{\Omega_{k},\Theta} dV_{\Omega_k} \leq 2\int \|f\|^2_{\Omega,\Theta} dV_\Omega,$$

by (3.24). If $l \geq k$, we get

$$\int |u_l|^2_{\Omega_k} dV_{\Omega_k} \leq 2\int \|f\|^2_{\Omega,\Theta} dV_\Omega$$

by (3.25) so we can choose a subsequence of u_l which converses weakly to u in L^2 with respect to any Ω_k. Then $\bar\partial u = f$ and

$$\int |u|^2_{\Omega_k} dV_{\Omega_k} \leq 2\int \|f\|^2_{\Omega,\Theta} dV_\Omega$$

and letting k tend to ∞, we get the statement of the theorem. ∎

Corollary 3.9.5 *Let M be a Stein manifold with some Kähler metric and let φ be plurisubharmonic on M. Suppose*

$$<[i\partial\bar\partial\varphi,\Lambda]\xi,\xi> \geq |\xi|^2$$

for any (n,q)-form ξ. Then for any (n,q)-form f with $\bar\partial f = 0$ there is a solution u to $\bar\partial u = f$ with

$$\int |u|^2 e^{-\varphi} \leq 2\int |f|^2 e^{-\varphi}.$$

Note that if M is a pseudoconvex domain in $\mathbb{C}^n$ with its standard metric, we get back the theorems of Section 1.3.

We end this section with one more application of Lemma 3.9.3. As mentioned in section 1.8, the first theorem of this type appears in [4].

Theorem 3.9.6 *Let M be a complex manifold with a complete Kähler metric Ω. Suppose $\Omega = i\partial\bar{\partial}\varphi$ where $|d\varphi|_\Omega$ is uniformly bounded. Then, for any (p,q)-form f in L^2 such that $\bar{\partial} f = 0$ there is a solution u to $\bar{\partial} u = f$ with*

$$\int |u|^2 \leq c \int |f|^2,$$

provided that $p + q \neq n$.

Proof. We shall consider two different metrics on the scalar-valued forms (i.e., forms with values in the trivial line bundle). The first one is the usual scalar product given by M's metric $<,>$, and the second one is $<,> e^{-\psi}$ where ψ is a certain weight function. Let $\bar{\partial}^*$ denote the adjoint of $\bar{\partial}$ with respect to the first metric and let $(\nabla'')^*$ denote the adjoint with respect to the second one. Clearly

$$(\nabla'')^* = e^{\psi} \bar{\partial}^* e^{-\psi}.$$

Theorem 3.8.2 gives if ξ is a test-form.

$$\int < [i\partial\bar{\partial}\psi, \Lambda]\xi, \xi > e^{-\psi} \leq \int |\bar{\partial}\xi|^2 e^{-\psi} + |(\nabla'')^*\xi|^2 e^{-\psi}.$$

Substitute $\xi = e^{\psi/2}\eta$. Then we get

$$\int [i\partial\bar{\partial}\psi, \Lambda]\eta, \eta > \leq \int |\bar{\partial}_\psi \eta|^2 + |\vartheta_\psi \eta|^2$$

where

$$\begin{aligned} \bar{\partial}_\psi &= e^{-\psi/2}\bar{\partial} e^{\psi/2} \quad \text{and} \\ \vartheta_\psi &= e^{\psi/2}(\nabla'')^* e^{-\psi/2}. \end{aligned}$$

It is easily seen that

$$|\bar{\partial}_\psi \eta|^2 \leq 2(|\partial\eta|^2 + |\partial\psi|^2|\eta|^2)$$

and

$$|\vartheta_\psi \eta|^2 \leq 2(|\bar{\partial}^*\eta|^2 + |\partial\psi|^2|\eta|^2).$$

Now we can choose $\psi = t\varphi$. Then

$$< [i\partial\bar{\partial}\psi, \Lambda]\eta, \eta >= t(p + q - n)|\eta|^2.$$

By hypothesis $|\partial\varphi|^2 \leq A$ for some A. Hence

$$t(p + q - n) \int |\eta|^2 \leq \int c|\bar{\partial}\eta|^2 + |\bar{\partial}^*\eta|^2 + 2At^2 \int |\eta|^2.$$

If $p + q - n > 0$, we choose t small but positive; if $p + q - n < 0$, we take t small and negative. Then we get

$$\int |\eta|^2 \leq c' \int |\bar{\partial}\eta|^2 + |\bar{\partial}^*\eta|^2.$$

The theorem now follows from Lemma 3.9.3. ■

3.10 The Hodge Theorem

In this section it becomes inevitable to go a bit further into the questions of regularity associated with the operators $\Box', \Box''$ and Δ. We shall start with a rather brief recapitulation of the necessary facts from PDE theory, after which Hodge's Theorem will be an easy consequence.

Let us consider a general second order differential operator L, acting on sections of a complex vector bundle F, over a compact Riemannian manifold. If $e_1, \ldots, e_r$ is a local frame, any smooth section $s \in C^\infty(U, F)$ can be written $s = \sum s_\nu e_\nu$ and

$$Ls = \sum (L_{\mu\nu} s_\nu) e_\mu \tag{3.26}$$

where $L_{\nu\mu}$ are scalar differential operators which we assume have smooth coefficients. In our case later, F will be the bundle of E-valued (p, q)-forms where E is a holomorphic bundle.

We now associate with L, its symbol $\sigma(L)$, which will be a quadratic form on each T_p^* with values in the linear maps from F_p to F_p. This may seem rather formidable, but let us see what it means concretely. Take a smooth real-valued function ϕ and consider

$$p(t) = e^{-it\phi} L(e^{it\phi} s),$$

where t is a positive parameter. This is a second degree polynomial in t and by definition the coefficient of t^2 is

$$\sigma(L)(d\phi, d\phi)s.$$

If, with respect to some local coordinates $x_1, \ldots, x_n$

$$L_{\mu\nu} = \sum A^{jk}_{\mu\nu} \frac{\partial^2}{\partial x_j \partial x_k} + \ldots,$$

where the dots indicate lower order terms, then

$$\sigma(L)(d\phi, d\phi)s = \sum A^{jk}_{\mu\nu} \phi_j \phi_k s_\nu e_\mu,$$

so $\sigma(L)$ only depends on $d\phi$ and is indeed a quadratic form with values in the space of linear maps from F to F.

Definition. L is elliptic if there is a positive constant c such that

$$|\sigma(L)(\xi, \xi)s| \geq c|\xi|^2|s|. \tag{3.27}$$

Clearly, the constant c depends on our choice of Riemannian metric and scalar product on F, but the property of being elliptic does not if our manifold is compact.

Let us fix a hermitean scalar product on F. Then we can define the formal adjoint of L by

$$\int < Ls, v > = \int < s, L^* v >$$

if s and v are smooth sections. Note that

$$\int < e^{-it\phi} L e^{it\phi} s, v > = \int < s, e^{-it\phi} L^* e^{it\phi} v >$$

so, identifying coefficients of t^2, we see that

$$\sigma(L)(d\phi, d\phi)^* = \sigma(L^*)(d\phi, d\phi).$$

In particular, we see that L is elliptic if and only if L^* is elliptic.

Next, we introduce the notation $L^2(M, F)$ for the space of sections in L^2, which is a Hilbert space with the natural scalar product. More generally, we need to introduce Sobolev norms on sections to F. If s is a section with support in a coordinate patch over which F is trivial, we can associate to s, a vector-valued function with compact support in an open set in $\mathbb{R}^n$. We then define the m:th Sobolev norm, $\|s\|_m$, as the sum of the Sobolev norms of the components of this vector-valued function. Clearly, the norm depends on our choice of coordinates and trivialization, but let us just fix one choice. A general section can, via a partition of unity, be decomposed into a sum of terms of the terms. Tedious verifications show that different choices of partition of unity will give equivalent norms. The space of L^2-sections with finite Sobolev m-norm is denoted W^m. We can consider L as a densely defined closed operator from $L^2(M, F)$ to itself in two different ways. Either we let the domain of L consist of all $s \in L^2(M, F)$ such that $Ls \in L^2$, where we compute Ls in the sense of distributions

$$\int < Ls, v > = \int < s, L^* v > \qquad v \in C^\infty(M, F).$$

The other choice is to extend the definition of L from C^∞ by closing the graph. Then $s \in \text{Dom}\ (L)$ if there is a sequence $v_\nu \in C^\infty(M, F)$ such that $v_\nu \to s$ and $Lv_\nu \to w$ in L^2. Then of course we put $Ls = w$. The second definition gives a domain which a priori is smaller, but as a matter of fact, the two definitions are equivalent, and the common domain is just W^2. This follows, among other things, from our next theorem which is called Gårdings inequality.

Theorem 3.10.1 *Let L be a second order elliptic operator acting on sections to a complex vector bundle F over a compact Riemannian manifold M. Let $s \in L^2(M, F)$ and suppose Ls (taken in the sense of distributions) lies in $L^2 = W^0$. Then $s \in W^2$ and*

$$\delta\|s\|_2 \le \|Ls\|_0 + \|s\|_0 \tag{3.28}$$

for some $\delta > 0$ depending only on L.

For the proof we refer to Warner,[9], Chapter 4.

We can now state the PDE-theorem behind Hodges Theorem.

Theorem 3.10.2 *Suppose L is elliptic. Then*

i) $N(L) =: \{s \in L^2(M, F);\ Ls = 0\}$ *is of finite dimension.*

ii) $R(L) =: \{Ls \in L^2(M, F);\ s \in L^2(M, F)\}$ *is closed and has finite co-dimension.*

iii) $L^2(M, F) = N(L) \oplus R(L^*)$.

Moreover, $N(L) \subseteq C^\infty(M, F)$ and we also have the decomposition.

iv) $C^\infty(M, F) = N(L) \oplus L^* C^\infty(M, F)$.

Proof. On $N(L)$ the inequality (3.28) takes the form

$$\|s\|_2 \le C\|s\|_0.$$

By the Rellich lemma this means that the unit ball in $N(L)$ is compact. Therefore $\dim N(L) < \infty$ since otherwise there would be an infinite orthonormal system which could not contain any convergent subsequence.

We now claim that if $s \in N(L)^\perp$ then

$$\|s\|_0 \leq c\|Ls\|_0, \tag{3.29}$$

for some constant c. Otherwise, we could find a sequence s_n with $\|s_n\|_0 = 1$ and $\|Ls_n\|_0$ tending to 0. Then (3.28) implies, again by the Rellich lemma, that there is a subsequence converging to s. Since L is closed, $Ls = 0$. Hence $s \in N(L) \cap N(L)^\perp$, so $s = 0$, contradicting $\|s\|_0 = 1$. But (3.29) implies immediately that $R(L)$ is closed, since if $v_n \in R(L)$, we can write $v_n = Ls_n$ with $s_n \in N(L)^\perp$, and then $\{s_n\}$ must be convergent if $\{v_n\}$ is convergent, so

$$\lim v_n = L(\lim s_n) \in R(L).$$

On the other hand, $R(L)^\perp \subseteq N(L^*)$ so codim $R(L) \leq \dim N(L^*) < \infty$, since L^* is also an elliptic operator. Moreover $R(L) \subseteq N(L^*)^\perp$ so, actually $R(L) = N(L^*)^\perp$ since $R(L)$ is closed. In the same way $R(L^*) = N(L)^\perp$ so, we have proved i), ii) and iii).

To prove iv), it suffices to show that if $Ls \in C^\infty(M, F)$, then $s \in C^\infty(M, F)$. So, suppose $Ls \in C^\infty$, and let X be any first order differential operator. First, note that (3.28) implies a bounde on $\|s\|_2$. Applying X to s, we get

$$L(Xs) = XL(s) + [L, X]s \tag{3.30}$$

so $L(X(s)) \in L^2$ since $[L, X]$ is of second order. Therefore we get a bound on $\|Xs\|_2$, and since this holds for any X, we can estimate $\|s\|_3$. But then we can let X be a second order operator in (3.30), and continuing in this way, we see that s lies in all Sobolev spaces. By the Sobolev lemma s is smooth, and we are done. ■

Proposition 3.10.3 *Let E be a holomorphic bundle over a Kähler manifold. Then*

$$\Box'' : C^\infty_{p,q}(M, E) \to C^\infty_{p,q}(M, E)$$

is elliptic and $(\Box'')^* = \Box''$. If E is the trivial line bundle so that $\Box'$ and Δ are defined, then $\Box'$ and Δ are also elliptic and formally self-adjoint.

Proof. We shall prove the statement concerning $\Box''$, the other being similar. Remember

$$\Box'' = \bar\partial(\nabla'')^* + (\nabla'')^*\bar\partial$$

and

$$(\nabla'')^* = i[\nabla', \Lambda],\ \nabla' = \partial + \theta,$$

(see Proposition 3.7.2). To compute $\sigma(\Box'')$, we consider

$$\int < e^{-it\phi}\Box'' e^{it\phi}s, s >= \int |\bar\partial e^{it\phi}s|^2 + |(\nabla'')^* e^{it\phi}s|^2.$$

Identifying coefficients of t^2, we get

$$\int < \sigma(\Box'')(d\phi, d\phi)s, s >= \int |\bar\partial\phi \wedge s|^2 + |\partial\phi \lrcorner s|^2$$

(cf. Proposition 3.5.3). Hence

$$\sigma(\Box'')(d\phi, d\phi)s = \partial\phi \lrcorner(\bar\partial\phi \wedge s) + \bar\partial\phi \wedge (\partial\phi \lrcorner s) = |\partial\phi|^2 s$$

by Proposition 3.5.1 thus $\sigma(\Box'')$ is just a multiple times identity operator, so $\Box''$ is clearly elliptic.

Lemma 3.10.4 *On a compact manifold $\Box''s = 0$ if and only if $\bar\partial s = 0$ and $(\nabla'')^* s = 0$. In the same way, if ξ is a differential form,*

$$\Box''\xi = 0 \quad \text{if and only if} \quad \bar\partial\xi = 0,\, (\nabla'')^*\xi = 0$$

and

$$\Delta\xi = 0 \quad \text{if and only if} \quad d\xi = 0,\, d^*\xi = 0.$$

Proof. $\Box''s = 0$ *implies*

$$0 = \int <\Box''s, s> = \int |\bar\partial\xi|^2 + |(\nabla'')^*\xi|^2.$$

The other statements are proved in the same way. ∎

Let E be a holomorphic vector bundle over a compact Kähler manifold. An E-valued (p,q)-form ξ, satisfying $\Box''\xi = 0$ is called a *harmonic form.* As we have seen, any harmonic form satisfies $\bar\partial\xi = 0$ and so defines an element in $H^{p,q}(M,E)$. We shall now see that all cohomology classes are represented by some harmonic form, and moreover the harmonic representative is unique.

By Theorem 3.10.2 applied to $L = \Box''$, we have

$$C^\infty_{p,q}(M,E) = \mathcal{H}_{p,q}(M,E) \oplus \Box'' C^\infty_{p,q}(M,E) \tag{3.31}$$

wher $\mathcal{H}_{p,q} = N(\Box'')$ is the space of harmonic forms. We now claim that

$$\Box'' C^\infty_{p,q} = \bar\partial C^\infty_{p,q-1} \oplus (\nabla'')^* C^\infty_{p,q+1}. \tag{3.32}$$

First, note that $\bar\partial C^\infty \perp (\nabla'')^* C^\infty$ since $\bar\partial^2 = 0$, and clearly $\Box'' C^\infty \subseteq \oplus \bar\partial C^\infty \oplus (\nabla'')^* C^\infty$. But, Lemma 3.10.4 shows that

$$\bar\partial C^\infty_{p,q-1} \oplus (\nabla'')^* C^\infty_{p,q+1} \perp \mathcal{H}_{p,q},$$

so (3.32) follows from (3.31).

Clearly, a smooth form ξ is $\bar\partial$-closed if and only if $\xi \perp (\nabla'')^* C^\infty_{p,q+1}$, so if $Z_{p,q}$ denotes the space of $\bar\partial$-closed forms, we have

$$Z_{p,q} = \mathcal{H}_{p,q} \oplus \bar\partial C^\infty_{p,q-1}.$$

this means that any cohomology class contains exactly one harmonic representative, so we have proved the first part of Hodge's Theorem.

Theorem 3.10.5 *Let E be a holomorphic vector bundle over a compact Kähler manifold. Then*

$$H^{p,q}(M,E) \simeq \mathcal{H}_{p,q}(M,E).$$

Let us now consider scalar valued forms. Then the same arguments as above apply to $\Delta = dd^* + d^*d$, so in particular, we have the orthogonal decomposition of the space of k-forms

$$C^\infty_k(M) = \mathcal{H}_k(M) \oplus \Delta C^\infty_k = \mathcal{H}_k \oplus dC^\infty_{k=1} \oplus d^* C^\infty_{k+1}, \tag{3.33}$$

when $\mathcal{H}_k$ stands for the space of Δ-harmonic k-forms. This has of course nothing to do with the complex structure of M and so is valid for any Riemannian manifold. From this it follows first the analog of Theorem 3.10.5 for de Rham cohomology.

Theorem 3.10.6 *Let M be a compact Riemannian manifold. Then*

$$H^k(M,\mathbb{C}) \simeq \mathcal{H}_k(M). \tag{3.34}$$

But we also know that $\frac{1}{2}\Delta = \Box''$ so

$$\mathcal{H}_k(M) = \oplus_{p+q=k}\mathcal{H}_{p,q}(M)$$

(a k-form is Δ harmonic if and only all the terms in tits decomposition after bidegre are $\Box''$-harmonic). Moreover $\Box'' = \bar{\Box}'' = \Box'$ is a real operator so $\overline{\mathcal{H}_{p,q}} = \mathcal{H}_{p,q}$. We collect this in the second part of Hodge's Theorem.

Theorem 3.10.7 *Let M be a compact Kähler manifold. Then*

$$H^k(M, \mathbb{C}) \simeq \oplus_{p+q=k}H^{p,q}(M)$$

and

$$H^{p,q} \simeq \overline{H^{p,q}}.$$

Thus the Dolbeault cohomology groups $H^{p,q}$, that are defined in terms of the analytic strucure, determine the toplogically defined de Rham groups. The second statement says in particular that $H^{0,1} \simeq \overline{H^{1,0}}$, i.e., any class in $H^{0,1}$ has a unique representative of the form $\bar{h}$ where h is a holomorphic $(1,0)$-form.

Hodge's Theorem has numerous applications in geometry. We close by giving a few of the most important.

Theorem 3.10.8 *(***Poincaré duality***) Let M be an N-dimensional Riemannian manifold. Then $\overline{H^k(M, \mathbb{C})} \simeq H^{N-k}(M, \mathbb{C})$*

Proof: By Theorem 3.10.6 this follows since the operator

$$\xi \to \overline{*\xi}$$

is an isomorphism between $\mathcal{H}_k(M)$ and $\mathcal{H}_{N-k}(M)$. (Remember Proposition 3.5.10 says that $*\Delta = \Delta *$.) ■

On a Kähler manifold the same argument gives

Theorem 3.10.9 (Serre duality) *Let M be an n-dimensional Kähler manifold. Then*

$$\overline{H^{p,q}(M)} \simeq H^{n-p,n-q}(M).$$

Finally we also have

Theorem 3.10.10 (Hard Lefschetz Theorem) *The operator*

$$L^k : H^{n-k}(M) \to H^{n+k}(M)$$

is an isomorphism.

Proof By Hodge's Theorem we just need to prove that if ξ is an $n-k$-form then ξ is harmonic if and only if L^{n-k} is harmonic. This follows from Theorem 3.6.9 and Proposition 3.5.9. ■

Bibliography

[1] L Hörmander *L^2-estimates and existence theorems for the $\bar{\partial}$-operator* Acta Math 113 (1965)

[2] L Hörmander *An Introduction to Complex Analysis in Several Variables* North-Holland (1973)

[3] D Catlin *Subelliptic estimates for the $\bar{\partial}$-Neumann problem on pseudoconvex domains* Ann Math 126 (1987).

[4] H Donelly and C Fefferman *L^2-cohomology and index theorem for the Bergman metric.* Ann Math 118 (1983)

[5] J-P Demailly *Estimations L^2 pour l'operateur $\bar{\partial}$ d'un Fibré vectoriel holomorphe semi-positif au-dessus d'une variété Kählerienne complete.* Ann Scient Éc Norm Sup 15 (1982).

[6] G B Folland and J J Kohn *The Neumann Problem for the Cauchy-Riemann complex* Annals of Mathematics Studies 75 (1972)

[7] J J Kohn and L Nirenberg *Non-coercive boundary value problems* Comm Pure Appl Math 18 (1965)

[8] R O Wells Differential Analysis on Complex Manifolds Prentice-Hall 1973

[9] F Warner Foundations of Differentiable Manifolds and Lie Groups Scott, Foresman and Company 1971