Computer practicals 2 – Weak approximation of SDEs and the Monte Carlo technique

Tasks

1. Consider the linear SDE
$$dS = rSdt + \sigma S dw, \quad S(0) = S_0, \tag{1}$$
where $r \geq 0$ is a risk-free interest rate and $\sigma > 0$ is volatility. This is an SDE for GBM written under the risk neutral measure. Consider a European call option and assume that the price process is described by GBM. The price of the call with maturity T and strike K at time $t = 0$ is equal to
$$\begin{aligned} u(0, S_0) &= e^{-rT} E(S(T) - K)_+ \\ &= S_0 \Phi\left(\frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}\right) \\ &\quad - K e^{-rT} \Phi\left(\frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right), \end{aligned} \tag{2}$$
where[2] $(x)_+ = \max(x, 0)$ and $\Phi(x)$ is the standard normal distribution function. Apply the three methods: the mean-square Euler, the weak Euler and 2nd-order weak method from the lecture notes to the price the call. Realise these methods accompanying them by the Monte Carlo technique and study their errors. By taking a large number of Monte Carlo runs, show that the Euler methods are of first weak and confirm the order of the 2nd order weak scheme.

2. By solving (1) explicitly, find the exact expression of $S(T)$. Apply the Monte Carlo technique to $u(0, S_0) = e^{-rT} E(K - S(T))_+$ with exact $S(T)$ and experimentally study convergence of the Monte Carlo technique.

3. For *physical pendulum with linear friction and additive noise*[3], Langevin equations can be written as
$$\begin{aligned} dP^i &= f^i(Q)\, dt - \nu P^i\, dt + \sqrt{2\nu/\beta}\, dw_i(t), \\ dQ^i &= P^i\, dt, \quad i = 1, 2, \end{aligned} \tag{3}$$
where $\nu > 0$ is a damping parameter, β is an inverse temperature, $w_1(t)$ and $w_2(t)$ are independent standard Wiener processes,
$$f(q) = -\frac{\partial U}{\partial q^i}(q)$$
and the potential
$$U(q) = \frac{1}{2}\left[\cos\left(\frac{2\pi q^1}{\lambda}\right) + \cos\left(\frac{2\pi q^2}{\lambda}\right)\right], \quad \lambda > 0. \tag{4}$$
Note that the relation
$$\frac{1}{\beta} = \frac{1}{2} E\left[\left(P^1\right)^2 + \left(P^2\right)^2\right] \tag{5}$$
between the temperature and average kinetic energy holds.

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[2] MatLab function subplus().
[3] See [K. Lindenberg, B.J. West. *The Nonequilibrium Statistical Mechanics of Open and Closed Systems*. VCH Publishers, New York, 1990] and [G.N. Milstein, M.V. Tretyakov. Computing ergodic limits for Langevin equations. *Phys. D* **229** (2007), 81–95].

Apply two second-order weak schemes to (3): the Heun method and the *second-order weak quasi-symplectic method*[4] has the form

$$\begin{aligned}
P_0 &= p, \quad Q_0 = q, \\
\mathcal{P}_{1,k} &= P_k \exp\left(-\nu\frac{h}{2}\right), \quad \mathcal{Q}_{1,k} = Q_k + \frac{h}{2}\mathcal{P}_{1,k}, \\
\mathcal{P}_{2,k} &= \mathcal{P}_{1,k} + hf(\mathcal{Q}_{1,k}) + h^{1/2}\sqrt{2\nu/\beta}\begin{pmatrix}\xi_{1k}\\ \xi_{2k}\end{pmatrix}, \\
P_{k+1} &= \mathcal{P}_{2,k}\exp\left(-\nu\frac{h}{2}\right), \; Q_{k+1} = \mathcal{Q}_{1,k} + \frac{h}{2}\mathcal{P}_{2,k}, \; k = 0,\dots,N-1,
\end{aligned} \tag{6}$$

where ξ_{lk} are i.i.d. random variables with the law

$$P(\xi = 0) = 2/3, \;\; P(\xi = \pm\sqrt{3}) = 1/6.$$

Using these two methods, compute the mean-square displacement of the particle:

$$D := E(Q^1(t) - Q^1(0))^2 + E(Q^2(t) - Q^2(0))^2.$$

Use (5) to control the accuracy of simulation. Compare the two method in the case of small ν, e.g. $\nu = 0.0004$, $\beta = 5$, $\lambda = 4$. The outcomes can be compared with the results in[5]. This experiment needs a bit of time to run and hence will be not considered in the class.

Remark 1 *In all exercises do not forget to report the Monte Carlo error: please remember that reporting outcomes of Monte Carlo simulation without providing the corresponding statistical errors essentially has no value.*

[4] See [G.N. Milstein, M.V. Tretyakov. Quasi-symplectic methods for Langevin-type equations. *IMA J. Numer. Anal.*, **23** (2003), 593–626] and [G.N. Milstein, M.V. Tretyakov. *Stochastic Numerics for Mathematical Physics*. Springer, 2004].

[5] [G.N. Milstein, M.V. Tretyakov. Computing ergodic limits for Langevin equations. *Phys. D* **229** (2007), 81–95].