Connectedness of Poisson cylinders in Euclidean space

Erik I. Broman* and Johan Tykesson†

May 9, 2014

Abstract

We consider the Poisson cylinder model in $\mathbb{R}^d$, $d \geq 3$. We show that given any two cylinders $\mathfrak{c}_1$ and $\mathfrak{c}_2$ in the process, there is a sequence of at most $d-2$ other cylinders creating a connection between $\mathfrak{c}_1$ and $\mathfrak{c}_2$. In particular, this shows that the union of the cylinders is a connected set, answering a question appearing in [13]. We also show that there are cylinders in the process that are not connected by a sequence of at most $d-3$ other cylinders. Thus, the diameter of the cluster of cylinders equals $d-2$.

1 Introduction

This paper is devoted to the study of the geometry of a random collection of bi-infinite cylinders in $\mathbb{R}^d$, $d \geq 3$. Before we give the precise definition of this model in Section 2, we describe it informally.

We start with a homogenous Poisson line process ω of intensity $u \in (0, \infty)$ in $\mathbb{R}^d$. As the parameter u will play a very little role in this paper, we will denote its associated probability measure by $\mathbb{P}$ and keep the dependence on u implicit. Around each line $L \in \omega$, we then center a bi-infinite cylinder $\mathfrak{c}(L)$ of base-radius 1. We will sometimes abuse notation and say that $\mathfrak{c}(L) \in \omega$. The union over ω of all cylinders is a random subset of $\mathbb{R}^d$ and we call it $\mathcal{C}$. We think of $\mathcal{C}$ as the *covered region* and its complement $\mathcal{V} := \mathbb{R}^d \setminus \mathcal{C}$ as the *vacant region*. We will refer to this model as the Poisson cylinder model, and before we move on to describe our results, we will discuss some previous results. The model was first suggested by I. Benjamini to the second author [1] and subsequently studied in [13]. In [13], the focus was on the existence of a non-degenerate percolative phase transition in $\mathcal{V}$ (see [8] for a general text on continuum percolation models). Indeed, letting

$$u_*(d) := \sup\{u : \mathcal{V} \text{ has unbounded connected components a.s.}\},$$

it was proved that $0 < u_*(d) < \infty$ for every $d \geq 4$, and that $u_*(3) < \infty$. Later, it was proved in [5] that $u_*(3) > 0$.

*Department of Mathematics, Uppsala University, Sweden. Supported by the Swedish Research Council. E-mail: broman@math.uu.se

†Department of Mathematics, Chalmers University of Technology and Gothenburg University, Sweden. E-mail: johan.tykesson@gmail.com

In the present paper, we focus on connectivity properties of $\mathcal{C}$. To that end, for any $\mathfrak{c}_a, \mathfrak{c}_b \in \omega$ we let the *cylinder distance* $\text{Cdist}(\mathfrak{c}_a, \mathfrak{c}_b)$ be the minimal number k such that there exist cylinders $\mathfrak{c}_1, \dots \mathfrak{c}_k \in \omega$ so that

$$\mathfrak{c}_a \cup \mathfrak{c}_b \bigcup_{i=1}^{k} \mathfrak{c}_k,$$

is a connected set. We then define the *diameter* of $\mathcal{C}$ as

$$\text{diam}(\mathcal{C}) = \sup\{\text{Cdist}(\mathfrak{c}_a, \mathfrak{c}_b) : \mathfrak{c}_a, \mathfrak{c}_b \in \omega\}.$$

Our main result is as follows.

Theorem 1.1. *For any $d \geq 3$,*

$$\mathbb{P}[\text{diam}(\mathcal{C}) = d - 2] = 1.$$

Remarks: We prove the case $d = 3$ in Section 3, while $d \geq 4$ is proved in Section 6. While a unified approach would be desirable, utilizing the method of proof for $d \geq 4$ also in the case $d = 3$, necessitates exceptions and the handling of special cases (see further the remark at the end of the paper). This defeats the purpose of a unified proof, and therefore we prefer to divide the proof into two cases. In order to keep the paper as short as possible, we will leave some of the long (but elementary) calculations in the proof of the case $d = 3$ to the reader.

When $d = 2$, every line in a Poisson line process a.s. intersects every other line in the process, so that trivially $\mathbb{P}[\text{diam}(\mathcal{C}) = 0] = 1$.

It is an easy consequence of scaling, that Theorem 1.1 holds also for cylinders with radius different from 1. Considering a model with random radii, it will still be the case that a.s. $\text{diam}(\mathcal{C}) \leq d - 2$ (unless the distribution of the radii are degenerate). This follows from an easy coupling argument.

It is interesting to note that if we define

$$u_c(d) := \inf\{u : \exists! \text{ component of } \mathcal{C}(u, \omega) \text{ containing infinitely many cylinders a.s.}\}$$

(which is very natural in the context of percolation models), then Theorem 1.1 implies that $u_c(d) = 0$ for every $d \geq 3$. In fact, Theorem 1.1 tell us that the union of the cylinders consists solely of a unique infinite component. This is in sharp contrast to similar results for other continuum percolation models as well as for discrete percolation models (see for example [8] and [3]). In those settings, the phase transition is non-trivial in that the critical parameter value is strictly bounded away from 0. However, proofs of such results usually rely on some sort of local dependencies and/or so-called finite energy conditions. Our case is quite different, since our model lack these features. Indeed, the long range dependence in $\mathcal{C}$ and $\mathcal{V}$ manifests itself in for example the following way (see [13] Equation (3.9)):

$$\frac{c(d, u)}{|x - y|^{d-1}} \leq \text{cov}_u(\mathbf{1}\{x \in \mathcal{V}\}, \mathbf{1}\{y \in \mathcal{V}\}) \leq \frac{c'(d, u)}{|x - y|^{d-1}} \tag{1.1}$$

as soon as $|x - y| > 2$, and for some constants $c(d, u), c'(d, u) \in (0, \infty)$ independent of x, y. This long range dependence creates challenges in the study of $\mathcal{C}$ and $\mathcal{V}$ as techniques developed for percolation models exhibiting only bounded range dependence are often not applicable. In fact, the lack of the mentioned features is one of the main motivations for studying the model.

The by far most difficult part of Theorem 1.1 is to prove that $\mathbb{P}[\text{diam}(\mathcal{C}) \leq d-2] = 1$. A naive approach to proving this upper bound, can informally be described as follows. Assume for way of explanation that $d = 4$, and consider two cylinders $\mathfrak{c}_a$ and $\mathfrak{c}_b$. Proceed by exploring the set of cylinders that intersect $\mathfrak{c}_a$, and number these cylinders $\mathfrak{c}^1, \mathfrak{c}^2, \ldots$. Then, explore the set of cylinders $\mathfrak{c}^{1,1}, \mathfrak{c}^{1,2}, \ldots$ that intersect $\mathfrak{c}^1$ and continue in the natural way. One could hope to prove that for a.e. sequence $\mathfrak{c}^1, \mathfrak{c}^2, \ldots$,

$$\sum_{i,j=1}^{\infty} \mathbb{P}[\mathfrak{c}^{i,j} \cap \mathfrak{c}_b \neq \emptyset] = \infty,$$

and from there prove that $\mathbb{P}[\exists i, j \geq 1 : \mathfrak{c}^{i,j} \cap \mathfrak{c}_b \neq \emptyset] = 1$. However, having the information whether there exists $\mathfrak{c}^{1,j}$ such that $\mathfrak{c}^{1,j} \cap \mathfrak{c}_b \neq \emptyset$ also give us partial information about the cylinders $\mathfrak{c}^{2,1}, \mathfrak{c}^{2,2}, \ldots$ that intersect $\mathfrak{c}^2$. One therefore have to somehow control the accumulated information that one gains when exploring the sets of cylinders $\mathfrak{c}^{i,1}, \mathfrak{c}^{i,2}, \ldots$ as i increases. This very much complicates the situation for this approach and others like it.

The study of these questions is partly inspired by some recent works on the *random interlacements process* on $\mathbb{Z}^d$, $d \geq 3$, introduced in [12]. The random interlacement is a discrete percolation model obtained by a Poissonian collection of bi-infinite random walk trajectories. For random interlacements, inequalities similar to (1.1) hold, but with $d-1$ replaced by $d-2$. It was shown in [10] and [9] that given any two trajectories in the random interlacement, there is some sequence of at most $\lceil d/2 \rceil - 2$ other trajectories connecting them. The key tool in the proofs of [9] was the notion of stochastic dimension introduced in [2]. However, for the Poisson cylinder model, it turns out that the concept of stochastic dimension is not applicable. To see this, we will provide a short intuitive explanation.

Let $\mathcal{R}$ be a random subset of $\mathbb{Z}^d \times \mathbb{Z}^d$. Here, $\mathcal{R}$ should be thought of as a random equivalence relation, but we will simply refer to $\mathcal{R}$ as a relation. For example, the relation $\mathcal{R}$ could correspond to two points belonging to the same cylinder in the Poisson cylinder model.

The precise definition of stochastic dimension of a relation $\mathcal{R}$, is given in Definition 2.2 of [2], while we here give an informal definition. Let $\alpha \in [0, d)$. If for all $x \neq y \in \mathbb{Z}^d$ and for some constants $c, c' \in (0, \infty)$,

$$c|x-y|^{-(d-\alpha)} \leq \mathbb{P}[(x, y) \in \mathcal{R}] \leq c'|x-y|^{-(d-\alpha)},$$

and a natural correlation inequality for the events $\{(x, y) \in \mathcal{R}\}$ and $\{(z, v) \in \mathcal{R}\}$ holds (see condition (2.2) in [2]), then we say that $\mathcal{R}$ has *stochastic dimension* α. The aforementioned correlation inequality essentially says that for some constant $c < \infty$ and all $x, y, z, v \in \mathbb{Z}^d$, $\mathbb{P}[(x, y) \in \mathcal{R}, (z, v) \in \mathcal{R}]$ can be at most $c|x-y|^{-(d-\alpha)}|z-v|^{-(d-\alpha)}$ plus smaller terms. Observe that if the relation $\mathcal{R}$ has stochastic dimension d, then $\inf_{x,y} \mathbb{P}[(x, y) \in \mathcal{R}] > 0$.

Now consider the case when $\mathcal{R}$ is the relation that two points belong to the same Poisson cylinder. Clearly, in view of the inequalities in (1.1), if $\mathcal{R}$ has stochastic dimension then this dimension must be 1. However, due to the rigidity of cylinders, the required correlation inequality in the definition of stochastic dimension does not hold. For example, let $r > 0$ be large and let $x = (r, 0, ..., 0)$, $y = (2r, 0, ..., 0)$, $z = (3r, 0, ..., 0)$ and $v = (4r, 0, ..., 0)$. Then, if there is a cylinder $\mathfrak{c} \in \omega$ which connects x and v, then this cylinder will also connect y and z. Hence, $\mathbb{P}[(x, v) \in \mathcal{R}, (y, z) \in \mathcal{R}]$ is of order $r^{-(d-1)}$. However, for $\mathcal{R}$ to have stochastic dimension 1, it is necessary that $\mathbb{P}[(x, v) \in \mathcal{R}, (y, z) \in \mathcal{R}]$ is at most of order $r^{-2(d-1)}$. This shows that $\mathcal{R}$ does not have any stochastic dimension.

Now if $\mathcal{R}$ would have had stochastic dimension 1, then our proof could have been made easier, mainly because of the following. Let $1 \leq n \leq d$ and let $\mathcal{R}^n$ be the set of all (x, y) for which there exist $z_1, ..., z_{n-1}$ such that $(x, z_1) \in \mathcal{R}, (z_1, z_2) \in \mathcal{R}, ..., (z_{n-1}, y) \in \mathcal{R}$. In other words, $(x, y) \in \mathcal{R}^n$ if and only if x and y are connected via a sequence of at most n cylinders. If $\mathcal{R}$ had stochastic dimension 1, one could have used Theorem 2.4 from [2] to easily show that for all $x \neq y$ and for some constants $c, c' \in (0, \infty)$,

$$c|x - y|^{-(d-n)} \leq \mathbb{P}[(x, y) \in \mathcal{R}^n] \leq c'|x - y|^{-(d-n)} \tag{1.2}$$

Observe that if $n = d$, then the probability in (1.2) is uniformly bounded away from 0. This would have been a major step in showing $\mathbb{P}[\text{diam}(\mathcal{C}) \leq d - 2] = 1$.

In the absence of stochastic dimension, we thus had to take other routes to show our results. The proof in the case $d = 3$ relies on a projection method combined with an integral formula from [11] to show that the number of lines intersecting any two cylinders $\mathfrak{c}_a, \mathfrak{c}_b$ is a.s. infinite. When $d \geq 4$, in order to prove the lower bound of Theorem 1.1, we adapted a method from [10]. Proving the upper bound of Theorem 1.1 when $d \geq 4$ is the main effort of the paper, and we provide an informal description of our approach at the beginning of Section 6.

The rest of the paper is organized as follows. In Section 2 we define the Poisson cylinder model precisely. In Section 3 we give the proof of Theorem 1.1 for $d = 3$. Some preliminary measure estimates needed for the proof of Theorem 1.1 when $d \geq 4$, are given in Section 4. Finally, the proofs of the lower and upper bounds of Theorem 1.1 are given in Section 5 and Section 6 respectively.

2 Notation and definitions

We let $A(d, 1)$ be the set of bi-infinite lines in $\mathbb{R}^d$. Let $G(d, 1)$ be the set of bi-infinite lines in $\mathbb{R}^d$ that pass through the origin. In other words, $A(d, 1)$ is the set of 1-dimensional affine subspaces of $\mathbb{R}^d$, and $G(d, 1)$ is the set of 1-dimensional linear subspaces of $\mathbb{R}^d$. Subsets of $G(d, 1)$ and $A(d, 1)$ will typically be denoted by scripted letters like $\mathcal{A}$ and $\mathcal{L}$. If $K \in \mathcal{B}(\mathbb{R}^d)$ we let $\mathcal{L}_K \subset A(d, 1)$ denote the set of lines that intersect K:

$$\mathcal{L}_K = \{L \in A(d, 1) \, : \, L \cap K \neq \emptyset\}.$$

Let $B_d(0, 1)$ denote the d-dimensional ball of radius 1 and let κ_d denote the volume of $B_d(0, 1)$. On $G(d, 1)$ there is a unique Haar measure $\nu_{d,1}$, normalized so that $\nu_{d,1}(G(d, 1)) = 1$, and on $A(d, 1)$, there is a unique Haar measure $\mu_{d,1}$ normalized so that $\mu_{d,1}(\mathcal{L}_{B^d(0,1)}) = \kappa_{d-1}$ (see for instance [11] Chapter 13). We let SO_d be the rotation group on $\mathbb{R}^d$. Typically, we think of the elements of SO_d as the orthogonal $d \times d$ matrices with determinant 1. For any subspace $H \subset \mathbb{R}^d$, and set $A \subset \mathbb{R}^d$, we let $\Pi_H(A) \subset \mathbb{R}^d$ denote the projection of A onto H. We will let $e_1, e_2, \ldots, e_d$ denote the generic orthonormal set of vectors that span $\mathbb{R}^d$.

2.1 The Poisson cylinder model

We consider the following space of point measures on $A(d,1)$:

$$\Omega = \{\omega = \sum_{i=0}^{\infty} \delta_{L_i} \text{ where } L_i \in A(d,1), \text{ and } \omega(\mathcal{L}_A) < \infty \text{ for all compact } A \subset \mathbb{R}^d\}.$$

Here, δ_L of course denotes point measure at L.

In what follows, we will often use the following standard abuse of notation: if ω is some point measure, the expression "$x \in \omega$" will stand for "$x \in \mathrm{supp}(\omega)$". If $\omega \in \Omega$ and $A \in \mathcal{B}(\mathbb{R}^d)$ we let ω_A denote the restriction of ω to $\mathcal{L}_A$. We will draw an element ω from Ω according to a Poisson point process with intensity measure $u\mu_{d,1}$ where $u > 0$. We call ω a *(homogeneous) Poisson line process* of intensity u.

If $L \in A(d,1)$, we denote by $\mathfrak{c}(L)$ the cylinder of base radius 1 centered around L:

$$\mathfrak{c}(L) = \{x \in \mathbb{R}^d \,:\, d(x,L) \leq 1\}.$$

Finally the object of main interest in this paper, the union of all cylinders is denoted by $\mathcal{C}$:

$$\mathcal{C} = \mathcal{C}(\omega) = \bigcup_{L \in \omega} \mathfrak{c}(L).$$

3 Proof of Theorem 1.1 when $d = 3$

The aim of this section is to prove the following theorem.

Theorem 3.1. *For $d = 3$,*

$$\mathbb{P}[\mathrm{diam}(\mathcal{C}) = 1] = 1.$$

In Section 3.1 we consider two arbitrary fixed cylinders $\mathfrak{c}_1, \mathfrak{c}_2$ and show that the $\mu_{3,1}$-measure of the set of lines that intersect both of them is infinite, see Proposition 3.2. It will then be straightforward to prove Theorem 3.1, which we do in Section 3.2.

3.1 Lines intersecting two cylinders in three dimensions

We write

$$L = \{t(l_1, l_2, l_3) \,:\, -\infty < t < \infty\}$$

for a line in $G(3,1)$, where $l_1^2 + l_2^2 + l_3^2 = 1$.

Proposition 3.2. *For any two lines $L_1, L_2 \in A(3,1)$,*

$$\mu_{3,1}(\mathcal{L}_{\mathfrak{c}(L_1)} \cap \mathcal{L}_{\mathfrak{c}(L_2)}) = \infty.$$

Proof. We will consider only the case $L_1, L_2 \in G(3,1)$, as the general case follows by an easy modification. By invariance of $\mu_{3,1}$ under translations and rotations of $\mathbb{R}^3$, we can without loss of generality assume that $L_1 = \{te_1 : -\infty < t < \infty\}$. Furthermore,

$$L_2 := \{t(k_1, k_2, k_3) \,:\, -\infty < t < \infty\},$$

for some $k_1^2 + k_2^2 + k_3^2 = 1$. By the representation of [11] Theorem 13.2.12 we have

$$\begin{aligned}\mu_{3,1}&(\mathcal{L}_{\mathfrak{c}(L_1)} \cap \mathcal{L}_{\mathfrak{c}(L_2)}) \\ &= \int_{G(d,1)} \int_{L^\perp} I(L + y \in \mathcal{L}_{\mathfrak{c}(L_1)} \cap \mathcal{L}_{\mathfrak{c}(L_2)}) \lambda_2(dy) \nu_{3,1}(dL),\end{aligned} \tag{3.1}$$

where λ_2 denotes two-dimensional Lebesgue measure and I is an indicator function. Observe that for fixed L, the set of $y \in L^\perp$ such that $y + L \in \mathcal{L}_{\mathfrak{c}(L_i)}$ is exactly $\Pi_{L^\perp}(\mathfrak{c}(L_i))$ for $i = 1, 2$. Hence,

$$\int_{L^\perp} I(L + y \in \mathcal{L}_{\mathfrak{c}(L_1)} \cap \mathcal{L}_{\mathfrak{c}(L_2)}) \lambda_2(dy) = \lambda_2 \left(\Pi_{L^\perp}(\mathfrak{c}(L_1)) \cap \Pi_{L^\perp}(\mathfrak{c}(L_2)) \right). \tag{3.2}$$

Let $K(L) := \Pi_{L^\perp}(\mathfrak{c}(L_1)) \cap \Pi_{L^\perp}(\mathfrak{c}(L_2))$. The sets $\Pi_{L^\perp}(\mathfrak{c}(L_1))$ and $\Pi_{L^\perp}(\mathfrak{c}(L_2))$ are two-dimensional cylinders of width 2 in $L^\perp$, with central axes $\Pi_{L^\perp}(L_1)$ and $\Pi_{L^\perp}(L_2)$ respectively. Therefore, $K(L)$ is a rhombus except when $\Pi_{L^\perp}(L_1) = \Pi_{L^\perp}(L_2)$ in which case $K(L)$ is an infinite strip or a disk. It is a straightforward exercise (although lengthy) to prove that

$$\lambda_2(K(L)) = \frac{4\sqrt{(l_2^2 + l_3^2)((k_3 l_2 - k_2 l_3)^2 + (k_1 l_2 - k_2 l_1)^2 + (k_3 l_1 - k_1 l_3)^2)}}{|k_2 l_3 - k_3 l_2|}.$$

As we will indicate below, it follows that

$$\int_{G(3,1)} \lambda_2(K(L)) \nu_{3,1}(dL) = \infty. \tag{3.3}$$

For simplicity, we first assume that $k_1 > 0$ and that $k_3/k_2 > 0$. Furthermore, we let $l_1 = \cos\theta$, $l_2 = \sin\theta \sin\varphi$ and $l_3 = \sin\theta\cos\varphi$ where $0 \le \theta \le \pi$ and $0 \le \varphi \le 2\pi$ and note that

$$\lambda_2(K(L)) \ge |\sin\theta| \left| \frac{k_1 l_2 - k_2 l_1}{k_2 l_3 - k_3 l_2} \right| = \left| \frac{k_1 \sin\theta \sin\varphi - k_2 \cos\theta}{k_2 \cos\varphi - k_3 \sin\varphi} \right|,$$

so that

$$\begin{aligned}\int_{G(3,1)} &\lambda_2(K(L)) \nu_{3,1}(dL) \\ &\ge \int_0^{2\pi} \int_0^\pi \left| \frac{k_1 \sin\theta \sin\varphi - k_2 \cos\theta}{k_2 \cos\varphi - k_3 \sin\varphi} \right| \sin\theta d\theta d\varphi \\ &\ge \int_0^{2\pi} \left| \int_0^\pi \frac{k_1 \sin\theta \sin\varphi - k_2 \cos\theta}{k_2 \cos\varphi - k_3 \sin\varphi} \sin\theta d\theta \right| d\varphi \\ &= c_1 \int_0^{2\pi} \left| \frac{k_1 \sin\varphi}{k_2 \cos\varphi - k_3 \sin\varphi} \right| d\varphi \\ &\ge c_2 \int_0^{\pi/2} \frac{\sin\varphi}{|\cos\varphi - a \sin\varphi|} d\varphi,\end{aligned}$$

where $a = k_3/k_2 > 0$ by assumption. Using the substitution $x = \cos\varphi$, the expression above becomes

$$c_2 \int_0^1 \frac{1}{|x - a\sqrt{1-x^2}|} dx.$$

The last integrand has a singularity at $x = a/\sqrt{1+a^2}$ and it is straightforward to verify that the integral diverges. Finally, (3.3) follows by similar calculations when either of the assumptions $k_1 > 0$ or $k_3/k_2 > 0$ does not hold.

Combining (3.1), (3.2) and (3.3) finishes the proof of the proposition. □

From Proposition 3.2, the following corollary is easy.

Corollary 3.3. *Let $d = 3$. Fix $L_1, L_2 \in A(3,1)$. For any $u > 0$, we have*

$$\mathbb{P}[\omega(\mathcal{L}_{\mathfrak{c}(L_1)} \cap \mathcal{L}_{\mathfrak{c}(L_2)}) = \infty] = 1.$$

Proof. Follows trivially from Proposition 3.2. □

3.2 Proof of Theorem 3.1

Proof of Theorem 3.1. For lines $L_1, L_2 \in A(3,1)$, let

$$E(L_1, L_2) = \{\omega(\mathcal{L}_{\mathfrak{c}(L_1)} \cap \mathcal{L}_{\mathfrak{c}(L_2)}) = \infty\}.$$

We know from Corollary 3.3 that

$$\mathbb{P}[E(L_1, L_2)] = 1 \text{ for all } L_1, L_2 \in A(3,1). \tag{3.4}$$

Let $D := \cap_{(L_1,L_2)\in\omega^2_{\neq}} E(L_1, L_2)$. Here $\omega^2_{\neq}$ denotes the set of all 2-tuples of distinct lines from ω. Observe that if D occurs, then $\mathcal{C}$ is connected, and moreover any two cylinders are connected via some other cylinder. Hence it suffices to show that $\mathbb{P}[D] = 1$. This is intuitively clear in view of (3.4), but we now make this precise. Observe that

$$D = \left\{ \sum_{(L_1,L_2)\in\omega^2_{\neq}} I(E(L_1, L_2)^c) = 0 \right\},$$

so that it suffices to show

$$\mathbb{E}\left[\sum_{(L_1,L_2)\in\omega^2_{\neq}} I(E(L_1, L_2)^c) \right] = 0.$$

Let $\mathbb{E}^{L_1,L_2}$ denote expectation with respect to $\omega + \delta_{L_1} + \delta_{L_2}$. According to the Slivnyak-Mecke formula (see [11] Corollary 3.2.3) we have

$$\begin{aligned}
&\mathbb{E}\left[\sum_{(L_1,L_2)\in\omega^2_{\neq}} I(E(L_1,L_2)^c)\right] \\
&= \int_{A(3,1)}\int_{A(3,1)} \mathbb{E}^{L_1,L_2}\left[I(E(L_1,L_2)^c)\right]\nu_{3,1}(dL_1)\nu_{3,1}(dL_2) \\
&= \int_{A(3,1)}\int_{A(3,1)} \mathbb{P}^{L_1,L_2}\left[E(L_1,L_2)^c\right]\nu_{3,1}(dL_1)\nu_{3,1}(dL_2) \\
&= \int_{A(3,1)}\int_{A(3,1)} \mathbb{P}\left[E(L_1,L_2)^c\right]\nu_{3,1}(dL_1)\nu_{3,1}(dL_2) \\
&= \int_{A(3,1)}\int_{A(3,1)} 0\,\nu_{3,1}(dL_1)\nu_{3,1}(dL_2) = 0,
\end{aligned}$$

where the penultimate equality follows from Corollary 3.3. This proves that $\mathbb{P}[\text{diam}(\mathcal{C}) \leq 1] = 1$.

It is an immediate consequence from the Poissonian nature of the model, that with probability 1 there exists two cylinders $\mathfrak{c}_1, \mathfrak{c}_2 \in \omega$ such that $\mathfrak{c}_1 \cap \mathfrak{c}_2 \neq \emptyset$. Therefore, $\mathbb{P}[\text{diam}(\mathcal{C}) > 0] = 1$. □

4 Preliminary results in d dimensions

In this section we estimate the $\mu_{d,1}$-measure of lines that intersect both a ball and a cylinder that are far apart. We say that a ball $B(x,1)$ and a cylinder $\mathfrak{c}$ is at distance r, if the distance between x and the centerline of $\mathfrak{c}$ is r.

We will use the following lemma from [13].

Lemma 4.1. *Consider any two balls B_1, B_2 with radii 1 and whose centers are at distance r. There exists constants $0 < c_1 < c_2 < \infty$ depending on d but not r, such that for any $r \geq 4$,*

$$\frac{c_1}{r^{d-1}} \leq \mu_{d,1}(\mathcal{L}_{B_1} \cap \mathcal{L}_{B_2}) \leq \frac{c_2}{r^{d-1}}. \tag{4.1}$$

Remark: For future reference, we note that Lemma 4.1 is easily generalised to hold for any pair of balls of arbitrary radii. Of course, the constants c_1, c_2 will then depend on these radii. Lemma 4.1 can easily be understood as follows. Suppose that B_1 is centered at the origin. Then B_2 is centered somewhere on $\partial B(0,r)$. We can cover $\partial B(0,r)$ by order $r^{(d-1)}$ balls of radius 1. Hence the measure of the lines intersecting both B_1 and B_2 should be of order $r^{-(d-1)}$ by symmetry. Finally, we remark that using the methods of Section 3.1, one can show a stronger statement than (4.1), namely that there exist constants $0 < c < c' < \infty$ dependent on d but not r such that for any $r \geq 4$,

$$\frac{c}{r^{d-1}} \leq \mu_{d,1}(\mathcal{L}_{B_1} \cap \mathcal{L}_{B_2}) \leq \frac{c}{r^{d-1}} + \frac{c'}{r^{d+1}}.$$

We have the following result.

Proposition 4.2. *For any $d \geq 3$, there exist constants $c = c(d) > 0$ and $c' = c'(d) < \infty$ such that for all $r \geq 1$, and $x \in \mathbb{R}^d$, $L \in A(d,1)$ at distance r from each other, we have that*

$$c\, r^{-(d-2)} \leq \mu_{d,1}(\mathcal{L}_{B(x,1)} \cap \mathcal{L}_{\mathfrak{c}(L)}) \leq c'\, r^{-(d-2)}. \tag{4.2}$$

Proof. We begin with the upper bound. By rotation and translation invariance of $\mu_{d,1}$, we can without loss of generality assume that $x = (r, 0, ..., 0)$ and $L = \{te_2 : t \in \mathbb{R}\}$. For $i \in \mathbb{Z}$, let $B_i := B((0, i, 0, ..., 0), 2)$. Observe that

$$\mathcal{L}_{\mathfrak{c}(L)} \subset \bigcup_{i \in \mathbb{Z}} \mathcal{L}_{B_i}$$

so that

$$\mathcal{L}_{B(x,1)} \cap \mathcal{L}_{\mathfrak{c}(L)} \subset \bigcup_{i \in \mathbb{Z}} \left(\mathcal{L}_{B(x,1)} \cap \mathcal{L}_{B_i}\right). \tag{4.3}$$

We now get for $r \geq 1$ that

$$\begin{aligned}
\mu_{d,1}(\mathcal{L}_{B(x,1)} \cap \mathcal{L}_{\mathfrak{c}(L)}) &\overset{(4.3)}{\leq} \sum_{i=-\infty}^{\infty} \mu_{d,1}(\mathcal{L}_{B(x,1)} \cap \mathcal{L}_{B_i}) \overset{(4.1)}{\leq} c \sum_{i=-\infty}^{\infty} (r^2 + i^2)^{-(d-1)/2} \\
&= cr^{-(d-1)} \sum_{i=-\infty}^{\infty} (1 + (i/r)^2)^{-(d-1)/2} \leq c' r^{-(d-1)} \int_{-\infty}^{\infty} (1 + (x/r)^2)^{-(d-1)/2}\, dx \\
&= c' r^{-(d-2)} \int_{-\infty}^{\infty} (1 + y^2)^{-(d-1)/2}\, dy = c'' r^{-(d-2)},
\end{aligned}$$

where the integral in the last step is convergent since $d \geq 3$. This finishes the proof of the upper bound in (4.2), and we proceed with the lower bound.

For proof-technical reasons we now assume without loss of generality that $r \geq 10$. For $i \in \{2, 3, ..., \lfloor r \rfloor\}$, let $D_i := B((0, i, 0, ..., 0), 1/8)$. Observe that

$$\bigcup_{i=2}^{\lfloor r \rfloor} \mathcal{L}_{D_i} \subset \mathcal{L}_{\mathfrak{c}(L)}$$

so that

$$\bigcup_{i=2}^{\lfloor r \rfloor} (\mathcal{L}_{D_i} \cap \mathcal{L}_{B(x,1)}) \subset \mathcal{L}_{\mathfrak{c}(L)} \cap \mathcal{L}_{B(x,1)}. \tag{4.4}$$

We will now show that

$$(\mathcal{L}_{D_i} \cap \mathcal{L}_{B(x,1)})_{i=2}^{\lfloor r \rfloor} \text{ is a sequence of pairwise disjoint sets of lines.} \tag{4.5}$$

Let $i, j \in \{2, ..., \lfloor r \rfloor\}$ where $i \neq j$ and assume that

$$L_1 \in \mathcal{L}_{D_i} \cap \mathcal{L}_{D_j} \tag{4.6}$$

and that

$$L_1 \in \mathcal{L}_{D_i} \cap \mathcal{L}_{B(x,1)}. \tag{4.7}$$

As usual, we write L_1 on the form $L_1 = \{t(k_1, ..., k_d) : t \in \mathbb{R}\} + v$ for some $v \in \mathbb{R}^d$. We observe that if (4.6) holds, then

$$\frac{k_1}{k_2} \geq \frac{-2/8}{|i-j| - 2/8} \geq \frac{-2/8}{1-2/8} \geq -\frac{1}{3},$$

while for (4.7) to be satisfied, then

$$\frac{k_1}{k_2} \leq -\frac{r-1-1/8}{i+1+1/8} \leq -\frac{r-1-1/8}{r+1+1/8} \leq -\frac{10-1-1/8}{10+1+1/8} = -\frac{71}{89}.$$

We conclude that (4.6) and (4.7) cannot both hold, which proves (4.5).

Proceeding, we have that

$$\mu_{d,1}(\mathcal{L}_{\mathfrak{c}(L)} \cap \mathcal{L}_{B(x,1)}) \overset{(4.4)}{\geq} \mu_{d,1}(\cup_{i=2}^{\lfloor r \rfloor}(\mathcal{L}_{D_i} \cap \mathcal{L}_{B(x,1)})) \overset{(4.5)}{=} \sum_{i=2}^{\lfloor r \rfloor} \mu_{d,1}(\mathcal{L}_{D_i} \cap \mathcal{L}_{B(x,1)})$$
$$\overset{(4.1)}{\geq} c \sum_{i=2}^{\lfloor r \rfloor}(r^2+i^2)^{-(d-1)/2} \geq c \sum_{i=2}^{\lfloor r \rfloor}(2r^2)^{-(d-1)/2} = c'(\lfloor r \rfloor - 1)r^{-(d-1)} \geq c'' r^{-(d-2)},$$

finishing the proof of the proposition in the case $r \geq 10$. The full statement follows easily. □

5 Proof of Theorem 1.1, the lower bound when $d \geq 4$

In this section we prove the following theorem.

Theorem 5.1. *For any* $d \geq 4$, $\mathbb{P}[\text{diam}(\mathcal{C}) \geq d-2] = 1$.

As a key step, we first show that the probability that two points x and y in $\mathbb{R}^d$ are connected via a sequence of at most $d-1$ cylinders tends to 0 as $|x-y| \to \infty$, see Proposition 5.3 below. We will think of the integer lattice $\mathbb{Z}^d$ as a subset $\mathbb{R}^d$, embedded in the natural way.

For each $y = (y_1, ..., y_d) \in \mathbb{R}^d$ let $\lfloor y \rfloor := (\lfloor y_1 \rfloor, ..., \lfloor y_d \rfloor) \in \mathbb{Z}^d$.

We will need the following lemma, which (as remarked in [10]) follows from (1.38) of Proposition 1.7 in [4].

Lemma 5.2. *For any positive integer* $n < d$ *and any* $z_0, z_n \in \mathbb{Z}^d$ *there exists a constant* $c = c(d) < \infty$ *such that*

$$\sum_{z_1,\dots,z_{n-1} \in \mathbb{Z}^d} \prod_{i=0}^{n-1} \min(1, |z_i - z_{i+1}|^{-(d-1)}) \leq c|z_0 - z_n|^{-(d-n)}. \tag{5.1}$$

For $x, y \in \mathbb{R}^d$ and $n \geq 1$, let $A_n(x,y)$ be the event that there exist *distinct* lines $L_1, ..., L_n \in \omega$ such that

1. $x \in \mathfrak{c}(L_1)$ and $y \in \mathfrak{c}(L_n)$.

2. $\mathfrak{c}(L_i) \cap \mathfrak{c}(L_{i+1}) \neq \emptyset$ for $i = 1, ..., n-1$.

In addition, let

$$\tilde{A}_n(x,y) = \cup_{i=1}^n A_i(x,y).$$

We can now state the first result of this section:

Proposition 5.3. *For $d \geq 3$, $n \in \{1, ..., d-1\}$ there exists a constant $c = c(u,d) < \infty$ such that for any $x, y \in \mathbb{R}^d$ with $|x-y| \geq 2d$,*

$$\mathbb{P}[\tilde{A}_n(x,y)] \leq c|x-y|^{-(d-n)}.$$

Proof. The proof follows the first part of the proof of Theorem 1 in [10] closely. Recall that we think of the integer lattice $\mathbb{Z}^d$ as embedded in $\mathbb{R}^d$. Fix $n \in \{1, ..., d-1\}$ and $x, y \in \mathbb{R}^d$ where $|x-y| \geq 2d$.

For $v, w \in \mathbb{Z}^d$, let

$$T(v,w) := \mathcal{L}_{B(v,\sqrt{d}+1)} \cap \mathcal{L}_{B(w,\sqrt{d}+1)}$$

and introduce the event

$$E(v,w) := \{\omega\left(T(v,w)\right) \geq 1\}.$$

For $z_0, \ldots, z_n \in \mathbb{Z}^d$ we let $E(z_0,z_1) \circ E(z_1,z_2) \circ \ldots \circ E(z_{n-1},z_n)$ denote the event that there exists distinct lines $L_1, \ldots, L_n$ such that $L_i \in T(z_{i-1}, z_i)$ for every $i = 1, \ldots, n$. If $A_n(x,y)$ occurs, then there exist distinct lines $L_1, ..., L_n$ in ω and points $x_1, ..., x_{n-1} \in \mathbb{R}^d$ such that

$$L_i \in \mathcal{L}_{B(x_{i-1},1)} \cap \mathcal{L}_{B(x_i,1)} \text{ for } i = 1, .., n,$$

where we put $x_0 := x$ and $x_n := y$. Since $|x - \lfloor x \rfloor| \leq \sqrt{d}$ for any $x \in \mathbb{R}^d$, it follows that we also have

$$L_i \in \mathcal{L}_{B(\lfloor x_{i-1} \rfloor, 1+\sqrt{d})} \cap \mathcal{L}_{B(\lfloor x_i \rfloor, 1+\sqrt{d})} \text{ for } i = 1, .., n.$$

Therefore, we have shown the inclusion

$$A_n(x,y) \subset \bigcup_{z_1, \ldots, z_{n-1} \in \mathbb{Z}^d} E(z_0,z_1) \circ E(z_1,z_2) \circ \ldots \circ E(z_{n-1},z_n), \tag{5.2}$$

where we let $z_0 := \lfloor x \rfloor$ and $z_n := \lfloor y \rfloor$. Let $\omega_{\neq}^n$ denote the set of all $n-$tuples of distinct lines $L_1, ..., L_n$ in ω. Then we have

$$I(E(z_0,z_1) \circ \ldots \circ E(z_{n-1},z_n)) \leq \sum_{\omega_{\neq}^n} \prod_{i=1}^n I(L_i \in T(z_{i-1}, z_i)). \tag{5.3}$$

Now a union bound together with (5.2) and (5.3) implies

$$\mathbb{P}[A_n(x,y)] \leq \sum_{z_1, \ldots, z_{n-1} \in \mathbb{Z}^d} \mathbb{E}\left[\sum_{\omega_{\neq}^n} \prod_{i=1}^n I(L_i \in T(z_{i-1}, z_i))\right]. \tag{5.4}$$

According to the Slivnyak-Mecke formula (see [11] Corollary 3.2.3), the expectation on the right hand side of (5.4) equals

$$\begin{aligned}\prod_{i=1}^{n}\mathbb{E}[\omega(T(z_{i-1},z_i))] &= u^n\prod_{i=1}^{n}\mu_{d,1}(T(z_{i-1},z_i)) \\ &\le c(u,d)\prod_{i=1}^{n}\min(1,|z_{i-1}-z_i|^{-(d-1)}),\end{aligned} \tag{5.5}$$

where we applied Lemma 4.1 (and the remark thereafter) in the last inequality. From (5.4) and (5.5) we get

$$\begin{aligned}\mathbb{P}[A_n(x,y)] &\le c(u,d)\sum_{z_1,\dots,z_{n-1}\in\mathbb{Z}^d}\prod_{i=1}^{n}\min(1,|z_{i-1}-z_i|^{-(d-1)}) \\ &\overset{(5.1)}{\le} c(u,d)|\lfloor x\rfloor-\lfloor y\rfloor|^{-(d-n)} \le c'(u,d)|x-y|^{-(d-n)},\end{aligned} \tag{5.6}$$

whenever $n\in\{1,...,d-1\}$. Finally we get

$$\begin{aligned}\mathbb{P}[\tilde{A}_n(x,y)] &\le \sum_{k=1}^{n}\mathbb{P}[A_k(x,y)] \\ &\overset{(5.6)}{\le} \sum_{k=1}^{n}c(u,d)|x-y|^{-(d-k)} \le c'(u,d)|x-y|^{-(d-n)}.\end{aligned}$$

□

For $n\le d-1$, consider the event

$$\bigcup_{x,y\in\mathbb{R}^d}\left(\tilde{A}_n(x,y)^c\cap\{x,y\in\mathcal{C}\}\right).$$

In words: there exist points $x,y\in\mathcal{C}$ which are not connected via any sequence of n cylinders if $n\le d-1$. From Proposition 5.3 it is quite intuitive that the probability of this event should be 1. We prove this in full detail in the next corollary, which is in the spirit of the final part of the proof of Theorem 2.1(ii) in [6].

Corollary 5.4. *For any $n\le d-1$ we have*

$$\mathbb{P}\left[\bigcup_{x,y\in\mathbb{R}^d}\left(\tilde{A}_n(x,y)^c\cap\{x,y\in\mathcal{C}\}\right)\right]=1. \tag{5.7}$$

Proof. Fix $n\le d-1$ and identify $\mathbb{Z}$ with the points along the e_1-axis with integer coordinates. For $R\ge 1$ let K_R^1 denote the set $\mathbb{Z}\cap[1,R+1]$, K_R^2 denote the set $\mathbb{Z}\cap[e^R,e^R+R]$ and define

$$H_R := \bigcup_{x\in K_R^1,y\in K_R^2}\left(\tilde{A}_n(x,y)^c\cap\{x,y\in\mathcal{C}\}\right).$$

We will show that

$$\mathbb{P}[H_R] \overset{R\to\infty}{\to} 1,$$

which implies

$$\mathbb{P}[\cup_{R\geq 1} H_R] = 1.$$

Then (5.7) follows since

$$\bigcup_{R\geq 1} H_R \subset \bigcup_{x,y\in\mathbb{R}^d} \left(\tilde{A}_n(x,y)^c \cap \{x,y\in\mathcal{C}\}\right).$$

Let E_R^1 be the event that there is no pair $x \in K_R^1$ and $y \in K_R^2$ for which $x, y \in \mathcal{C}$. That is, we let

$$E_R^1 := \bigcap_{x\in K_R^1} \bigcap_{y\in K_R^2} \{x,y\in\mathcal{C}\}^c = \{\mathcal{C}\cap K_R^1 = \emptyset\} \cup \{\mathcal{C}\cap K_R^2 = \emptyset\}. \tag{5.8}$$

Also, introduce the event

$$E_R^2 := \bigcup_{x\in K_R^1} \bigcup_{y\in K_R^2} \tilde{A}_n(x,y),$$

which is the event that there exists $x \in K_R^1$ and $y \in K_R^2$ such that they are connected via at most n cylinders. We have

$$H_R^c = \bigcap_{x\in K_R^1} \bigcap_{y\in K_R^2} (\{x,y\in\mathcal{C}\} \cap \tilde{A}_n(x,y)^c)^c = \bigcap_{x\in K_R^1} \bigcap_{y\in K_R^2} \left(\{x,y\in\mathcal{C}\}^c \cup \tilde{A}_n(x,y)\right). \tag{5.9}$$

From (5.9) we see that

$$H_R^c \cap E_R^1 = E_R^1 \text{ and } H_R^c \cap (E_R^1)^c \subset E_R^2.$$

The second inclusion follows since if $(E_R^1)^c$ occurs, then there must exist $x \in K_R^1$ and $y \in K_R^2$ such that $x, y \in \mathcal{C}$, and for H_R^c to occur, $\tilde{A}_n(x,y)$ must occur for these x, y. Hence

$$H_R^c \subset E_R^1 \cup E_R^2.$$

We now argue that

$$\lim_{R\to\infty} \mathbb{P}[E_R^1] = 0. \tag{5.10}$$

For $i \in \mathbb{Z}$ let B_i be the ball of radius 1 centered at $(i, 0, \dots, 0)$ and $F_i = \mathcal{L}_{B_i} \setminus (\mathcal{L}_{B_{i-1}} \cup \mathcal{L}_{B_{i+1}})$. It is straightforward to see that if $L \in F_i \subset \mathcal{L}_{B_i}$, then $L \notin \mathcal{L}_{B_j}$ for every $j \neq i$. Hence $(F_i)_{i\in\mathbb{Z}}$ is a sequence of disjoint sets of lines. It is also easy to see that $\mu_{d,1}(F_i) = c_1(d) > 0$. We now get

$$\begin{aligned}\mathbb{P}[\mathcal{C}\cap K_R^1 = \emptyset] &= \mathbb{P}[\cap_{i\in K_R^1}\{\omega(\mathcal{L}_{B_i}) = 0\}] \leq \mathbb{P}[\cap_{i\in K_R^1}\{\omega(F_i) = 0\}] \\ &= \Pi_{i\in K_R^1}\mathbb{P}[\omega(F_i) = 0] = e^{-|K_R^1|c_1(d)} = e^{-cR}.\end{aligned}$$

In the same way, we get $\mathbb{P}[\mathcal{C}\cap K_R^2 = \emptyset] \leq e^{-cR}$ so (5.8), a union bound, and letting $R \to \infty$ gives (5.10).

For the event E_R^2 we have

$$\mathbb{P}[E_R^2] \leq \sum_{x\in K_R^1} \sum_{y\in K_R^2} \mathbb{P}[\tilde{A}_n(x,y)] \overset{(5.6)}{\leq} cR^2 e^{-R} \to 0,$$

as $R \to \infty$. Hence,

$$\lim_{R\to\infty} \mathbb{P}[H_R^c] \leq \lim_{R\to\infty} \mathbb{P}[E_R^1] + \lim_{R\to\infty} \mathbb{P}[E_R^2] = 0,$$

as required. □

Proof of Theorem 5.1: Let $k \leq d-3$. According to Corollary 5.4, we can a.s. find $x, y \in \mathcal{C}$ such that x and y are not connected via any sequence of $k+2$ cylinders. Since $x, y \in \mathcal{C}$, this means that there is a cylinder $\mathfrak{c}_1 \in \omega$ (which contains x) and a cylinder $\mathfrak{c}_2 \in \omega$ (which contains y) such that $\mathfrak{c}_1$ and $\mathfrak{c}_2$ are not connected via any sequence of k cylinders. □

6 Proof of Theorem 1.1, the upper bound when $d \geq 4$

In this section we prove

Theorem 6.1. *For any $d \geq 4$, $\mathbb{P}[\mathrm{diam}(\mathcal{C}) \leq d-2] = 1$.*

Obviously, Theorem 1.1 follows from Theorems 3.1, 5.1 and 6.1. Only the proof of Theorem 6.1 remains.

The proof of Theorem 6.1 is fairly long. In order to facilitate the reading, we will try to provide a short intuitive and very informal description of the main underlying idea. We let a cylinder-path of length k from $\mathfrak{c}_1$ to $\mathfrak{c}_2$ be a collection $\mathfrak{c}^1, \ldots, \mathfrak{c}^k$ of cylinders such that $\mathfrak{c}_1 \cap \mathfrak{c}^1 \neq \emptyset$, $\mathfrak{c}^1 \cap \mathfrak{c}^2 \neq \emptyset$ and so on. Assuming Theorem 6.1, there should be plenty of such cylinder-paths from $\mathfrak{c}_1$ to $\mathfrak{c}_2$ using $d-2$ cylinders. We will therefore look for collections of boxes $B_1, \ldots, B_{d-3}$ (of small sidelength) such that $\mathfrak{c}^i$ and $\mathfrak{c}^{i+1}$ "meets" in B_i, that is $\mathfrak{c}^i \cap \mathfrak{c}^{i+1} \cap B_i \neq \emptyset$. Finding such collections are complicated by the longe-range dependencies of the line-process ω. Therefore, we will have to be very careful in the way we look for the boxes, in order to have enough independence for the proof to work.

To that end, we divide the cylinders $\mathfrak{c}_1, \mathfrak{c}_2$ into smaller parts $\{\mathfrak{c}_{1,m}\}_{m\geq 1}$ and $\{\mathfrak{c}_{2,m}\}_{m\geq 1}$ (see below for exact definition). For fixed m, we look for a cylinder intersecting $\mathfrak{c}_{1,m}$ and some small box B_1 inside a larger box $B_{R^m}^1$ of sidelength R^m (again, see below for exact definition). We then look for another cylinder connecting B_1 to $B_2 \subset B_{R^m}^2$ and so on until finally we try to find a cylinder connecting $B_{d-3} \subset B_{R^m}^{d-3}$ to $\mathfrak{c}_{2,m}$. By being very careful in how we choose the placements of the boxes $B_{R^m}^i$ we gain independence in scales. That is, whether there is a path on scale m is independent of whether there is a path on scale $m+1$.

Before presenting the proof, we start with some definitions. For simplicity, we assume in this section that the radius of a cylinder is $\sqrt{d}$, the reason for this will be clear shortly and can be made without loss of generality.

Consider two arbitrary cylinders $\mathfrak{c}_1, \mathfrak{c}_2$ with centerlines $L_1, L_2 \in A(d,1)$ respectively. Since for any two lines in $G(d,1)$, there is a plane that they belong to, we can without loss of generality (due to the invariances of $\mu_{d,1}$) assume that $L_1 = \{te_1 : -\infty < t < \infty\}$ and that $L_2 = \{p + t(l_1, l_2, 0, \ldots, 0) : -\infty < t < \infty\}$ where $p = (0, 0, p_3, p_4, \ldots, p_d)$.

For any integers $m, R \geq 0$ consider the boxes $B^1_{R^m}, B^2_{R^m}, \ldots, B^{d-3}_{R^m}$ where $B^i_{R^m} = q_{i,m} + [-R^m/2, R^m/2]^d$, $q_{i,m} = p + NR^m e_{i+3}$ and $N = 10d + 1$. The reason for this choice of N will become clear later. We will assume throughout that $R \geq 2 \max_{i=1,\ldots,d} |p_i|$ and also that $R > 20\sqrt{d} + 1$.

We can tile the boxes $B^i_{R^m}$ in the canonical way with smaller boxes of sidelength 1. We denote such boxes by $B_{i,m}$, that is $B_{i,m} \subset B^i_{R^m}$. Note that if the centerlines L_a, L_b of cylinders $\mathfrak{c}_a, \mathfrak{c}_b$ both intersect a box B of sidelength 1, then since the radii of the cylinders are $\sqrt{d}$, we have $\mathfrak{c}_a \cap \mathfrak{c}_b \neq \emptyset$. This is the reason for our choice of radius.

For any two sets $E_1, E_2 \subset \mathbb{R}^d$, we let $E_1 \leftrightarrow E_2$ denote the event that the Poisson process ω includes an element L in the set $\mathcal{L}_{E_1} \cap \mathcal{L}_{E_2}$. We will say that E_1, E_2 are connected, and that L connects E_1 and E_2. Furthermore, we let $E_1 \overset{n}{\leftrightarrow} E_2$ denote the event that there are exactly n such connecting lines. It will greatly facilitate our analysis to consider disjoint parts of the cylinders $\mathfrak{c}_1, \mathfrak{c}_2$. Therefore, we define for every $m \geq 1$,

$$\mathfrak{c}_{1,m} := \{x \in \mathfrak{c}_1 : R^{m-1}/2 \leq d(\Pi_{L_1}(x), o) < R^m/2 - 10\sqrt{d}\},$$

and

$$\mathfrak{c}_{2,m} := \{x \in \mathfrak{c}_2 : R^{m-1}/2 \leq d(\Pi_{L_2}(x), p) < R^m/2 - 10\sqrt{d}\}.$$

Let $\vec{B}_m := (B_{1,m}, \ldots, B_{d-3,m})$ and let $\mathcal{P}_m(\vec{B}_m)$ be shorthand for $\{\mathfrak{c}_{1,m} \leftrightarrow B_{1,m} \leftrightarrow \ldots \leftrightarrow B_{d-3,m} \leftrightarrow \mathfrak{c}_{2,m}\}$. We define

$$X_{R,m} = \sum_{\vec{B}_m} I(\mathcal{P}_m(\vec{B}_m)),$$

where the sum is over all choices of $\vec{B}_m$ and I is as before an indicator function. Obviously, if $X_{R,m} > 0$, then $\mathfrak{c}_1, \mathfrak{c}_2$ are connected via a cylinder-path of length $d-2$.

We will prove the following two lemmas.

Lemma 6.2. *There exists a constant $c = c(u,d) > 0$ such that for all R large enough, $\mathbb{P}[X_{R,m} > 0] \geq c$ for every $m \geq 1$.*

Lemma 6.3. *For any R large enough, the sets*

$$\begin{aligned}
&\mathcal{L}_{\mathfrak{c}_{1,1}} \cap \mathcal{L}_{B^1_{R^1}}, \mathcal{L}_{B^1_{R^1}} \cap \mathcal{L}_{B^2_{R^1}}, \ldots, \mathcal{L}_{B^{d-4}_{R^1}} \cap \mathcal{L}_{B^{d-3}_{R^1}}, \mathcal{L}_{B^{d-3}_{R^1}} \cap \mathcal{L}_{\mathfrak{c}_{2,1}},\\
&\vdots\\
&\mathcal{L}_{\mathfrak{c}_{1,m}} \cap \mathcal{L}_{B^1_{R^m}}, \mathcal{L}_{B^1_{R^m}} \cap \mathcal{L}_{B^2_{R^m}}, \ldots, \mathcal{L}_{B^{d-4}_{R^m}} \cap \mathcal{L}_{B^{d-3}_{R^m}}, \mathcal{L}_{B^{d-3}_{R^m}} \cap \mathcal{L}_{\mathfrak{c}_{2,m}},\\
&\vdots
\end{aligned}$$

are mutually disjoint.

Before presenting the proofs of these lemmas, we will show how Theorem 6.1 follows from them.

Proof of Theorem 6.1 We observe that Lemma 6.3 implies that $\{X_{R,m}\}_{m\geq 1}$ is a sequence of independent random variables, as $X_{R,m}$ is defined only in terms of the restriction of ω to the sets

$$\mathcal{L}_{\mathfrak{c}_{1,m}} \cap \mathcal{L}_{B^1_{Rm}}, \mathcal{L}_{B^1_{Rm}} \cap \mathcal{L}_{B^2_{Rm}}, \ldots, \mathcal{L}_{B^{d-4}_{Rm}} \cap \mathcal{L}_{B^{d-3}_{Rm}}, \mathcal{L}_{B^{d-3}_{Rm}} \cap \mathcal{L}_{\mathfrak{c}_{2,m}}. \tag{6.1}$$

From this and Lemma 6.2 it follows via Borel-Cantelli, that

$$\mathbb{P}[\exists m \geq 1 : X_{R,m} > 0] = 1,$$

so that $\mathfrak{c}_1$ and $\mathfrak{c}_2$ are connected via at most $d-2$ cylinders almost surely. It only remains to show that a.s every pair of cylinders $\mathfrak{c}_1, \mathfrak{c}_2 \in \omega$ are also connected by at most $d-2$ cylinders. However, this is completely analogous to the proof in the case $d = 3$, so we will be brief. For lines $L_1, L_2 \in A(d, 1)$, let

$$E(L_1, L_2) = \{\text{Cdist}(\mathfrak{c}(L_1), \mathfrak{c}(L_2)) \leq d-2\},$$

and let

$$D = \cap_{(L_1,L_2)\in\omega^2_{\neq}} E(L_1, L_2).$$

As in the proof of the case $d = 3$ in Section 3.2, we can show that $\mathbb{P}[D] = 1$, which implies the theorem. $\square$

We proceed by proving Lemma 6.3 as it will also be useful in proving Lemma 6.2.

Proof of Lemma 6.3. Throughout the proof, we keep in mind that $|p_i| \leq R/2$ for $i = 1, ..., d$. The lemma will follow if we show the following six statements for $1 \leq i, j \leq d-4$, $m, n \geq 1$ and R large enough:

$$\mathcal{L}_{\mathfrak{c}_{1,m}} \cap \mathcal{L}_{B^1_{Rm}} \text{ and } \mathcal{L}_{\mathfrak{c}_{1,n}} \cap \mathcal{L}_{B^1_{Rn}} \text{ are disjoint when } m \neq n. \tag{6.2}$$

$$\mathcal{L}_{\mathfrak{c}_{2,m}} \cap \mathcal{L}_{B^{d-3}_{Rm}} \text{ and } \mathcal{L}_{\mathfrak{c}_{2,n}} \cap \mathcal{L}_{B^{d-3}_{Rn}} \text{ are disjoint when } m \neq n. \tag{6.3}$$

$$\mathcal{L}_{B^i_{Rm}} \cap \mathcal{L}_{B^{i+1}_{Rm}} \text{ and } \mathcal{L}_{B^j_{Rn}} \cap \mathcal{L}_{B^{j+1}_{Rn}} \text{ are disjoint unless } m = n \text{ and } i = j. \tag{6.4}$$

$$\mathcal{L}_{\mathfrak{c}_{1,m}} \cap \mathcal{L}_{B^1_{Rm}} \text{ and } \mathcal{L}_{B^i_{Rn}} \cap \mathcal{L}_{B^{i+1}_{Rn}} \text{ are disjoint.} \tag{6.5}$$

$$\mathcal{L}_{\mathfrak{c}_{2,m}} \cap \mathcal{L}_{B^{d-3}_{Rm}} \text{ and } \mathcal{L}_{B^i_{Rn}} \cap \mathcal{L}_{B^{i+1}_{Rn}} \text{ are disjoint.} \tag{6.6}$$

$$\mathcal{L}_{\mathfrak{c}_{1,n}} \cap \mathcal{L}_{B^1_{Rn}} \text{ and } \mathcal{L}_{\mathfrak{c}_{2,m}} \cap \mathcal{L}_{B^{d-3}_{Rm}} \text{ are disjoint.} \tag{6.7}$$

We start with (6.2). It suffices to show that $\mathcal{L}_{\mathfrak{c}_{1,m}} \cap \mathcal{L}_{\mathfrak{c}_{1,n}}$ and $\mathcal{L}_{\mathfrak{c}_{1,m}} \cap \mathcal{L}_{B^1_{Rm}}$ are disjoint when $m \neq n$. Suppose that $L \in \mathcal{L}_{\mathfrak{c}_{1,m}} \cap \mathcal{L}_{\mathfrak{c}_{1,n}}$. Let $(k_1, ..., k_d)$ be a directional vector of L. Observe that since L intersects both $\mathfrak{c}_{1,m}$ and $\mathfrak{c}_{1,n}$, we have that for some $z \in \mathfrak{c}_{1,m}$ and some $z' \in \mathfrak{c}_{1,n}$,

$$\frac{k_4}{k_1} = \frac{z_4 - z'_4}{z_1 - z'_1}.$$

Hence, (recall that the radii of the cylinders are $\sqrt{d}$)

$$\left|\frac{k_4}{k_1}\right| = \left|\frac{z_4 - z'_4}{z_1 - z'_1}\right| \leq \frac{\sqrt{d} - (-\sqrt{d})}{10\sqrt{d}} = 1/5. \tag{6.8}$$

Now suppose also that $L \in \mathcal{L}_{\mathfrak{c}_{1,m}} \cap \mathcal{L}_{B^1_{R^m}}$. We have that for some $z \in \mathfrak{c}_{1,m}$ and some $z' \in B^1_{R^m}$,

$$\left|\frac{k_4}{k_1}\right| = \left|\frac{z_4 - z_4'}{z_1 - z_1'}\right| \geq \left|\frac{p_4 + (N-1/2)R^m - \sqrt{d}}{-R^m/2 - (R^m/2 - 10\sqrt{d})}\right| \geq N - 2, \tag{6.9}$$

provided that R is large enough. Since (6.8) and (6.9) cannot both hold, we get (6.2).

The proof of (6.3) is similar to, but just slightly more technical than the proof of (6.2) since $\mathfrak{c}_2$ does not run along a coordinate axis. The details are left to the reader.

Next we establish (6.4). To this end, suppose that $L_1 \in \mathcal{L}_{B^i_{R^m}} \cap \mathcal{L}_{B^{i+1}_{R^m}}$, $L_2 \in \mathcal{L}_{B^j_{R^n}} \cap \mathcal{L}_{B^{j+1}_{R^n}}$ and $L_3 \in \mathcal{L}_{B^i_{R^m}} \cap \mathcal{L}_{B^j_{R^n}}$. We will show that L_1, L_2 and L_3 cannot all be the same line, by showing that at least one of their corresponding directional vectors is linearly independent of the two others. Then (6.4) follows. Let $x \in L_1 \cap B^i_{R^m}$, $x' \in L_1 \cap B^{i+1}_{R^m}$, $y \in L_2 \cap B^j_{R^n}$, $y' \in L_2 \cap B^{j+1}_{R^n}$, $z \in L_3 \cap B^i_{R^m}$ and $z' \in L_3 \cap B^j_{R^n}$. Let $v_1 = x - x'$, $v_2 = y - y'$ and $v_3 = z - z'$. Then v_i is a directional vector of L_i. Observe that for some $\alpha, \alpha' \in [-R^m/2, R^m/2]^d$

$$v_1 = (\alpha_1 - \alpha_1', ..., \alpha_{i+3} - \alpha_{i+3}' + NR^m, \alpha_{i+4} - \alpha_{i+4}' - NR^m, ..., \alpha_d - \alpha_d'),$$

and for some $\beta, \beta' \in [-R^n/2, R^n/2]^d$,

$$v_2 = (\beta_1 - \beta_1', ..., \beta_{j+3} - \beta_{j+3}' + NR^n, \beta_{j+4} - \beta_{j+4}' - NR^n, ..., \beta_d - \beta_d').$$

We will only make use of the vector v_3 in the case $i = j$. If $i = j$, then for some $\gamma \in [-R^m/2, R^m/2]^d$ and $\gamma' \in [-R^n/2, R^n/2]^d$,

$$v_3 = (\gamma_1 - \gamma_1', ..., \gamma_{i+3} - \gamma_{i+3}' + NR^m, \gamma_{i+4} - \gamma_{i+4}' - NR^n, ..., \gamma_d - \gamma_d').$$

We will now consider different cases.
Case $i \neq j$, m, n arbitrary: Without loss of generality, suppose $i > j$. Then

$$\left|\frac{(v_2)_{i+4}}{(v_1)_{i+4}}\right| = \left|\frac{\beta_{i+4} - \beta_{i+4}'}{\alpha_{i+4} - \alpha_{i+4}' - NR^m}\right| \leq \frac{R^n}{NR^m - R^m} = \frac{R^{n-m}}{N-1}. \tag{6.10}$$

On the other hand

$$\left|\frac{(v_2)_{j+3}}{(v_1)_{j+3}}\right| = \left|\frac{\beta_{j+3} - \beta_{j+3}' + NR^n}{\alpha_{j+3} - \alpha_{j+3}'}\right| \geq \frac{NR^n - R^n}{R^m} = (N-1)R^{n-m} \tag{6.11}$$

From (6.10) and (6.11) it follows that

$$\left|\frac{(v_2)_{i+4}}{(v_1)_{i+4}}\right| \neq \left|\frac{(v_2)_{j+3}}{(v_1)_{j+3}}\right|, \tag{6.12}$$

implying that v_1 and v_2 are linearly independent. Hence, L_1 and L_2 are different lines.
Case $i = j$, $m \neq n$: Without loss of generality assume that $n > m$. We get

$$\left|\frac{(v_2)_{j+4}}{(v_3)_{j+4}}\right| = \left|\frac{\beta_{j+4} - \beta_{j+4}' - NR^n}{\gamma_{j+4} - \gamma_{j+4}' - NR^n}\right| \leq \frac{NR^n + R^n}{NR^n - R^n/2 - R^m/2} \leq \frac{N+1}{N-1}, \tag{6.13}$$

using $n > m$ in the last inequality. We also have

$$\left|\frac{(v_2)_{j+3}}{(v_3)_{j+3}}\right| = \left|\frac{\beta_{j+3} - \beta'_{j+3} + NR^n}{\gamma_{j+3} - \gamma'_{j+3} + NR^m}\right| \geq \frac{NR^n - R^n}{NR^m + R^m/2 + R^n/2} \geq \frac{(N-1)R^n}{NR^m + R^n} \geq 0.9(N-1), \tag{6.14}$$

when R is large enough, since $n > m$. Hence, when R is large and by the choice of N,

$$\left|\frac{(v_2)_{j+4}}{(v_3)_{j+4}}\right| \neq \left|\frac{(v_2)_{j+3}}{(v_3)_{j+3}}\right|. \tag{6.15}$$

It follows that v_2 and v_3 are linearly independent for R large enough, implying that L_2 and L_3 are not the same line.

We move on to show (6.5). Let $L_4 \in \mathcal{L}_{\mathfrak{c}_{1,m}} \cap \mathcal{L}_{B^1_{R^m}}$ and $L_5 \in \mathcal{L}_{B^i_{R^n}} \cap \mathcal{L}_{B^{i+1}_{R^n}}$. Let $x \in L_4 \cap \mathfrak{c}_{1,m}$, $x' \in L_4 \cap B^1_{R^m}$, $y \in L_5 \cap B^i_{R^n}$ and $y' \in L_5 \cap B^{i+1}_{R^n}$. Let $v_4 = x - x'$ and $v_5 = y - y'$ be directional vectors for L_4 and L_5 respectively. Then, since $\mathfrak{c}_{1,m} \subset [-R^m/2, R^m/2]^d$, for some $\alpha, \alpha' \in [-R^m/2, R^m/2]^d$,

$$v_4 = (\alpha_1 - p_1 - \alpha'_1, ..., \alpha_4 - p_4 - \alpha'_4 - NR^m, ..., \alpha_d - p_d - \alpha'_d)$$

and for some $\beta, \beta' \in [-R^n/2, R^n/2]^d$,

$$v_5 = (\beta_1 - \beta'_1, ..., \beta_{i+3} + NR^n - \beta'_{i+3}, \beta_{i+4} - NR^n - \beta'_{i+4}, ..., \beta_d - \beta'_d).$$

Suppose first that $i = 1$. Then

$$\left|\frac{(v_4)_4}{(v_5)_4}\right| = \left|\frac{\alpha_4 - p_4 - \alpha'_4 - NR^m}{\beta_4 + NR^n - \beta'_4}\right| \geq \frac{(N-2)R^m}{(N+1)R^n} = \left(\frac{N-2}{N+1}\right) R^{m-n} \tag{6.16}$$

and

$$\left|\frac{(v_4)_5}{(v_5)_5}\right| = \left|\frac{\alpha_5 - \alpha'_5 - p_5}{\beta_5 - NR^n - \beta'_5}\right| \leq \frac{2R^m}{(N-1)R^n} = \left(\frac{2}{N-1}\right) R^{m-n}. \tag{6.17}$$

By the choice of N, we see that (6.16) and (6.17) are mutually exclusive, so (6.5) follows in the case $i = 1$. Suppose instead $2 \leq i \leq d-4$. Then for R large enough,

$$\left|\frac{(v_4)_4}{(v_5)_4}\right| = \left|\frac{\alpha_4 - p_4 - \alpha'_4 - NR^m}{\beta_4 - \beta'_4}\right| \geq (N-2)R^{m-n}, \tag{6.18}$$

and

$$\left|\frac{(v_4)_{i+3}}{(v_5)_{i+3}}\right| = \left|\frac{\alpha_{i+3} - p_{i+3} - \alpha'_{i+3}}{\beta_{i+3} + NR^n - \beta'_{i+3}}\right| \leq \frac{1.5R^{m-n}}{N-1}. \tag{6.19}$$

Since (6.18) and (6.19) are mutually exclusive, we get (6.5) also in the case $i \neq 1$.

The statement (6.6) follows analogously.

Next, we show (6.7). We do this by showing that

$$\mathcal{L}_{\mathfrak{c}_{1,n}} \cap \mathcal{L}_{B^1_{R^n}} \text{ and } \mathcal{L}_{\mathfrak{c}_{1,n}} \cap \mathcal{L}_{B^{d-3}_{R^m}} \text{ are disjoint.}$$

Suppose that $L_6 \in \mathcal{L}_{\mathfrak{c}_{1,n}} \cap \mathcal{L}_{B^1_{R^n}}$ and $L_7 \in \mathcal{L}_{\mathfrak{c}_{1,n}} \cap \mathcal{L}_{B^{d-3}_{R^m}}$. Let $x \in L_6 \cap \mathfrak{c}_{1,n}$, $x' \in L_6 \cap B^1_{R^n}$, $y \in L_7 \cap \mathfrak{c}_{1,n}$, and $y' \in L_7 \cap B^{d-3}_{R^m}$. Let $v_6 = x - x'$ and $v_7 = y - y'$ be directional vectors of L_6 and L_7 respectively. Observe that for some $\alpha \in [-R^n/2, R^n/2]^d$ and some $\beta \in [-R^m/2, R^m/2]^d$ we have

$$v_6 = (x_1 - p_1 - \alpha_1, ..., x_4 - p_4 - \alpha_4 - NR^n, ..., x_d - p_d - \alpha_d)$$

and

$$v_7 = (y_1 - p_1 - \beta_1, ..., y_d - p_d - \beta_d - NR^m).$$

Since $|x_i|, |y_i| \le \sqrt{d}$ for $i = 2, ..., d$, we get

$$\left|\frac{(v_6)_4}{(v_7)_4}\right| = \left|\frac{x_4 - p_4 - \alpha_4 - NR^n}{y_4 - p_4 - \beta_4}\right| \ge \frac{(N-2)R^n - \sqrt{d}}{\sqrt{d} + R^m} \ge (N-3)R^{n-m}, \tag{6.20}$$

provided that R is large enough. We also get

$$\left|\frac{(v_6)_d}{(v_7)_d}\right| = \left|\frac{x_d - p_d - \alpha_d}{y_d - p_d - \beta_d - NR^m}\right| \le \frac{R^n + \sqrt{d}}{(N-2)R^m - \sqrt{d}} \le \frac{R^{n-m}}{N-3}, \tag{6.21}$$

when R is large enough. Since (6.20) and (6.21) cannot both hold, we get (6.7). This completes the proof of the lemma. □

In much of what follows, whenever m can be considered fixed, we will simply write $B_1, B_i, \ldots$ instead of $B_{1,m}, B_{i,m}, \ldots$ Furthermore, we will say that $f(R) = \Omega(R^\alpha)$, if there exists two constants $0 < c < C < \infty$, such that $cR^\alpha \le f(R) \le CR^\alpha$ for all R large enough.

We will need the following lemma. We formulate it in exactly the way that we will use it, rather than in the most general way possible.

Lemma 6.4. *For any $d \ge 4, m \ge 1$ and boxes $B_1 \subset B^1_{R^m}, B_{d-3} \subset B^{d-3}_{R^m}$ of sidelength 1,*

$$\mathbb{P}[\mathfrak{c}_{1,m} \leftrightarrow B_1] = \Omega(R^{-m(d-2)}), \text{ and } \mathbb{P}[\mathfrak{c}_{2,m} \leftrightarrow B_{d-3}] = \Omega(R^{-m(d-2)}). \tag{6.22}$$

Furthermore, for any $d \ge 5, i = 1, \ldots, d-4$ and pair of boxes (B_i, B_{i+1}) of sidelengths 1 such that $B_i \subset B^i_{R^m}$ and $B_{i+1} \subset B^{i+1}_{R^m}$

$$\mathbb{P}[B_i \leftrightarrow B_{i+1}] = \Omega(R^{-m(d-1)}). \tag{6.23}$$

Remark: There are obvious similarities between this lemma and Lemma 4.1 and Proposition 4.2. These results will also be used explicitly in the proof.

Proof. We begin by proving (6.23). We note that the distance $d(B_i, B_{i+1})$ between the centers of B_i and B_{i+1} can be bounded by

$$\begin{aligned} &d(B_i, q_{i,m}) + d(q_{i,m}, q_{i+1,m}) + d(q_{i+1,m}, B_{i+1}) \\ &\quad \le \sqrt{d}R^m/2 + \sqrt{2}NR^m + \sqrt{d}R^m/2 = R^m(\sqrt{2}N + \sqrt{d}). \end{aligned}$$

As the boxes contain balls of radius 1, we can use Lemma 4.1 to conclude that

$$\mu_{d,1}(\mathcal{L}_{B_i} \cap \mathcal{L}_{B_{i+1}}) \ge \frac{c_1}{\Big(R^m(\sqrt{2}N + \sqrt{d})\Big)^{(d-1)}} = c_2 R^{-m(d-1)}.$$

Furthermore, since $N = 10d+1$, the constant c_2 depends only on d. Using that $1-e^{-x} \geq x/2$ for x small enough, we get that

$$\mathbb{P}[B_i \leftrightarrow B_{i+1}] = 1 - e^{-u\mu_{d,1}(\mathcal{L}_{B_i} \cap \mathcal{L}_{B_{i+1}})} \geq c_3 R^{-m(d-1)},$$

for R large enough. Here, c_3 depends only on d and u. A similar comment applies to all numbered constants below.

The distance $d(B_i, B_{i+1})$ can be bounded from below by

$$\begin{aligned} &d(q_{i,m}, q_{i+1,m}) - d(B_i, q_{i,m}) - d(q_{i+1,m}, B_{i+1}) \\ &\quad \geq \sqrt{2}NR^m - \sqrt{d}R^m/2 - \sqrt{d}R^m/2 = R^m(\sqrt{2}N - \sqrt{d}). \end{aligned}$$

Since the boxes B_i and B_{i+1} can be covered by a constant number of balls of radius 1, we get, using Lemma 4.1, that for R large enough,

$$\mu_{d,1}(\mathcal{L}_{B_i} \cap \mathcal{L}_{B_{i+1}}) \leq \frac{c_4}{\Big(R^m(\sqrt{2}N - \sqrt{d})\Big)^{(d-1)}} = c_5 R^{-m(d-1)},$$

so that

$$\mathbb{P}[B_i \leftrightarrow B_{i+1}] = 1 - e^{-u\mu_{d,1}(\mathcal{L}_{B_i} \cap \mathcal{L}_{B_{i+1}})} \leq c_5 R^{-m(d-1)}.$$

We proceed by proving (6.22) for the event $\{\mathfrak{c}_{1,m} \leftrightarrow B_1\}$. Trivially, $\mathbb{P}[\mathfrak{c}_{1,m} \leftrightarrow B_1] \leq \mathbb{P}[\mathfrak{c}_1 \leftrightarrow B_1]$. Furthermore, the distance between the center of B_1 and the centerline L_1 of $\mathfrak{c}_1$ is bounded below by

$$d(o, q_{1,m}) - d(q_{1,m}, B_1) \geq NR^m - \sqrt{d}R^m/2 = R^m(N - \sqrt{d}/2).$$

We can therefore use Proposition 4.2 to conclude that

$$\mu_{d,1}(\mathcal{L}_{\mathfrak{c}_1} \cap \mathcal{L}_{B_1}) \leq \frac{c_6}{\Big(R^m(N - \sqrt{d}/2)\Big)^{d-2}} = c_7 R^{-m(d-2)}.$$

Using that $1 - e^{-x} \leq x$ for every x, we get that for R large enough

$$\mathbb{P}[\mathfrak{c}_{1,m} \leftrightarrow B_1] \leq 1 - e^{-u\mu_{d,1}(\mathcal{L}_{\mathfrak{c}_1} \cap \mathcal{L}_{B_1})} \leq c_7 R^{-m(d-2)}.$$

In order to establish a lower bound for $\mathbb{P}[\mathfrak{c}_{1,m} \leftrightarrow B_1]$, we will use a similar technique to that of the proof of Proposition 4.2. To that end, consider the collection of balls $\mathcal{D}_m$, which is the set of balls $D_i \subset \mathfrak{c}_{1,m}$ of radius 1/8 with center $(i, 0 \dots, 0)$ for $i \in \mathbb{Z}$. Much as in the proof of Proposition 4.2, we note that

$$\bigcup_{D_i \in \mathcal{D}_m} \mathcal{L}_{D_i} \subset \mathcal{L}_{\mathfrak{c}_{1,m}},$$

so that

$$\bigcup_{D_i \in \mathcal{D}_m} (\mathcal{L}_{D_i} \cap \mathcal{L}_{B_1}) \subset \mathcal{L}_{\mathfrak{c}_{1,m}} \cap \mathcal{L}_{B_1}. \tag{6.24}$$

We will now show that

$$(\mathcal{L}_{D_i} \cap \mathcal{L}_{B_1})_{D_i \in \mathcal{D}_m} \text{ is a disjoint collection of sets of lines.} \tag{6.25}$$

Let $i, j \in \mathcal{D}_m$ where $i \neq j$ and assume that

$$L \in \mathcal{L}_{D_i} \cap \mathcal{L}_{D_j} \tag{6.26}$$

and

$$L \in \mathcal{L}_{D_i} \cap \mathcal{L}_{B_1}. \tag{6.27}$$

As usual, we write L on the form $L = \{t(k_1, ..., k_d) \, : \, -\infty < t < \infty\} + v$ for some $v \in \mathbb{R}^d$. As in the proof of Lemma 6.3, by considering the first and fourth coordinates of the intersections of L with D_i, D_j, we observe that if (6.26) holds, then

$$\left|\frac{k_4}{k_1}\right| \leq \frac{2/8}{|i-j| - 2/8} \leq \frac{2/8}{1 - 2/8} = \frac{1}{3}. \tag{6.28}$$

Similarly, in order for (6.27) to be satisfied, then

$$\left|\frac{k_4}{k_1}\right| \geq \frac{(N-1)R^m}{R^m} = N - 1. \tag{6.29}$$

We conclude that as $N = 10d + 1$, (6.28) and (6.29) cannot both hold, which proves (6.25). Furthermore, we note that for any $D_i \in \mathcal{D}_m$,

$$\begin{aligned} d(B_1, D_i) &\leq d(B_1, q_{1,m}) + d(q_{1,m}, p) + d(p, o) + d(o, D_i) \\ &\leq \sqrt{d}R^m/2 + \sqrt{d}R/2 + NR^m + R^m/2 \leq \sqrt{d}R^m + NR^m + R^m/2 \leq 2NR^m, \end{aligned}$$

where we use that $R \geq 2\max_{i=1,...,d} |p_i|$. Therefore, by Lemma 4.1 (and the remark that follows it), we get that

$$\mu_{d,1}(\mathcal{L}_{D_i} \cap \mathcal{L}_{B_1}) \geq \frac{c_8}{(2NR^m)^{d-1}}. \tag{6.30}$$

Proceeding, we have that

$$\begin{aligned} \mu_{d,1}(\mathcal{L}_{\mathfrak{c}_{1,m}} \cap \mathcal{L}_{B_1}) &\overset{(6.24)}{\geq} \mu_{d,1}\left(\bigcup_{D_i \in \mathcal{D}_m} (\mathcal{L}_{D_i} \cap \mathcal{L}_{B_1})\right) \\ &\overset{(6.25)}{=} \sum_{D_i \in \mathcal{D}_m} \mu_{d,1}(\mathcal{L}_{D_i} \cap \mathcal{L}_{B_1}) \overset{(6.30)}{\geq} |\mathcal{D}_m| \frac{c_8}{(2NR^m)^{d-1}} \geq c_9 R^{-m(d-2)}. \end{aligned}$$

It follows that $\mathbb{P}[\mathfrak{c}_{1,m} \leftrightarrow B_1] \geq c_9 R^{-m(d-2)}$. The corresponding statement for the event $\{\mathfrak{c}_{2,m} \leftrightarrow B_{d-3}\}$ follows analogously. $\square$

We proceed by proving Lemma 6.2. The proof itself contains an elementary geometric claim. The claim is very natural, but nevertheless requires a proof. In order not to disturb the flow of the proof proper, we will defer the proof of this claim till later. In what follows, we write $f(R) = O(R^\alpha)$ iff there exists a constant $C < \infty$, such that $|f(R)| \leq CR^\alpha$ for all R large enough. In particular, $O(1)$ refers to a function which is bounded for all R.

Proof of Lemma 6.2.

Fix $m \geq 1$. We will use the second moment method, i.e. that

$$\mathbb{P}[X_{R,m} > 0] \geq \frac{\mathbb{E}[X_{R,m}]^2}{\mathbb{E}[X_{R,m}^2]},$$

and proceed by bounding $\mathbb{E}[X_{R,m}^2]$. Letting $\vec{B'}_m := (B'_{1,m} \ldots, B'_{d-3,m})$ we have that

$$\mathbb{E}[X_{R,m}^2] = \mathbb{E}\left[\sum_{\vec{B}_m}\sum_{\vec{B'}_m} I(\mathcal{P}(\vec{B}_m))I(\mathcal{P}(\vec{B'}_m))\right]. \tag{6.31}$$

For fixed $\vec{B}_m$, we write ω as $\omega = \eta \cup \xi$, where $\eta = \eta_{\vec{B}_m}$ is a Poisson process of intensity measure $u\mu_{d,1}$ on the set $\left(\mathcal{L}_{\mathfrak{c}_{1,m}} \cap \mathcal{L}_{B_{1,m}}\right) \cup \left(\mathcal{L}_{B_{1,m}} \cap \mathcal{L}_{B_{2,m}}\right) \cup \cdots \cup \left(\mathcal{L}_{B_{d-3,m}} \cap \mathcal{L}_{\mathfrak{c}_{2,m}}\right)$ and where $\xi = \xi_{\vec{B}_m}$ is an independent Poisson process with the same intensity measure on the complement in $A(d,1)$. Furthermore, from now on the dependence on m will be dropped from the notation in order to avoid it from being overly cumbersome. That is, we will write $\vec{B}, B_1, B'_1, \ldots$, instead of $\vec{B}_m, B_{1,m}, B'_{1,m}, \ldots$ However, we will keep the notation $\mathfrak{c}_{i,m}$ as dropping the emphasis on m changes the meaning.

For any η, let $\eta(\mathfrak{c}_{1,m}, B_1), \eta(B_1, B_2), \ldots, \eta(B_{d-3}, \mathfrak{c}_{2,m})$ denote the restrictions of η onto $\mathcal{L}_{\mathfrak{c}_{1,m}} \cap \mathcal{L}_{B_1}, \mathcal{L}_{B_1} \cap \mathcal{L}_{B_2}, \ldots, \mathcal{L}_{B_{d-3}} \cap \mathcal{L}_{\mathfrak{c}_{2,m}}$ respectively. We define

$$S_1(B_1, \eta) := \{B'_1 \subset B^1_{R^m} : \mathcal{L}_{B'_1} \cap \eta(\mathfrak{c}_{1,m}, B_1) \neq \emptyset\},$$

$$S_2(B_1, \eta) := \{B'_1 \subset B^1_{R^m} : \mathcal{L}_{B'_1} \cap \eta(B_1, B_2) \neq \emptyset\},$$

and for $i = 2, \ldots, d-4$,

$$S_1(B_i, \eta) := \{B'_i \subset B^i_{R^m} : \mathcal{L}_{B'_i} \cap \eta(B_{i-1}, B_i) \neq \emptyset\},$$

$$S_2(B_i, \eta) := \{B'_i \subset B^i_{R^m} : \mathcal{L}_{B'_i} \cap \eta(B_i, B_{i+1}) \neq \emptyset\},$$

and finally

$$S_1(B_{d-3}, \eta) := \{B'_{d-3} \subset B^{d-3}_{R^m} : \mathcal{L}_{B'_{d-3}} \cap \eta(B_{d-4}, B_{d-3}) \neq \emptyset\},$$

$$S_2(B_{d-3}, \eta) := \{B'_{d-3} \subset B^{d-3}_{R^m} : \mathcal{L}_{B'_{d-3}} \cap \eta(B_{d-3}, \mathfrak{c}_{2,m}) \neq \emptyset\}.$$

Furthermore, for fixed $\vec{B}, \vec{B'}$ we define

$$\mathcal{S}_{\vec{B}}(\vec{B'}, \eta) := \{(B'_i, B'_{i+1}) : i \in \{1, \ldots, d-4\}, B'_i \in S_2(B_i, \eta) \text{ and } B'_{i+1} \in S_1(B_{i+1}, \eta)\}.$$

Thus, if there exists $L \in \eta(B_i, B_{i+1})$ connecting B'_i to B'_{i+1}, then $(B'_i, B'_{i+1}) \in \mathcal{S}_{\vec{B}}(\vec{B'}, \eta)$. Note that the reverse implication is not true. The reason is that it is geometrically possible that there exist lines $L_1 \in \eta(B_i, B_{i+1})$ such that $L_1 \cap B'_i \neq \emptyset$, $L_1 \cap B'_{i+1} = \emptyset$ and $L_2 \in \eta(B_i, B_{i+1})$ such that $L_2 \cap B'_i = \emptyset$, $L_2 \cap B'_{i+1} \neq \emptyset$. In this case, $B'_i \in S_2(B_i, \eta)$ because of L_1, while $B'_{i+1} \in S_1(B_{i+1}, \eta)$ because of L_2. Therefore, $|\mathcal{S}_{\vec{B}}(\vec{B'}, \eta)|$ provides an upper bound on the number of pairs (B'_i, B'_{i+1}) that are connected in η.

Furthermore, we let $\mathcal{E}_{\vec{B}}(\vec{B'}, \eta)$ be the subset of $\{B'_1, B'_{d-3}\}$ that includes B'_1 iff $B'_1 \in S_1(B_1, \eta)$ and similarly includes B'_{d-3} iff $B'_{d-3} \in S_2(B_{d-3}, \eta)$. Informally, $\mathcal{E}_{\vec{B}}(\vec{B'}, \eta)$ includes

B_1' iff B_1' intersects the lines in η connecting $\mathfrak{c}_{1,m}$ to B_1, so that also B_1' is connected to $\mathfrak{c}_{1,m}$. Given $\vec{B}, \vec{B}'$ and η, we let $N_{\vec{B}}(\vec{B}', \eta) := |\mathcal{E}_{\vec{B}}(\vec{B}', \eta)| + |\mathcal{S}_{\vec{B}}(\vec{B}', \eta)|$. We observe that in the special case $d = 4$, a straightforward adjustment of the above definitions is needed, since then we only consider one box $B^1_{R^m}$. We will make no further comment on this.

Noting that the event $\mathcal{P}(\vec{B})$ is determined by η alone, we get from (6.31) that

$$\begin{aligned}\mathbb{E}[X_{R,m}^2] &= \sum_{\vec{B}} \mathbb{E}\left[\sum_{\vec{B}'} I(\mathcal{P}(\vec{B}))I(\mathcal{P}(\vec{B}'))\right] \\ &= \sum_{\vec{B}} \mathbb{P}[\mathcal{P}(\vec{B})]\mathbb{E}\left[\sum_{\vec{B}'} I(\mathcal{P}(\vec{B}'))\middle| \eta \in \mathcal{P}(\vec{B})\right] \\ &= \sum_{\vec{B}} \mathbb{P}[\mathcal{P}(\vec{B})]\mathbb{E}\left[\mathbb{E}\left[\sum_{\vec{B}'} I(\mathcal{P}(\vec{B}'))\middle|\eta\right]\middle| \eta \in \mathcal{P}(\vec{B})\right] \\ &= \sum_{\vec{B}} \mathbb{P}[\mathcal{P}(\vec{B})]\mathbb{E}\left[\sum_{\vec{B}':N_{\vec{B}}(\vec{B}',\eta)=0} \mathbb{P}[\mathcal{P}(\vec{B}') \mid \eta] + \sum_{\vec{B}':N_{\vec{B}}(\vec{B}',\eta)>0} \mathbb{P}[\mathcal{P}(\vec{B}') \mid \eta]\middle| \eta \in \mathcal{P}(\vec{B})\right].\end{aligned}$$

Note that when we condition on η, the only randomness left is in ξ.

We observe that for $\eta \in \mathcal{P}(\vec{B})$ and when $N_{\vec{B}}(\vec{B}', \eta) = 0$,

$$\mathbb{P}[\mathcal{P}(\vec{B}') \mid \eta] \leq \mathbb{P}[\mathcal{P}(\vec{B}')]. \tag{6.32}$$

The inequality follows since when $N_{\vec{B}}(\vec{B}', \eta) = 0$, there does not exist $L \in \eta$ such that $L \in \big(\mathcal{L}_{\mathfrak{c}_{1,m}} \cap \mathcal{L}_{B_1'}\big) \cup \big(\mathcal{L}_{B_1'} \cap \mathcal{L}_{B_2'}\big) \cup \cdots$. However, it could be that η gives partial knowledge of the *absence* of lines in ω connecting for instance B_1' to B_2'. This happens if there exists $L \in \mathcal{L}_{B_1} \cap \mathcal{L}_{B_2}$ such that $L \notin \eta$ but $L \in \mathcal{L}_{B_1'} \cap \mathcal{L}_{B_2'}$. Continuing, we see that

$$\begin{aligned}&\mathbb{E}[X_{R,m}^2] \qquad\qquad (6.33)\\ &\overset{(6.32)}{\leq} \sum_{\vec{B}} \mathbb{P}[\mathcal{P}(\vec{B})]\mathbb{E}\left[\sum_{\vec{B}'} \mathbb{P}[\mathcal{P}(\vec{B}')] + \sum_{\vec{B}':N_{\vec{B}}(\vec{B}',\eta)>0} \mathbb{P}[\mathcal{P}(\vec{B}') \mid \eta]\middle| \eta \in \mathcal{P}(\vec{B})\right] \\ &= \mathbb{E}[X_{R,m}]^2 + \sum_{\vec{B}} \mathbb{P}[\mathcal{P}(\vec{B})]\mathbb{E}\left[\sum_{\vec{B}':N_{\vec{B}}(\vec{B}',\eta)>0} \mathbb{P}[\mathcal{P}(\vec{B}') \mid \eta]\middle| \eta \in \mathcal{P}(\vec{B})\right].\end{aligned}$$

We will proceed by analysing and bounding

$$\mathbb{E}\left[\sum_{\vec{B}':N_{\vec{B}}(\vec{B}',\eta)>0} \mathbb{P}[\mathcal{P}(\vec{B}') \mid \eta]\middle| \eta \in \mathcal{P}(\vec{B})\right],$$

for any fixed $\vec{B}$.

For $k_1, \ldots, k_{d-2} \geq 1$, using Lemma 6.3 and Lemma 6.4, and that the number of lines are Poisson distributed,

$$
\begin{aligned}
&\mathbb{P}[\mathfrak{c}_{1,m} \overset{k_1}{\leftrightarrow} B_1 \overset{k_2}{\leftrightarrow}, \ldots, \overset{k_{d-3}}{\leftrightarrow} B_{d-3} \overset{k_{d-2}}{\leftrightarrow} \mathfrak{c}_{2,m} \mid \mathcal{P}(\vec{B})] \qquad (6.34)\\
&\quad = \mathbb{P}[\mathfrak{c}_{1,m} \overset{k_1}{\leftrightarrow} B_1 \mid \mathfrak{c}_{1,m} \leftrightarrow B_1]\mathbb{P}[B_1 \overset{k_2}{\leftrightarrow} B_2 \mid B_1 \leftrightarrow B_2] \times \cdots \times \mathbb{P}[B_{d-3} \overset{k_{d-2}}{\leftrightarrow} \mathfrak{c}_{2,m} \mid B_{d-3} \leftrightarrow \mathfrak{c}_{2,m}]\\
&\quad = O(R^{-m(k_1-1)(d-2)})O(R^{-m(k_2-1)(d-1)}) \times \cdots \times O(R^{-m(k_{d-2}-1)(d-2)}),
\end{aligned}
$$

where we used that

$$
\begin{aligned}
&\mathbb{P}[\mathfrak{c}_{1,m} \overset{k_1}{\leftrightarrow} B_1 \mid \mathfrak{c}_{1,m} \leftrightarrow B_1]\\
&\quad = \frac{(u\mu_{d,1}(\mathcal{L}_{\mathfrak{c}_{1,m}} \cap \mathcal{L}_{B_1}))^{k_1} \exp(-u\mu_{d,1}(\mathcal{L}_{\mathfrak{c}_{1,m}} \cap \mathcal{L}_{B_1}))}{k_1!(1 - \exp(-u\mu_{d,1}(\mathcal{L}_{\mathfrak{c}_{1,m}} \cap \mathcal{L}_{B_1}))}\\
&\quad \leq \frac{2(u\mu_{d,1}(\mathcal{L}_{\mathfrak{c}_{1,m}} \cap \mathcal{L}_{B_1}))^{k_1-1}}{k_1!} = O(R^{-m(k_1-1)(d-2)}).
\end{aligned}
$$

The inequality above holds (as in the proof of Lemma 6.4) for R large enough since $1 - e^{-x} \geq x/2$ holds for x small enough. Let

$$
\mathcal{P}_d(\vec{B}) := \bigcup_{1 \leq k_i \leq d \forall i} \{\mathfrak{c}_{1,m} \overset{k_1}{\leftrightarrow} B_1 \overset{k_2}{\leftrightarrow}, \ldots, \overset{k_{d-3}}{\leftrightarrow} B_{d-3} \overset{k_{d-2}}{\leftrightarrow} \mathfrak{c}_{2,m}\}.
$$

Using (6.34), we note that

$$
\mathbb{P}[\mathcal{P}_d(\vec{B})^c \mid \mathcal{P}(\vec{B})] = O(R^{-md(d-2)}).
$$

Therefore, for any positive random variable $Z(\eta)$ bounded above by some finite number $|Z|$, we get that

$$
\begin{aligned}
&\mathbb{E}[Z \mid \eta \in \mathcal{P}(\vec{B})]\\
&\quad = \mathbb{E}[Z \mid \eta \in \mathcal{P}_d(\vec{B})]\mathbb{P}[\mathcal{P}_d(\vec{B}) \mid \mathcal{P}(\vec{B})] + \mathbb{E}[Z \mid \eta \in \mathcal{P}_d(\vec{B})^c, \eta \in \mathcal{P}_d(\vec{B})]\mathbb{P}[\mathcal{P}_d(\vec{B})^c \mid \mathcal{P}(\vec{B})]\\
&\quad \leq \mathbb{E}[Z \mid \eta \in \mathcal{P}_d(\vec{B})] + |Z|O(R^{-md(d-2)}).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{\vec{B}' : N_{\vec{B}}(\vec{B}', \eta) > 0} \mathbb{P}[\mathcal{P}(\vec{B}') \mid \eta] \middle| \eta \in \mathcal{P}(\vec{B})\right] \qquad (6.35)\\
&\quad \leq \mathbb{E}\left[\sum_{\vec{B}' : N_{\vec{B}}(\vec{B}', \eta) > 0} \mathbb{P}[\mathcal{P}(\vec{B}') \mid \eta] \middle| \eta \in \mathcal{P}_d(\vec{B})\right] + O(R^{-md(d-2)}) \sum_{\vec{B}'} 1\\
&\quad = O(R^{-md}) + \mathbb{E}\left[\sum_{\vec{B}' : N_{\vec{B}}(\vec{B}', \eta) > 0} \mathbb{P}[\mathcal{P}(\vec{B}') \mid \eta] \middle| \eta \in \mathcal{P}_d(\vec{B})\right],
\end{aligned}
$$

since the number of sequences $\vec{B}'$ equals $R^{md(d-3)}$.

Consider now

$$\sum_{\vec{B}':N_{\vec{B}}(\vec{B}',\eta)>0} \mathbb{P}[\mathcal{P}(\vec{B}') \mid \eta] \tag{6.36}$$

for some fixed $\vec{B}$ and $\eta \in \mathcal{P}_d(\vec{B})$. As before, after conditioning on η, the randomness left is in ξ. In order to get a sufficiently good estimate of (6.36), we will have to divide the sum into parts depending on the values of $|\mathcal{E}_{\vec{B}}(\vec{B}',\eta)|$ and $|\mathcal{S}_{\vec{B}}(\vec{B}',\eta)|$. We will then proceed by counting the number of configurations $\vec{B}'$ corresponding to specific values of $|\mathcal{E}_{\vec{B}}(\vec{B}',\eta)|$ and $|\mathcal{S}_{\vec{B}}(\vec{B}',\eta)|$, and estimate the corresponding probability $\mathbb{P}[\mathcal{P}(\vec{B}') \mid \eta]$. We will start with the latter as that is the easiest part.

Let $l = |\mathcal{S}_{\vec{B}}(\vec{B}',\eta)|$ and $k = |\mathcal{E}_{\vec{B}}(\vec{B}',\eta)|$. Informally, if $k = 0$, then neither of the events $\mathfrak{c}_{1,m} \leftrightarrow B'_1$ and $B'_{d-3} \leftrightarrow \mathfrak{c}_{2,m}$ occur in η. If $k = 1$, then exactly one of them occur in η, while if $k = 2$, both of them occur in η. Similarly, the number of connections $B'_i \leftrightarrow B'_{i+1}$ that occur in η are at most l. Therefore, in order for $\mathcal{P}(\vec{B}')$ to occur, the remaining connections must occur in ξ. Hence, using Lemma 6.4, we get that

$$\mathbb{P}[\mathcal{P}(\vec{B}') \mid \eta] \leq \Omega(R^{-m(2-k)(d-2)})\Omega(R^{-m(d-4-l)(d-1)}), \tag{6.37}$$

where we made use of Lemma 6.3 again. The reason that there is an inequality rather than an equality follows much as in (6.32).

In order to bound the number of configurations $\vec{B}'$ such that $l = |\mathcal{S}_{\vec{B}}(\vec{B}',\eta)|$ and $k = |\mathcal{E}_{\vec{B}}(\vec{B}',\eta)|$, we will use the following claim.

Claim: For $\eta \in \mathcal{P}_d(\vec{B})$, $j = 1,2$ and $i = 1,\dots,d-3$ we have that $|S_j(B_i,\eta)| = O(R^m)$. Furthermore, for $N = 10d+1$, $|S_1(B_i,\eta) \cap S_2(B_i,\eta)| = O(1)$ for every $i = 1,\dots,d-3$.

This claim is very natural. Consider for instance the box B_1. Since $\eta \in \mathcal{P}_d(\vec{B})$ there are at least one and at most d lines in both $\eta(\mathfrak{c}_{1,m}, B_1)$ and $\eta(B_1,B_2)$. From this, it follows that there can only be a linear number (in the sidelength of $B^1_{R^m}$) of other boxes B'_1 that intersects $\eta(\mathfrak{c}_{1,m}, B_1)$ *or* $\eta(B_1,B_2)$. Furthermore, due to the positions of the boxes $B^1_{R^m}$ and $B^2_{R^m}$, the lines in $\eta(\mathfrak{c}_{1,m}, B_1)$ will have a large angle to the lines in $\eta(B_1,B_2)$. Therefore, there cannot be more than some constant number of boxes $B'_1 \subset B^1_{R^m}$ that intersects the lines in $\eta(\mathfrak{c}_{1,m}, B_1)$ *and* $\eta(B_1,B_2)$.

We will have to consider the different cases $k = 0,1,2$ separately. Therefore, assume first that $k = 0$. Recall that we are only considering $N_{\vec{B}}(\vec{B}',\eta) > 0$ and thus $l > 0$ when $k = 0$. We have that

$$\begin{aligned} &|\{\vec{B}' : |\mathcal{E}_{\vec{B}}(\vec{B}')| = 0, |\mathcal{S}_{\vec{B}}(\vec{B}')| = l\}| \\ &\quad = O(R^{md(d-3)})O(R^{-m(l+1)(d-1)}R^{-m(l-1)}) = O(R^{m(d^2+2-d(l+4))}). \end{aligned} \tag{6.38}$$

To see this, consider first $l = 1$ and assume that only $(B'_1, B'_2) \in \mathcal{S}_{\vec{B}}(\vec{B}')$. Then, B'_1 and B'_2 must be placed along the line(s) in η connecting B_1 to B_2. By the claim, the number of ways that the pair (B'_1, B'_2) can be chosen decreases from $O(R^{2md})$ to $O(R^{2m})$, thus decreasing the total number of ways that $\vec{B}'$ can be chosen by a factor of $O(R^{2m}/R^{2md}) = O(R^{-2m(d-1)})$.

Consider now $l = 2$ and assume again that $(B'_1, B'_2) \in \mathcal{S}_{\vec{B}}(\vec{B}')$. If it is the case that $(B'_3, B'_4) \in \mathcal{S}_{\vec{B}}(\vec{B}')$ then the total number of ways that the entire sequence $\vec{B}'$ can be chosen must be of order $O(R^{md(d-3)})O(R^{-4m(d-1)})$. However, if instead it is the case that $(B_2', B_3') \in \mathcal{S}_{\vec{B}}(\vec{B'})$, then the total number of ways that the entire sequence $\vec{B'}$ can be chosen must be of order $O(R^{md(d-3)})O(R^{-3m(d-1)})O(R^{-m})$. The first two factors are explained as above, while the third factor reflects that the box B_2' must in fact belong to a collection of at most constant size (again using the claim). Continuing in the same way gives (6.38).

Hence, we conclude, using (6.37) and (6.38), that

$$\sum_{\vec{B'}:|\mathcal{E}_{\vec{B}}(\vec{B'})|=0,|\mathcal{S}_{\vec{B}}(\vec{B'})|=l} \mathbb{P}[\mathcal{P}(\vec{B'}) \mid \eta]$$
$$= O(R^{m(d^2+2-d(l+4))})O(R^{-2m(d-2)})O(R^{-m(d-4-l)(d-1)}) = O(R^{m(2-l-d)}),$$

so that

$$\sum_{l=1}^{d-4} \sum_{\vec{B'}:|\mathcal{E}_{\vec{B}}(\vec{B'})|=0,|\mathcal{S}_{\vec{B}}(\vec{B'})|=l} \mathbb{P}[\mathcal{P}(\vec{B'}) \mid \eta] = O(R^{m(1-d)}). \tag{6.39}$$

Consider now $k = 1$ and assume without loss of generality that $\mathcal{E}_{\vec{B}}(\vec{B'}) = \{B_1'\}$. The number of sequences $\vec{B'}$ satisfying $l = 0$, is of course $O(R^{md(d-3)})O(R^{-m(d-1)})$. Furthermore, arguing as in the case $k = 0$, we see that

$$|\{\vec{B'} : |\mathcal{E}_{\vec{B}}(\vec{B'})| = 1, |\mathcal{S}_{\vec{B}}(\vec{B'})| = l\}| \tag{6.40}$$
$$= O(R^{md(d-3)})O(R^{-m(l+1)(d-1)}R^{-ml}) = O(R^{m(d^2+1-d(l+4))}).$$

Therefore, using (6.37) and (6.40),

$$\sum_{\vec{B'}:|\mathcal{E}_{\vec{B}}(\vec{B'})|=1,|\mathcal{S}_{\vec{B}}(\vec{B'})|=l} \mathbb{P}[\mathcal{P}(\vec{B'}) \mid \eta]$$
$$= O(R^{m(d^2+1-d(l+4))})O(R^{-m(d-2)})O(R^{-m(d-4-l)(d-1)}) = O(R^{-m(l+1)}),$$

so that

$$\sum_{l=0}^{d-4} \sum_{\vec{B'}:|\mathcal{E}_{\vec{B}}(\vec{B'})|=1,|\mathcal{S}_{\vec{B}}(\vec{B'})|=l} \mathbb{P}[\mathcal{P}(\vec{B'}) \mid \eta] = O(R^{-m}). \tag{6.41}$$

Finally, we consider $k = 2$. The number of sequences $\vec{B'}$ satisfying $l = 0$, is of course $O(R^{md(d-3)})O(R^{-2m(d-1)})$. Furthermore, arguing as in the other cases, we see that for $l < d - 4$,

$$|\{\vec{B'} : |\mathcal{E}_{\vec{B}}(\vec{B'})| = 2, |\mathcal{S}_{\vec{B}}(\vec{B'})| = l\}| \tag{6.42}$$
$$= O(R^{md(d-3)})O(R^{-m(l+2)(d-1)}R^{-ml}) = O(R^{m(d^2+2-d(l+5))}).$$

Using (6.37) and (6.42)

$$\sum_{\vec{B'}:|\mathcal{E}_{\vec{B}}(\vec{B'})|=2,|\mathcal{S}_{\vec{B}}(\vec{B'})|=l} \mathbb{P}[\mathcal{P}(\vec{B'}) \mid \eta]$$
$$= O(R^{m(d^2+2-d(l+5))})O(R^{-m(d-4-l)(d-1)}) = O(R^{-m(l+2)}).$$

Furthermore,

$$|\{\vec{B}' : |\mathcal{E}_{\vec{B}}(\vec{B}')| = 2, |\mathcal{S}_{\vec{B}}(\vec{B}')| = d-4\}| \\ = O(R^{md(d-3)})O(R^{-m(d-3)(d-1)}R^{-m(d-3)}) = O(1),$$

so that

$$\sum_{l=0}^{d-4} \sum_{\vec{B}':|\mathcal{E}_{\vec{B}}(\vec{B}')|=2,|\mathcal{S}_{\vec{B}}(\vec{B}')|=l} \mathbb{P}[\mathcal{P}(\vec{B}') \mid \eta] = O(1) + \sum_{l=0}^{d-5} O(R^{-m(l+2)}) = O(1). \tag{6.43}$$

Combining (6.39),(6.41) and (6.43), we get that

$$\sum_{\vec{B}':N_{\vec{B}}(\vec{B}',\eta)>0} \mathbb{P}[\mathcal{P}(\vec{B}') \mid \eta] = O(1). \tag{6.44}$$

Combining (6.33),(6.35) and (6.44) we see that there exists a constant C such that for all R large enough,

$$\mathbb{E}[X_{R,m}^2] \leq \mathbb{E}[X_{R,m}]^2 + C\mathbb{E}[X_{R,m}].$$

Furthermore, by Lemma 6.4 there exists a constant $C' > 0$ such that for every $\vec{B}$, $\mathbb{P}[\mathcal{P}(\vec{B})] \geq C'R^{-2m(d-2)}R^{-m(d-4)(d-1)}$. Therefore,

$$\mathbb{E}[X_{R,m}] \geq R^{md(d-3)}C'R^{-2m(d-2)}R^{-m(d-4)(d-1)} = C',$$

so that

$$\frac{\mathbb{E}[X_{R,m}]^2}{\mathbb{E}[X_{R,m}^2]} \geq \frac{\mathbb{E}[X_{R,m}]^2}{\mathbb{E}[X_{R,m}]^2 + C\mathbb{E}[X_{R,m}]} = \frac{1}{1 + C\mathbb{E}[X_{R,m}]^{-1}} \geq \frac{1}{1 + C/C'} > 0,$$

for all R large enough. This proves the statement. $\square$

We will now prove the claim.

Proof of Claim. Recall that $\eta \in \mathcal{P}_d(\vec{B})$, so that the first part, i.e. $|S_j(B_i, \eta)| = O(R^m)$ follows from the fact that the number of lines in $\eta(\mathfrak{c}_{1,m}, B_1), \eta(B_1, B_2), \ldots$ are bounded by d.

Let R be so large that $R/2 > \max_{i=3,\ldots,d} |p_i| + 1$. Consider first any pair of lines L_1, L_2 such that $L_1 \in \mathcal{L}_{\mathfrak{c}_{1,m}} \cap \mathcal{L}_{B_1}$ and $L_2 \in \mathcal{L}_{B_1} \cap \mathcal{L}_{B_2}$ where $B_1 \subset B^1_{R^m}$ and $B_2 \subset B^2_{R^m}$. Let $x \in L_1 \cap \mathfrak{c}_{1,m}$ and $x' \in L_1 \cap B_1$, and note that $v_1 = x - x'$ is a directional vector for L_1. In the same way, letting $y \in L_2 \cap B_1$ and $y' \in L_2 \cap B_2$, $v_2 = y - y'$ becomes a directional vector for L_2. Furthermore, we can write $x' = q_{1,m} + \alpha$ where $\alpha \in [-R^m/2, R^m/2]^d$, $y = q_{1,m} + \alpha + \gamma$ where $\gamma \in [-1, 1]^d$ and $y' = q_{2,m} + \beta$ where $\beta \in [-R^m/2, R^m/2]^d$.

Considering the angle θ between the lines L_1 and L_2, we have that

$$\cos\theta = \frac{\langle v_1, v_2 \rangle}{|v_1||v_2|}. \tag{6.45}$$

By showing that $|\cos\theta|$ is uniformly bounded away from 1 when $N = 10d + 1$ and R is large, the second part of the claim is established. We note that

$$|v_1|^2 = |p + NR^m e_4 + \alpha - x| \geq |\alpha_4 + p_4 + NR^m - x_4|^2 \geq (N-1)^2 R^{2m},$$

and similarly,

$$|v|^2 = |q_{2,m}+\beta-q_{1,m}-\alpha-\gamma|^2 \geq |\beta_4-NR^m-\alpha_4-\gamma_4|^2+|\beta_5+NR^m-\alpha_5-\gamma_5|^2 \geq 2((N-1)R^m-1)^2.$$

Furthermore,

$$\begin{aligned}\langle v_1, v_2\rangle &= \langle q_{1,m} + \alpha - x, q_{2,m} + \beta - q_{1,m} - \alpha - \gamma\rangle \\ &= \sum_{i\neq 4,5} (p_i + \alpha_i - x_i)(\beta_i - \alpha_i - \gamma_i) + (p_4 + NR^m + \alpha_4 - x_4)(\beta_4 - \alpha_4 - NR^m - \gamma_4) \\ &\quad +(p_5 + \alpha_5 - x_5)(\beta_5 + NR^m - \alpha_5 - \gamma_5),\end{aligned}$$

so that using $|x_1| \leq R^m/2$ and $|x_i| \leq 1$ for $i \neq 1$,

$$\begin{aligned}&|\langle v_1, v_2\rangle| \\ &\quad \leq 2R^m(R^m + 1) + (d - 3)(R^m + 1)^2 + ((N + 1)R^m + 1)^2 + (R^m + 1)((N + 1)R^m + 1) \\ &\quad = R^{2m}(d + (N + 1)^2 + N) + O(R^m).\end{aligned}$$

Therefore, we get from (6.45) and that $N = 10d + 1$,

$$\begin{aligned}|\cos\theta| &\leq \frac{R^{2m}(d + (N + 1)^2 + N) + O(R^m)}{(N - 1)R^m\sqrt{2}((N - 1)R^m - 1)} \\ &= \frac{(d + (10d + 2)^2 + 10d + 1) + O(R^{-m})}{10d\sqrt{2}(10d - O(R^{-m}))} \\ &= \frac{(100d^2 + 51d + 5) + O(R^{-m})}{100d^2\sqrt{2}} \left(1 + O(R^{-m})\right).\end{aligned} \tag{6.46}$$

Since $(100d^2 + 51d + 5)/100d^2$ is decreasing in d, we can estimate the RHS by inserting $d = 4$ in which case we get the bound

$$\frac{1809}{1600\sqrt{2}} \left(1 + O(R^{-m})\right) \leq \frac{1.14}{\sqrt{2}} \left(1 + O(R^{-m})\right), \tag{6.47}$$

which is uniformly bounded away from 1 for R large enough. Therefore, the angle between L_1 and L_2 must be uniformly (in d and in the choice of L_1, L_2) bounded away from 0 for all R large enough. From this, the claim follows for all such L_1 and L_2.

The remaining cases (i.e. when $L_2 \in \mathcal{L}_{B_1} \cap \mathcal{L}_{B_2}$ and $L_3 \in \mathcal{L}_{B_2} \cap \mathcal{L}_{B_3}$ etc) are handled in the same way. The final case when $L_{d-3} \in \mathcal{L}_{B_{d-4}} \cap \mathcal{L}_{B_{d-3}}$ and $L_{d-2} \in \mathcal{L}_{B_{d-3}} \cap \mathfrak{c}_{2,m}$ is somewhat *more* technical than the current case due to the position of the cylinder $\mathfrak{c}_2$. However, the approach is completely analogous. □

Remark: As noted already in the introduction, a unified approach to the proof of Theorem 1.1 could be considered desirable. It would then be natural to define $X_{R,m} = I(\mathfrak{c}_{1,m} \leftrightarrow \mathfrak{c}_{2,m})$ and try to proceed along the lines of this section. Consider therefore first Lemma 6.3. When trying to prove a version of this lemma for $d = 3$, we note that $\mathcal{L}_{\mathfrak{c}_{1,m}} \cap \mathcal{L}_{\mathfrak{c}_{2,m}}$ are *not* necessarily disjoint for different values of m. This happens when the centerlines of $\mathfrak{c}_1$ and $\mathfrak{c}_2$ are close. Therefore, one would have to deal with this in some different way. Furthermore, when proving a version of Lemma 6.2 for $d = 3$, one could attempt an approach similar to that of the proof of (6.22). However, when trying to prove something analogous to the statement that (6.26) and (6.27) cannot both hold, we see again that in fact they *can* both be true when the centerlines of $\mathfrak{c}_1$ and $\mathfrak{c}_2$ are close. Instead, one would probably need to use an argument in the spirit of the proof of Theorem 3.1. Thus, it seems that the easiest way to obtain a proof for $d = 3$ is by Theorem 3.1.

Acknowledgments: The authors would like to thank an anonymous referee for many valuable comments and suggestions, and I. Benjamini for suggesting the model.

This paper was written while J.T. worked at the Department of Mathematics at Uppsala University.

References

[1] I. Benjamini. Personal communication.

[2] I. Benjamini, H. Kesten, Y. Peres and O. Schramm. Geometry of the uniform spanning forest: Transitions in dimensions 4, 8, 12,... *Ann. of Math. (2),* 160(2):465-491, 2004.

[3] G. Grimmett. Percolation (Springer), 1999.

[4] T. Hara, R. van der Hofstad and G. Slade. Critical two-point functions and the lace expansion spread-out high-dimensional percolation and related models. *Ann. Probab.*, 31:349-408.

[5] M. Hilario, V. Sidoravicius, A. Teixeira. Cylinders' percolation in three dimensions. Preprint, 2012.

[6] H. Lacoin and J. Tykesson. On the easiest way to connect k points in the random interlacements process. *ALEA*, 10:505-524, 2013.

[7] S. Li. Concise formulas for the area and volume of a hyperspherical cap. *Asian Journal of Mathematics and Statistics,* 4(1):66-70, 2011.

[8] R. Meester and Rahul Roy. Continuum percolation (Cambridge University Press), 1996.

[9] E. Procaccia and J. Tykesson. Geometry of the random interlacement. *Electron. Comm. Probab.*, 16:528-544, 2011.

[10] B. Ráth and A. Sapozhnikov. Connectivity properties of random interlacement and intersection of random walks. *ALEA*, 9:67-83.

[11] R. Schneider and W. Weil, Stochastic and Integral Geometry (Springer), 2008.

[12] A.-S. Sznitman. Vacant set of random interlacements and percolation. *Ann. of Math. (2),* 171(3):2039-2087, 2010.

[13] J. Tykesson and D. Windisch. Percolation in the vacant set of Poisson cylinders. *Probab. Theory Relat. Fields,* 154:165-191, 2012.