SDEs

Des Higham
Department of Mathematics and Statistics
University of Strathclyde
http://www.maths.strath.ac.uk/~aas96106/

SDEs

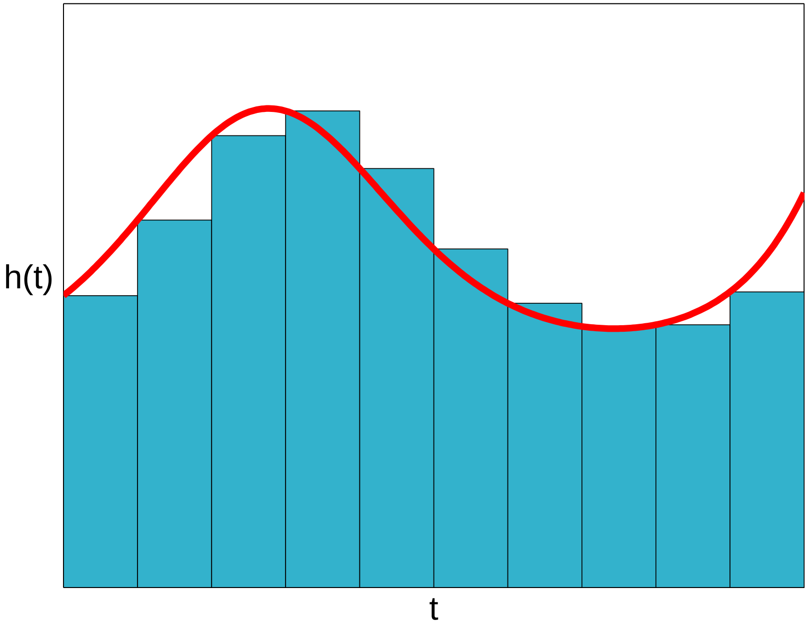
- Ito stochastic integrals
- Ito SDEs
- Examples of SDEs

Integration

For deterministic $h : \mathbb{R} \rightarrow \mathbb{R}$,
$t_i = i\delta t$, $\delta t = T/L$,
Riemann sum based on left endpoints

$$\sum_{i=0}^{L-1} h(t_i)(t_{i+1} - t_i)$$

Integral $\int_0^T h(t)\, dt$ defined by $\delta t \rightarrow 0$



Stochastic Integration

Integrate with respect to Brownian motion:
Riemann sum based on left endpoints

$$\sum_{i=0}^{L-1} h(t_i)(\mathbf{W}(t_{i+1}) - \mathbf{W}(t_i))$$

Integral $\int_0^T h(t)\, d\mathbf{W}(t)$ defined by $\delta t \to 0$

Digression: $\sum_{i=0}^{L-1} \delta \mathbf{W}_i^2$

$$\mathbb{E}\left[\sum_{i=0}^{L-1} \delta \mathbf{W}_i^2\right] = \sum_{i=0}^{L-1} \mathbb{E}\left[\delta \mathbf{W}_i^2\right] = L\delta t = T$$

$$\begin{aligned}
\mathbb{E}\left[\left(\sum_{i=0}^{L-1} \delta \mathbf{W}_i^2\right)^2\right] &= \sum_{i=0}^{L-1}\sum_{j=0}^{L-1} \mathbb{E}\left[\delta \mathbf{W}_i^2 \delta \mathbf{W}_j^2\right] \\
&= \sum_{i=0, i\neq j}^{L-1}\sum_{j=0}^{L-1} \mathbb{E}\left[\delta \mathbf{W}_i^2 \delta \mathbf{W}_j^2\right] + \sum_{i=0}^{L-1} \mathbb{E}\left[\delta \mathbf{W}_i^4\right] \\
&= \sum_{i=0, i\neq j}^{L-1}\sum_{j=0}^{L-1} \mathbb{E}\left[\delta \mathbf{W}_i^2\right] \mathbb{E}\left[\delta \mathbf{W}_j^2\right] + \sum_{i=0}^{L-1} \mathbb{E}\left[\delta \mathbf{W}_i^4\right] \\
&= L(L-1)\delta t^2 + 3L\delta t^2 \\
&= T^2 + O(\delta t)
\end{aligned}$$

Digression continued: $\sum_{i=0}^{L-1} \delta \mathbf{W}_i^2$

Hence,

$$\begin{aligned} \text{var}\left[\sum_{i=0}^{L-1} \delta \mathbf{W}_i^2\right] &:= \mathbb{E}\left[\left(\sum_{i=0}^{L-1} \delta \mathbf{W}_i^2\right)^2\right] - \left(\mathbb{E}\left[\sum_{i=0}^{L-1} \delta \mathbf{W}_i^2\right]\right)^2 \\ &= T^2 + O(\delta t) - T^2 \\ &= O(\delta t) \end{aligned}$$

The sum $\sum_{i=0}^{L-1} \delta \mathbf{W}_i^2$ has mean T and variance $O(\delta t)$.

Hence, as $\delta t \to 0$ it looks like the constant T.

Reminder: Stochastic Integration

Integrate with respect to Brownian motion:
Riemann sum based on left endpoints

$$\sum_{i=0}^{L-1} h(t_i)(\mathbf{W}(t_{i+1}) - \mathbf{W}(t_i))$$

Integral $\int_0^T h(t)\, d\mathbf{W}(t)$ defined by $\delta t \to 0$

Example: $h(t) = \mathbf{W}(t)$

$$
\begin{aligned}
\sum_{i=0}^{L-1} \mathbf{W}(t_i)\left(\mathbf{W}(t_{i+1}) - \mathbf{W}(t_i)\right) &= \tfrac{1}{2} \sum_{i=0}^{L-1} \Big(\mathbf{W}(t_{i+1})^2 - \mathbf{W}(t_i)^2 \\
&\qquad - \left(\mathbf{W}(t_{i+1}) - \mathbf{W}(t_i)\right)^2\Big) \\
&= \tfrac{1}{2} \sum_{i=0}^{L-1} \Big(\mathbf{W}(t_{i+1})^2 - \mathbf{W}(t_i)^2\Big) \\
&\qquad - \tfrac{1}{2} \sum_{i=0}^{L-1} \delta \mathbf{W}_i^2 \\
&\rightarrow \tfrac{1}{2}\mathbf{W}(T)^2 - \tfrac{1}{2} T
\end{aligned}
$$

So

$$
\int_0^T \mathbf{W}(t)\, d\mathbf{W}(t) = \tfrac{1}{2}\mathbf{W}(T)^2 - \tfrac{1}{2} T
$$

Warning!

Similar analysis for the midpoint Riemann sum:

$$\sum_{i=0}^{L-1} \mathbf{W}\left(\tfrac{1}{2}(t_i + t_{i+1})\right)\left(\mathbf{W}(t_{i+1}) - \mathbf{W}(t_i)\right) \to \tfrac{1}{2}\mathbf{W}(T)^2$$

We will always use the left endpoint definition: Ito

Properties of the Ito Integral

We assume integrand is **non-anticipative**:
$h(t)$ independent of $\{W(s)\}_{s>t}$
Now

$$\mathbb{E}\left[\sum_{i=0}^{L-1} h(t_i)\delta\mathbf{W}_i\right] = \sum_{i=0}^{L-1}\mathbb{E}\left[h(t_i)\delta\mathbf{W}_i\right]$$
$$= \sum_{i=0}^{L-1}\mathbb{E}[h(t_i)]\mathbb{E}[\delta\mathbf{W}_i]$$
$$= 0$$

$$\Rightarrow \mathbb{E}\left[\int_0^T h(t)\,d\mathbf{W}(t)\right] = 0$$

martingale property

Properties of the Ito Integral

$$\mathbb{E}\left[\left(\sum_{i=0}^{L-1} h(t_i)\delta\mathbf{W}_i\right)^2\right] = 2\sum_{i<j}^{L-1}\mathbb{E}\left[h(t_i)h(t_j)\delta\mathbf{W}_i\delta\mathbf{W}_j\right]$$

$$+\sum_{i=0}^{L-1}\mathbb{E}\left[h(t_i)^2\delta\mathbf{W}_i^2\right]$$

$$= 2A + B$$

$A = \sum_{i<j}^{L-1}\mathbb{E}\left[h(t_i)h(t_j)\delta\mathbf{W}_i\right]\mathbb{E}\left[\delta\mathbf{W}_j\right] = 0,$

$B = \sum_{i=0}^{L-1}\mathbb{E}\left[h(t_i)^2\right]\mathbb{E}\left[\delta\mathbf{W}_i^2\right] = \delta t\sum_{i=0}^{L-1}\mathbb{E}\left[h(t_i)^2\right]$

$\Rightarrow \mathbb{E}\left[\left(\int_0^T h(t)\, d\mathbf{W}(t)\right)^2\right] = \int_0^T \mathbb{E}\left[h(t)^2\right]\, dt$ Ito isometry

ODE

Given $f : \mathbb{R} \to \mathbb{R}$:

$$\frac{dx(t)}{dt} = f(x(t))$$

Typically, $x(0)$ given, solution required over $[0, T]$
Fundamental Theorem of Calculus $\Rightarrow$

$$x(t) - x(0) = \int_0^t f(x(s))\, ds$$

Can extend this to define an SDE . . .

SDE

Given functions f and g, the stochastic process $\mathbf{X}(t)$ is a solution of the SDE

$$d\mathbf{X}(t) = f(\mathbf{X}(t))dt + g(\mathbf{X}(t))d\mathbf{W}(t)$$

if $\mathbf{X}(t)$ solves the integral equation

$$\mathbf{X}(t) - \mathbf{X}(0) = \int_0^t f(\mathbf{X}(s))\, ds + \int_0^t g(\mathbf{X}(s))\, d\mathbf{W}(s)$$

Note 1: $d\mathbf{X}(t)$ and $d\mathbf{W}(t)$ are just **shorthand**
Note 2: $d\mathbf{W}(t)$ not diff'ble, so we cannot write $d\mathbf{W}(t)/dt$

Repeat this:

If $\mathbf{X}(t)$ satisfies

$$\mathbf{X}(t) - \mathbf{X}(0) = \int_0^t f(\mathbf{X}(s))\, ds + \int_0^t g(\mathbf{X}(s))\, d\mathbf{W}(s)$$

then we say that $\mathbf{X}(t)$ solves the SDE

$$d\mathbf{X}(t) = f(\mathbf{X}(t))dt + g(\mathbf{X}(t))d\mathbf{W}(t)$$

Typically, $\mathbf{X}(0)$ given, solution required over $[0, T]$

We say $f(\cdot)$ is the **drift** and $g(\cdot)$ is the **diffusion**

Note: $\mathbf{X}(t)$ is a **random variable** at each time t

SDE Example: $f(x) = \mu x$ and $g(x) = \sigma x$

$$dX(t) = \mu X(t)dt + \sigma X(t)dW(t)$$

Here μ and σ are real constants
Used to model **asset prices** in finance
Arises in Black–Scholes theory for option valuation
Solution:

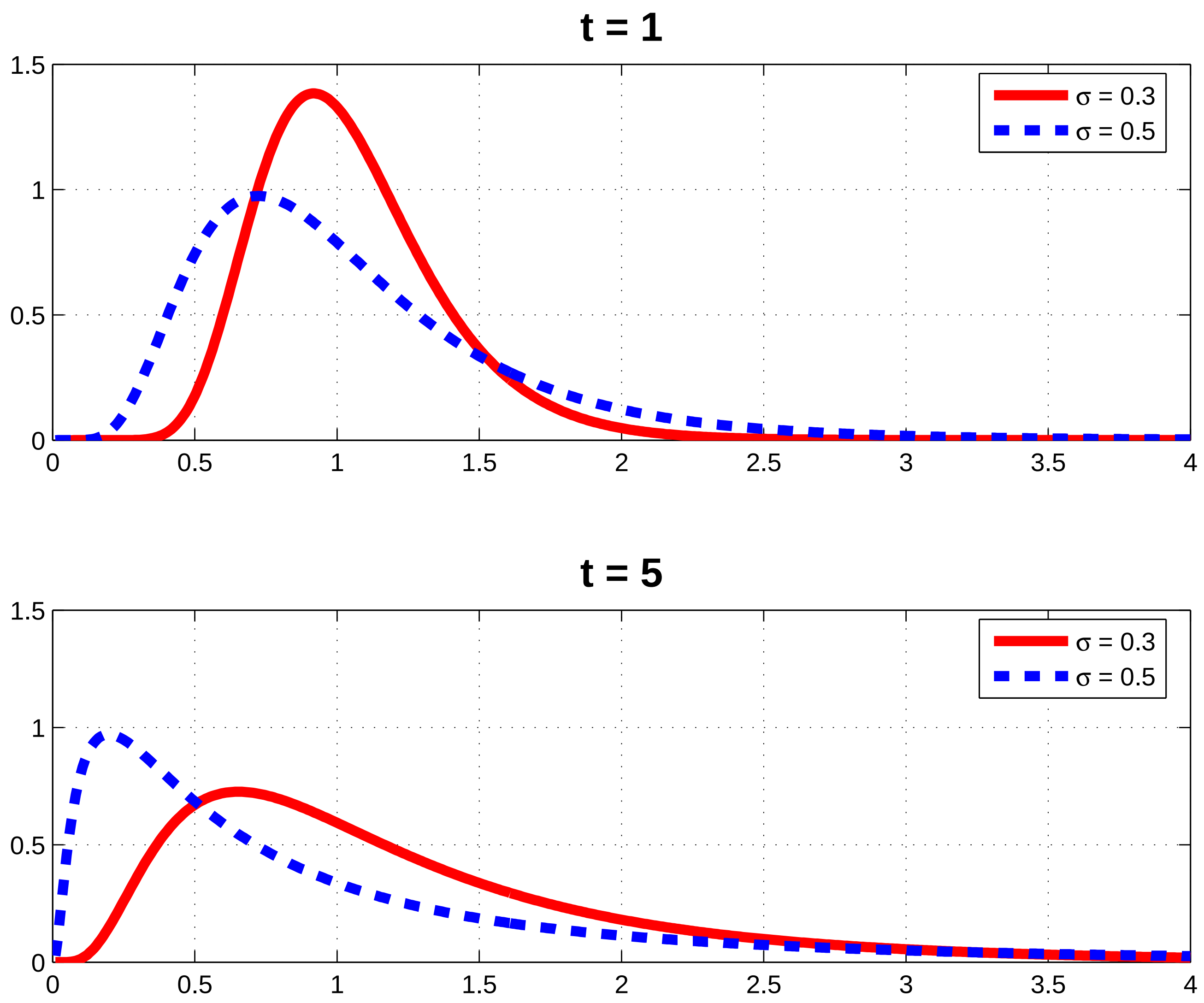
$$X(t) = X(0)e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)}$$

Satisfies

$$\mathbb{E}\left[X(t)\right] = \mathbb{E}\left[X(0)\right]e^{\mu t}$$

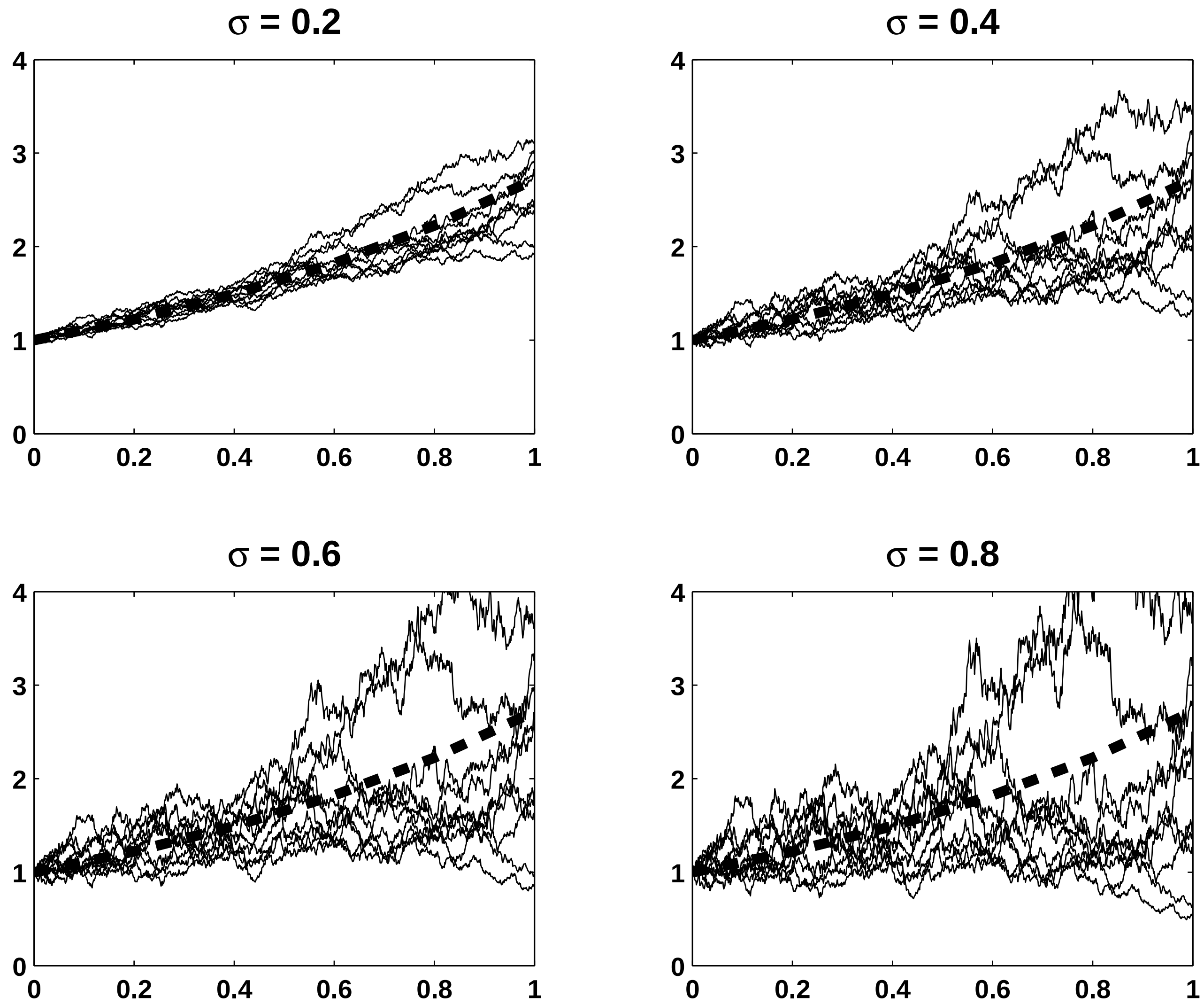
$$\text{var}\left[X(t)^2\right] = \mathbb{E}\left[X(0)^2\right]e^{(2\mu + \sigma^2)t}$$

$f(x) = \mu x$ and $g(x) = \sigma x$: Density



$f(x) = \mu x$ and $g(x) = \sigma x$: paths

10 discretized Brownian paths, $\mu = 1$



SDE Example: Interest Rates

Mean-reverting square root process

$$d\mathbf{X}(t) = \lambda\left(\mu - \mathbf{X}(t)\right)dt + \sigma\sqrt{\mathbf{X}(t)}\,d\mathbf{W}(t)$$

Also named after **Cox, Ingersoll and Ross**, 1985

For $\mathbf{X}(0) \geq 0$, if $\sigma^2 > 2\lambda\mu$ then $\mathbf{X}(t) \geq 0$ for all $t > 0$.

SDE Example: Population Dynamics

$$d\mathbf{X}(t) = r\mathbf{X}(t)\left(K - \mathbf{X}(t)\right)dt + \beta\mathbf{X}(t)d\mathbf{W}(t)$$

- $\mathbf{X}(t)$ is population size
- $1/r$ is a characteristic timescale
- K is carrying capacity
- β is environmental noise strength

Solution:

$$\mathbf{X}(t) = \frac{\mathbf{X}(0)\, e^{(rK - \frac{1}{2}\beta^2)t + \beta\mathbf{W}(t)}}{1 + \mathbf{X}(0)\, r \int_0^t e^{(rK - \frac{1}{2}\beta^2)s + \beta\mathbf{W}(s)}\, ds}$$

Note: positivity of the initial data is preserved

SDE Example: Political Opinions, Cobb (1981)

$$d\mathbf{X}(t) = r\left(G - \mathbf{X}(t)\right)dt + \sqrt{\epsilon\mathbf{X}(t)(1-\mathbf{X}(t))}\,d\mathbf{W}(t)$$

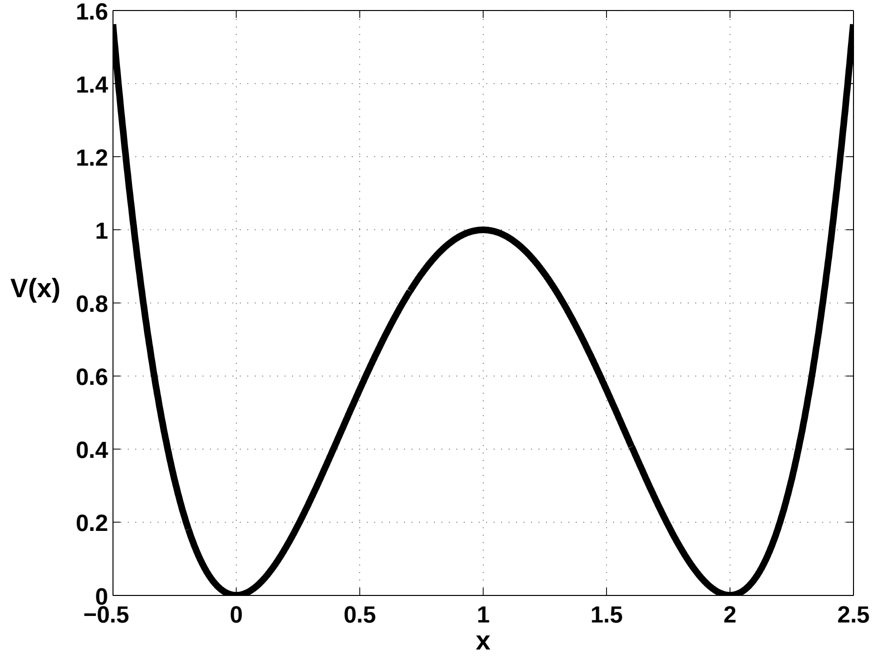
$\mathbf{X}(t)$ is political opinion of an individual at time t

$\mathbf{X}(t) = 0$ means ultra-liberal
$\mathbf{X}(t) = 1$ means ultra-conservative

- G is the long term average ($\mathbb{E}[\mathbf{X}(t)] \to G$)
- r is the rate at which $\mathbb{E}[\mathbf{X}(t)]$ approaches G
- ϵ is a noise strength

Idea: Extreme views $\Rightarrow$ less likely to change mind

Double-Well Potential: $V(x) = x^2(x-2)^2$



ODE version: $\frac{dx(t)}{dt} = -V'(x(t))$ satisfies

$$\frac{d}{dt}V(x(t)) = V'(x(t))\frac{dx(t)}{dt} = -\left(V'(x(t))\right)^2$$

SDE version: $d\mathbf{X}(t) = -V'(\mathbf{X}(t))dt + \sigma d\mathbf{W}(t)$

stint.m

```
%STINT  Approximate stochastic integral
%
% Ito integral of W dW

randn('state',100)                    % set the state of randn
T = 1; N = 500; dt = T/N;

dW = sqrt(dt)*randn(1,N);             % increments
W = cumsum(dW);                       % cumulative sum

ito = sum([0,W(1:end-1)].*dW)

itoerr = abs(ito - 0.5*(W(end)^2-T))
```