Geometric finite difference schemes for the generalised hyperelastic-rod wave equation

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August 29, 2008

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Geometric integrators are presented for a class of nonlinear dispersive equations which includes the Camassa-Holm equation, the BBM equation and the hyperelastic-rod wave equation. One group of schemes is designed to preserve a global property of the equations: the conservation of energy; while the other one preserves a more local feature of the equations: the multi-symplecticity.

Key words: *Hyperelastic-rod wave, Camassa–Holm equation, BBM equation, Conservative schemes, Finite Difference, Discrete gradients, Multi-symplecticity, Euler box scheme.*

1 Introduction

We consider the generalised hyperelastic-rod wave equation

$$u_t - u_{xxt} + \frac{1}{2}g(u)_x - \gamma(2u_x u_{xx} + uu_{xxx}) = 0, \quad u|_{t=0} = u_0, \tag{1}$$

with periodic boundary conditions and where $u = u(x,t)$ and g is a given smooth function. The generalised hyperelastic-rod wave was first introduced in [8]; it defines a whole class of equations, depending on the function g and the value of γ, which contains several well-known nonlinear dispersive equations. Taking $\gamma = 1$ and $g(u) = 2\kappa u + 3u^2$ (with $\kappa \geq 0$), equation (1) reduces to the Camassa–Holm equation:

$$u_t - u_{xxt} + \kappa u_x + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0. \tag{2}$$

Since its apparition in [4] in the context of water wave propagation where u represents the height's free surface above a flat bottom while κ is a parameter, the Camassa–Holm equation has been extensively studied, mainly because of its rich mathematical structure. The Camassa–Holm equation possesses a Lax pair which allows for a scattering and inverse scattering analysis, showing that the equation is integrable ([4, 12, 16, 26]). It is a geodesic on the group of diffeomorphisms for a given metric ([27, 15]). In addition, the Camassa–Holm equation is bi-Hamiltonian (see Section 2 for definitions and references). The bi-Hamiltonian structure of the equation will be used in this article to derive energy preserving numerical schemes (see Section 3). For $g(u) = 3u^2$, equation (1) becomes the hyperelastic-rod wave:

$$u_t - u_{xxt} + 3uu_x - \gamma(2u_x u_{xx} + uu_{xxx}) = 0, \tag{3}$$

which was introduced by Dai [18] in 1998. The equation models the propagation of nonlinear waves in cylindrical axially symmetric hyperelastic-rod. The parameter $\gamma \in \mathbb{R}$ is a constant depending on the material and prestress of the rod. The well-posedness of the Cauchy problem for (3) is established in [17, 38]. For $g(u) = 2u + u^2$ and for $\gamma = 0$, equation (1) leads to the Benjamin-Bona-Mahony (BBM) equation (or regularised long wave) [1]:

$$u_t - u_{xxt} + u_x + uu_x = 0, \tag{4}$$

which describes surface wave in a channel. While the solutions of the BBM equation are unique and globally defined in time, the solutions of the Camassa–Holm and hyperelastic-rod wave equations may break down in finite time. Due to the particular circumstances in which this occurs, this situation is also refered as wave breaking (see [13, 14] for more details). After wave breaking, the solutions are no longer unique and, in this article, only solutions before wave breaking will be considered.

We now briefly review – without intending to be exhaustive – the numerical schemes related to the generalised hyperelastic-rod wave equation that can be found in the literature. For the Camassa–Holm equation, schemes using pseudospectral discretisation have been used in [5, 25]. Methods based on multipeakons, a special class of solutions of the Camassa–Holm equation, can be found in [7, 6, 24, 23]. Finite difference schemes with convergence proof are studied in [9, 22]. In [37], the authors use a finite element method to derive a scheme which is high order accurate and nonlinearly stable. The Camassa-Holm equation admits a multi-symplectic formulation which can be used to derive multi-symplectic numerical schemes, see [10]. For the BBM equation, conservative finite difference schemes were proposed in [36] with a convergence and stability analysis. We also refer to [30, 28]. As far as the hyperelastic-rod wave equation, the authors are only aware of the numerical scheme given in [32] which is based on a Galerkin approximation and preserves a discretisation of the energy.

In this article we derive finite difference schemes for the generalised hyperelastic-rod equation which preserve some of the geometric properties of the equation. The first property is a global one, namely the preservation of the energy, while the second is local and corresponds to the preservation of multi-symplecticity. In Section 2, we look at the Hamiltonian (or "Hamiltonian-like") formulations of (1) and explain how methods for ordinary differential equations based on discrete gradients that have been developed in [33] can be applied to equation (1). In Section 3, the discrete gradients are computed and the corresponding energy preserving schemes are derived. In Section 4, we review some of the general theory of multi-symplectic PDEs following the approach of Bridges and Reich [3] and based on the work in [10], we derive a multi-symplectic scheme for the generalised hyperelastic-rod wave equation (1). Finally, we illustrate the behaviour of these new schemes by numerical experiments in Section 5.

2 The discrete gradient approach

In this section we review the Hamiltonian formulation for partial differential equations, give some "Hamiltonian like formulations" for our various equations and finally present the discrete gradient methods for ODEs of [33].

We first consider the Camassa–Holm equation (2) in his limiting case $\kappa = 0$:

$$u_t - u_{xxt} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0.$$

Defining $m = u - u_{xx}$, this equation can be rewritten as a Hamiltonian partial differential equation, that is,

$$m_t = \mathcal{D}(m)\frac{\delta\mathcal{H}}{\delta m}, \tag{5}$$

where the functional $\mathcal{H}(m)$ is the Hamiltonian and $\frac{\delta\mathcal{H}}{\delta m}$ denotes the variational derivative of $\mathcal{H}$ with respect to m defined as

$$\left\langle \frac{\delta\mathcal{H}}{\delta m}, \tilde{m} \right\rangle_{L^2} = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \mathcal{H}(m + \varepsilon\tilde{m}) \quad \text{for all } \tilde{m}$$

(here $\langle v, w\rangle_{L^2} = \int v(x)w(x)\,\mathrm{d}x$ denotes the L^2-scalar product). Equation (5) defines a Hamiltonian equation if in addition the operator $\mathcal{D}(m)$ is antisymmetric with respect to the L^2-scalar product, that is,

$$\langle v, \mathcal{D}(m)w\rangle_{L^2} = -\langle \mathcal{D}(m)v, w\rangle_{L^2},$$

and its Lie-Poisson bracket

$$\{F, H\}(m) = \left\langle \frac{\delta F}{\delta m}, \mathcal{D}(m)\frac{\delta H}{\delta m} \right\rangle_{L^2}$$

satisfies the Jacobi identity

$$\{\{F, G\}, H\} + \{\{G, H\}, F\} + \{\{H, F\}, G\} = 0. \tag{6}$$

The Camassa-Holm equation has a bi-Hamiltonian structure (see [35] for the definition and [4, 11] for the proofs): It is Hamiltonian for the two following pairs of antisymmetric operator and Hamiltonian function,

$$\begin{aligned} \mathcal{D}_1(m)(\cdot) &= -(u - u_{xx})(\cdot)_x - ((u - u_{xx})(\cdot))_x, \\ \mathcal{H}_1(m) &= \frac{1}{2}\int (u^2 + u_x^2)\,\mathrm{d}x \end{aligned} \tag{7}$$

and

$$\begin{aligned} \mathcal{D}_2(m)(\cdot) &= -(\partial_x(1-\partial_{xx}))(\cdot), \\ \mathcal{H}_2(m) &= \frac{1}{2}\int (u^3 + uu_x^2)\,\mathrm{d}x. \end{aligned} \tag{8}$$

For the other partial differential equations considered in the introduction, it is not clear if they also possess a Hamiltonian structure (the issue here being the Jacobi identity (6)), nevertheless we have the following "Hamiltonian-like" formulations. For the hyperelastic-rod wave (3), there exist, at least, two functionals $\mathcal{H}_1(m)$, which corresponds to the energy of the problem, and $\mathcal{H}_2(m)$ and two antisymmetric operators $\mathcal{D}_1(m)$ and $\mathcal{D}_2(m)$ such that this equation can be written as a Hamiltonian problem as in (5). They are given by

$$\begin{aligned} \mathcal{D}_1(m)(\cdot) &= -(u-\gamma u_{xx})(\cdot)_x - ((u-\gamma u_{xx})(\cdot))_x, \\ \mathcal{H}_1(m) &= \frac{1}{2}\int (u^2 + u_x^2)\,\mathrm{d}x \end{aligned} \tag{9}$$

and

$$\begin{aligned} \mathcal{D}_2(m)(\cdot) &= -(\partial_x(1-\partial_{xx}))(\cdot), \\ \mathcal{H}_2(m) &= \frac{1}{2}\int (u^3 + \gamma uu_x^2)\,\mathrm{d}x. \end{aligned} \tag{10}$$

For the Camassa–Holm equation given by (2), we obtain

$$\begin{aligned} \mathcal{D}_1(m)(\cdot) &= -(u-u_{xx}+\tfrac{\kappa}{2})(\cdot)_x - ((u-u_{xx}+\tfrac{\kappa}{2})(\cdot))_x, \\ \mathcal{H}_1[m] &= \frac{1}{2}\int (u^2 + u_x^2)\,\mathrm{d}x \end{aligned} \tag{11}$$

and

$$\begin{aligned} \mathcal{D}_2(m)(\cdot) &= -(\partial_x(1-\partial_{xx}))(\cdot), \\ \mathcal{H}_2[m] &= \frac{1}{2}\int (u^3 + \kappa u^2 + uu_x^2)\,\mathrm{d}x. \end{aligned} \tag{12}$$

For the generalised hyperelastic-rod wave (1), the formulation equivalent to (9) is not available and we only have a Hamiltonian like formulation given by

$$\begin{aligned} \mathcal{D}_2(m)(\cdot) &= -(\partial_x(1-\partial_{xx}))(\cdot), \\ \mathcal{H}_2(m) &= \frac{1}{2}\int (G(u) + \gamma uu_x^2)\,\mathrm{d}x, \end{aligned} \tag{13}$$

where G is an integral of g, i.e., $G' = g$. Finally, for the BBM equation (4), we have

$$\begin{aligned} \mathcal{D}_1(m)(\cdot) &= -(\tfrac{u}{3}+\tfrac{1}{2})(\cdot)_x - ((\tfrac{u}{3}+\tfrac{1}{2})(\cdot))_x, \\ \mathcal{H}_1(m) &= \frac{1}{2}\int (u^2 + u_x^2)\,\mathrm{d}x, \end{aligned} \tag{14}$$

and

$$\mathcal{D}_2(m)(\cdot) = -(\partial_x(1-\partial_{xx}))(\cdot),$$
$$\mathcal{H}_2(m) = \frac{1}{2}\int (u^2 + \frac{u^3}{3})\,\mathrm{d}x. \tag{15}$$

A remarkable feature of a Hamiltonian partial differential equation is the fact that the Hamiltonian functional $\mathcal{H}$ is conserved along the exact solution of the problem. Indeed, we have

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \Big\langle \frac{\delta\mathcal{H}}{\delta m}, \frac{\mathrm{d}m}{\mathrm{d}t}\Big\rangle = \Big\langle \frac{\delta\mathcal{H}}{\delta m}, \mathcal{D}(m)\frac{\delta\mathcal{H}}{\delta m}\Big\rangle = 0, \tag{16}$$

using the fact that the operator $\mathcal{D}(m)$ is antisymmetric. The Hamiltonians $\mathcal{H}_1$ and $\mathcal{H}_2$ are thus conserved along the exact solution of the partial differential equations considered here. Our goal in the next section will be to exploit this feature of the exact solution to design numerical schemes that exactly preserve a discretised version of these Hamiltonians. To do so, we first have to find appropriate discretisations of the partial differential equations (see Section 3 for the details) and then integrate the obtained differential equations in time by the *discrete gradient* approach.

We now review the discrete gradient approach used in the numerical integration of ODEs proposed in [33] (see also references therein). For a given smooth function $H : \mathbb{R}^n \to \mathbb{R}$ and a skew-symmetric matrix $D(y)$ depending on y, we consider the differential equation in $\mathbb{R}^n$ given by

$$\dot{y} = f(y) = D(y)\nabla H(y). \tag{17}$$

We say that $\overline{\nabla}H$ is a *discrete gradient* of H if

$$H(y') - H(y) = \overline{\nabla}H(y,y')\cdot(y'-y) \quad \text{for all } y, y' \in \mathbb{R}^n \tag{18}$$

and the consistency relation $\overline{\nabla}H(y,y) = \nabla H(y)$ is satisfied. Given a discrete gradient $\overline{\nabla}H$, one can construct schemes of the form

$$\frac{y_{n+1}-y_n}{\Delta t} = \tilde{D}(y_n, y_{n+1}, \Delta t)\overline{\nabla}H(y_n, y_{n+1}), \tag{19}$$

where we impose that the operator $v \mapsto \tilde{D}(y, y', \Delta t)(v)$ is antisymmetric for all $y, y', \Delta t$ and, for consistency reason, $\tilde{D}(y, y, 0) = D(y)$. There exist several discrete gradients of the same function H and one of them is given by the *mean value discrete gradient*, see [21, 33], which is given by

$$\overline{\nabla}H(y_n, y_{n+1}) = \int_0^1 \nabla H((1-\zeta)y_n + \zeta y_{n+1})\mathrm{d}\zeta. \tag{20}$$

In the next section, we will introduce another discrete gradient which can be applied to the type of Hamiltonians we will be considering.

Schemes which takes the form (19) exactly preserve the value of $H(y_n)$, as we have

$$\begin{aligned} H(y_{n+1}) - H(y_n) &= \overline{\nabla}H(y_n, y_{n+1})\cdot(y_{n+1}-y_n) \\ &= \Delta t\overline{\nabla}H(y_n, y_{n+1})\cdot\tilde{D}(y_n, y_{n+1}, h)\overline{\nabla}H(y_n, y_{n+1}) = 0. \end{aligned} \tag{21}$$

3 Energy preserving schemes

We consider periodic solutions on the interval $[0,T]$. We introduce the partition of $[0,T]$ in points separated by a distance $\Delta x = 1/n$ denoted $x_i = i\Delta x$ for $i = 0,\dots,n-1$. We consider the time step discretisation step Δt and $t_j = j\Delta t$. At $x = x_i$ and $t = t_j$, the value of u is approximated by u_i^j. We define the right and left discrete derivatives with respect to space at (x_i, t_j) as

$$(\delta_x^{\pm} u)_i^j = \frac{\pm 1}{\Delta x}(u_{i\pm1}^j - u_i^j).$$

and the symmetric derivative as

$$\delta_x = \frac{1}{2}(\delta_x^+ + \delta_x^-).$$

In order to derive energy-preserving schemes, we have to define all the continuous operations at the discrete level. The L^2-scalar product in the continuous case becomes the following discrete one

$$\langle u, v\rangle = \Delta x \sum_{i=0}^{n-1} u_i v_i \tag{22}$$

for which the following discrete summation by part rules hold:

$$\left\langle \delta_x^{\pm} u, v\right\rangle = -\left\langle u, \delta_x^{\mp} v\right\rangle \text{ and } \langle \delta_x u, v\rangle = -\langle u, \delta_x v\rangle\,. \tag{23}$$

We have to discretise the Hamiltonians $\mathcal{H}_1$ and $\mathcal{H}_2$. We will only consider in details the hyperelastic-rod wave equation, the results for the other equations being listed below. We approximate $\mathcal{H}_1$ and $\mathcal{H}_2$ by

$$H_1(m) = \frac{\Delta x}{2}\sum_{i=0}^{n-1}\big(u_i^2 + (\delta_x u_i)^2\big) \tag{24}$$

and

$$H_2(m) = \frac{\Delta x}{2}\sum_{i=0}^{n-1}\big(u_i^3 + \gamma u_i(\delta_x u_i)^2\big), \tag{25}$$

respectively. Here $m = (1-\delta_x^2)u$. Several methods to compute discrete gradients are given in [33]. In this section, we present another method which can be used in the case where the Hamiltonians consist only of sums and products of the unknown variables (i.e. $\{u_i\}_{i=0}^{n-1}$), as in (24) and (25). For a scalar function f, we denote the difference $f(m')-f(m)$ by $\delta[f]$ and the average $\frac{f(m')+f(m)}{2}$ by $\mu[f]$. A straightforward computation shows that, for any m and m', we have

$$f(m')g(m')-f(m)g(m) = \frac{1}{2}(f(m')-f(m))(g(m')+g(m))+\frac{1}{2}(g(m')-g(m))(f(m')+f(m))$$

which rewrites with our new notation as

$$\delta[(f\cdot g)] = \delta[f]\cdot\mu[g] + \delta[g]\cdot\mu[f]. \tag{26}$$

Note the similarity between (26) and the Leibniz rule $(fg)' = f'g + g'f$ and it becomes clear that the operator μ is introduced to account for the failure of a simple difference to fulfill the Leibnitz rule. By recursively applying the product rule (26), we obtain

$$\begin{aligned}\delta[H_1] &= \frac{\Delta x}{2}\sum_{i=0}^{n-1}\delta[(u_i)^2 + (\delta_x u_i)^2] \\ &= \frac{\Delta x}{2}\sum_{i=0}^{n-1}(2\delta[u_i]\mu[u_i] + 2\delta[\delta_x u_i]\mu[\delta_x u_i]).\end{aligned}$$

We use the fact that δ and μ commute with δ_x (which follows from the linearity of δ), the summation by part rule, and we obtain

$$\begin{aligned}\delta[H_1] &= \Delta x\sum_{i=0}^{n-1}(\delta[u_i]\mu[u_i] - \mu[u_i]\delta[\delta_x^2 u_i]) \\ &= \Delta x\sum_{i=0}^{n-1}\mu[u_i](\delta[u_i] - \delta[\delta_x^2 u_i]) \\ &= \langle \mu[u], \delta[m]\rangle\,,\end{aligned}$$

by the definition of the discrete scalar product (22). Hence, using the fact that $m = (1-\delta_x^2)u$, we get

$$H_1(m') - H_1(m) = \left\langle \frac{u'+u}{2}, m' - m\right\rangle \tag{27}$$

and therefore

$$\overline{\nabla} H_1(m,m') = \frac{u+u'}{2} = (1-\delta_x^2)^{-1}\left(\frac{m+m'}{2}\right). \tag{28}$$

For the second Hamiltonian of the hyperelastic-rod wave given by (25) , we obtain

$$\begin{aligned}\delta[H_2] &= \frac{\Delta x}{2}\sum_{i=0}^{n-1}\delta[u_i^3 + \gamma u_i(\delta_x u_i)^2] \\ &= \frac{\Delta x}{2}\sum_{i=0}^{n-1}(\mu[u_i^2]\delta[u_i] + \mu[u_i]\delta[u_i^2] + \gamma\delta[u_i]\mu[(\delta_x u_i)^2] + 2\gamma\mu[u_i]\mu[\delta_x u_i]\delta[\delta_x u_i]) \\ &= \frac{\Delta x}{2}\sum_{i=0}^{n-1}\Big(\Big(\mu[u_i^2] + 2\mu[u_i]^2 + \gamma\mu[(\delta_x u_i)^2]\Big)\delta[u_i] + 2\gamma\mu[u_i]\mu[\delta_x u_i]\delta[\delta_x u_i]\Big) \\ &= \frac{\Delta x}{2}\sum_{i=0}^{n-1}\Big(\mu[u_i^2] + 2\mu[u_i]^2 + \gamma\mu[(\delta_x u_i)^2] - 2\gamma\delta_x(\mu[u_i]\delta_x\mu[u_i])\Big)\delta[u_i] \\ &= \Big\langle \frac{1}{2}\mu[u^2] + \mu[u]^2 + \frac{\gamma}{2}\mu[(\delta_x u)^2] - \gamma\delta_x(\mu[u]\delta_x\mu[u]), \delta[u]\Big\rangle\,.\end{aligned}$$

Hence,

$$\overline{\nabla} H_2(m,m') = (1-\delta_x^2)^{-1}\big(\tfrac{1}{2}\mu[u^2] + \mu[u]^2 + \tfrac{\gamma}{2}\mu[(\delta_x u)^2] - \gamma\delta_x(\mu[u]\delta_x\mu[u])\big), \tag{29}$$

or

$$\begin{aligned}\overline{\nabla} H_2(m,m') &= \tfrac{1}{4}(1-\delta_x^2)^{-1}\Big(2u^2+2u'^2+2uu'+\gamma((\delta_x u)^2+(\delta_x u')^2)\\ &- \gamma\delta_x\big((u+u')(\delta_x u+\delta_x u')\big)\Big). \end{aligned} \tag{30}$$

Note that, if we take μ equals to the identity in (27) and (29) (so that the product rule holds exactly) and replace the discrete spatial derivative δ_x by its continuous counterpart ∂_x, then we obtain $\frac{\delta\mathcal{H}_1}{\delta m}$ and $\frac{\delta\mathcal{H}_2}{\delta m}$, respectively and in this way we check the consistency of the approximation.

Let us now compute the mean value discrete gradient, which we now denote $\overline{\nabla}^m H_j(m,m')$ (for $j=1,2$), as given by (20), that is,

$$\overline{\nabla}^m H_j(m,m') = \int_0^1 \nabla H_j((1-\zeta)m+\zeta m')\mathrm{d}\zeta. \tag{31}$$

Here the gradient ∇H is defined with respect to the discrete scalar product (22) and we have, for all $\tilde{m}$,

$$\begin{aligned}\langle \nabla H_1(m),\tilde{m}\rangle &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} H_1(m+\varepsilon\tilde{m})\\ &= \Delta x\sum_{i=0}^{n-1}(u_i\tilde{u}_i+\delta_x u_i\delta_x\tilde{u}_i) = \Delta x\sum_{i=0}^{n-1}(u_i(\tilde{u}_i-\delta_x^2\tilde{u}_i) = \langle u,\tilde{m}\rangle\,,\end{aligned}$$

after one summation by part, so that

$$\nabla H_1(m) = u. \tag{32}$$

In the same way, we obtain

$$\begin{aligned}\langle \nabla H_2(m),\tilde{m}\rangle &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} H_2(m+\varepsilon\tilde{m})\\ &= \frac{\Delta x}{2}\sum_{i=0}^{n-1}(3u_i^2\tilde{u}_i+\gamma\tilde{u}_i(\delta_x u_i)^2+2\gamma u_i\delta_x u_i\delta_x\tilde{u}_i)\end{aligned}$$

so that

$$\nabla H_2(m) = (1-\delta_x^2)^{-1}\Big(\frac{3}{2}u^2+\frac{\gamma}{2}(\delta_x u)^2-\gamma\delta_x(u\delta_x u)\Big) \tag{33}$$

(the multiplications are meant componentwise). From (31) and (32), we get

$$\overline{\nabla}^m H_1(m,m') = \int_0^1((1-\zeta)u+\zeta u')\mathrm{d}\zeta = \frac{u+u'}{2}$$

and the mean value discrete gradient coincides with the discrete gradient computed earlier in (28). For the second Hamiltonian, from (31) and (33), we obtain

$$\begin{aligned}
\overline{\nabla}^m H_2(m,m') &= (1-\delta_x^2)^{-1}\Big(\int_0^1 \Big(\frac{3}{2}\big((1-\zeta)u+\zeta u'\big)^2 + \frac{\gamma}{2}\big(\delta_x((1-\zeta)u+\zeta u')\big)^2 \\
&\qquad - \gamma\delta_x\big(((1-\zeta)u+\zeta u')(\delta_x(1-\zeta)u+\zeta u')\big)\Big)\mathrm{d}\zeta\Big) \\
&= (1-\delta_x^2)^{-1}\Big(\frac{1}{2}\big(u^2+uu'+u'^2\big) + \frac{\gamma}{6}\big((\delta_x u)^2+\delta_x u\delta_x u'+(\delta_x u')^2\big) \\
&\qquad - \frac{\gamma}{3}\delta_x\big(u\delta_x u + \frac{1}{2}u\delta_x u' + \frac{1}{2}u'\delta_x u + u'\delta_x u'\big)\Big) \qquad (34)
\end{aligned}$$

which differs from the discrete gradient computed earlier in (30). It remains to discretise the operators $\mathcal{D}_1$ and $\mathcal{D}_2$. We use the following approximations:

$$D_1(m)(v) = -((u-\gamma\delta_x^2 u)\delta_x v) - \delta_x((u-\gamma\delta_x^2 u)v) \qquad (35)$$

and

$$D_2(m)(v) = -\delta_x(1-\delta_x^2)(v). \qquad (36)$$

Using the summation by part rule (26), it can be checked that the discrete operators D_1 and D_2 are antisymmetric for the discrete scalar product (22). The discrete gradients (28), (30) and (34) are symmetric in m and m', that is, $\overline{\nabla}H(m,m') = \overline{\nabla}H(m',m)$ for any m and m'. For the extensions of the operators D_1 and D_2, we take

$$\tilde{D}_1(m,m',\Delta t)(v) = -((\frac{1}{2}(u+u') - \frac{\gamma}{2}\delta_x^2(u+u'))\delta_x v) - \delta_x((\frac{1}{2}(u+u') - \frac{\gamma}{2}\delta_x^2(u+u'))v) \qquad (37)$$

and

$$\tilde{D}_2(m,m',\Delta t) = D_2(m), \qquad (38)$$

respectively. With these special choices, both operators are symmetric in time, that is,

$$\tilde{D}_j(m,m',\Delta t) = \tilde{D}_j(m',m,-\Delta t) \qquad (39)$$

for $j=1,2$ and for all m, m', Δt. Finally, we obtain three schemes which all preserve one of the Hamiltonians, see (21). The first scheme is given by

$$\frac{m^{j+1}-m^j}{\Delta t} = \tilde{D}_1(m^{j+1},m^j,\Delta t)\overline{\nabla}H_1(m^{j+1},m^j)$$

or, more explicitely,

$$\begin{aligned}
u^{j+1} = u^j - \frac{\Delta t}{4}(1-\delta_x^2)^{-1}\Big(&\big(u^{j+1}+u^j-\gamma\delta_x^2(u^{j+1}+u^j)\big)\delta_x(u^{j+1}+u^j) \\
&+ \delta_x\Big(\big(u^{j+1}+u^j-\gamma\delta_x^2(u^{j+1}+u^j)\big)(u^{j+1}+u^j)\Big)\Big). \qquad (40)
\end{aligned}$$

It preserves the discrete energy H_1. The second scheme is given by

$$\frac{m^{j+1}-m^j}{\Delta t} = D_2(m^j)\overline{\nabla}H_2(m^{j+1},m^j)$$

or, more explicitely,

$$u^{j+1} = u^j - \frac{\Delta t}{4}\delta_x(1-\delta_x^2)^{-1}\Big(2\big((u^{j+1})^2 + u^{j+1}u^j + (u^j)^2\big) + \gamma((\delta_x u^{j+1})^2 + (\delta_x u^j)^2) - \gamma\delta_x\big(u^{j+1}\delta_x u^{j+1} + u^{j+1}\delta_x u^j + u^j\delta_x u^{j+1} + u^j\delta_x u^j\big)\Big). \quad (41)$$

The third scheme is given by

$$\frac{m^{j+1}-m^j}{\Delta t} = D_2(m^j)\overline{\nabla}^m H_2(m^{j+1}, m^j)$$

or more explicitely

$$u^{j+1} = u^j - \frac{\Delta t}{4}\delta_x(1-\delta_x^2)^{-1}\Big(2\big((u^{j+1})^2 + u^{j+1}u^j + (u^j)^2\big) + \frac{2\gamma}{3}\big((\delta_x u^{j+1})^2 + \delta_x u^{j+1}\delta_x u^j + (\delta_x u^j)^2\big) - \frac{2\gamma}{3}\delta_x\big(2u^{j+1}\delta_x u^{j+1} + u^{j+1}\delta_x u^j + u^j\delta_x u^{j+1} + 2u^j\delta_x u^j\big)\Big). \quad (42)$$

The schemes (41) and (42) preserve the discrete Hamiltonian H_2. The three schemes are second-order in time since they are symmetric in time by equation (39), see [33]. For the Camassa–Holm equation and the BBM equation, the schemes corresponding to (40) are

$$u^{j+1} = u^j - \frac{\Delta t}{4}(1-\delta_x^2)^{-1}\Bigg(\big((1-\delta_x)^2(u^{j+1}+u^j) + \kappa\big)\delta_x(u^{j+1}+u^j) + \delta_x\Big(\big((1-\delta_x^2)(u^{j+1}+u^j)\big)(u^{j+1}+u^j)\Big)\Bigg)$$

and

$$u^{j+1} = u^j - \frac{\Delta t}{6}(1-\delta_x^2)^{-1}\Big(\big((u^{j+1}+u^j)\delta_x(u^{j+1}+u^j)\big) + \delta_x\big((u^{j+1})^2 + (u^j)^2 + 2u^{j+1}u^j + 3u^{j+1} + 3u^j\big)\Big),$$

respectively. For any scalar function, and in particular G, the discrete gradient is unique as we have $\overline{\nabla}G(u,u') = \frac{G(u')-G(u)}{u'-u}$. For the generalised hyperelastic-rod wave equation, the schemes (41) and (42) rewrites

$$u^{j+1} = u^j - \frac{\Delta t}{4}\delta_x(1-\delta_x^2)^{-1}\Big(2\overline{\nabla}G(u^{j+1}, u^j) + \gamma((\delta_x u^{j+1})^2 + (\delta_x u^j)^2) - \gamma\delta_x\big(u^{j+1}\delta_x u^{j+1} + u^{j+1}\delta_x u^j + u^j\delta_x u^{j+1} + u^j\delta_x u^j\big)\Big)$$

and

$$u^{j+1} = u^j - \frac{\Delta t}{4}\delta_x(1-\delta_x^2)^{-1}\Big(2\overline{\nabla}G(u^{j+1},u^j) \\ + \frac{2\gamma}{3}\big((\delta_x u^{j+1})^2 + \delta_x u^{j+1}\delta_x u^j + (\delta_x u^j)^2\big) \\ - \frac{2\gamma}{3}\delta_x\big(2u^{j+1}\delta_x u^{j+1} + u^{j+1}\delta_x u^j + u^j\delta_x u^{j+1} + 2u^j\delta_x u^j\big)\Big).$$

In the particular cases of the Camassa–Holm equation and the BBM equation, we have

$$\overline{\nabla}G(u,u') = \kappa(u+u') + u^2 + u'^2 + uu'$$

and

$$\overline{\nabla}G(u,u') = u + u' + \frac{1}{3}(u^2 + u'^2 + uu'),$$

respectively.

4 Multi-symplectic integrators

We begin this section by reviewing the concept of multi-symplecticity in a general context, for more details, see e.g. [2, 3, 34]. A partial differential equation of the form $F(u, u_t, u_x, u_{tx}, \ldots) = 0$ is said to be *multi-symplectic* if it can be written as a system of first order equations:

$$M\, z_t + K\, z_x = \nabla_z S(z), \tag{43}$$

with $z \in \mathbb{R}^d$ a vector of state variables, typically consisting of the original variable u as one of its components. The matrices M and K are skew-symmetric $d \times d$-matrices, and S is a smooth scalar function depending on z. Equation (43) is not necessarily unique and the dimension d of the state vector may differ from one expression to another. A key observation for the multi-symplectic formulation (43) is that the matrices M and K define symplectic structures on subspaces of $\mathbb{R}^d$,

$$\omega = \mathrm{d}z \wedge M\mathrm{d}z, \qquad \kappa = \mathrm{d}z \wedge K\mathrm{d}z.$$

Considering any pair of solutions to the variational equation associated with (43), we have, see [3], that the following *multi-symplectic conservation law* applies

$$\partial_t\omega + \partial_x\kappa = 0. \tag{44}$$

With the two skew-symmetric matrices M and K, one can also define the density functions

$$\widetilde{E}(z) = S(z) - \frac{1}{2}z_x^T K z\,, \qquad \widetilde{F}(z) = \frac{1}{2}z_t^T K z,$$
$$\widetilde{G}(z) = S(z) - \frac{1}{2}z_t^T M z\,, \qquad \widetilde{I}(z) = \frac{1}{2}z_x^T M z,$$

which immediately yield the *local conservation laws*

$$\partial_t\widetilde{E}(z) + \partial_x\widetilde{F}(z) = 0 \qquad \text{and} \qquad \partial_t\widetilde{I}(z) + \partial_x\widetilde{G}(z) = 0,$$

for any solution to (43). Thus, under the usual assumption on vanishing boundary terms for the functions $\widetilde{F}(z)$ and $\widetilde{G}(z)$ one obtains the globally conserved quantities of (*energy* and *momentum*)

$$\mathcal{E}(z) = \int \widetilde{E}(z)\,\mathrm{d}x \qquad \text{and} \qquad \mathcal{I}(z) = \int \widetilde{I}(z)\,\mathrm{d}x.$$

Since the multi-symplectic conservation law (44) is a *local* conservation law, the multi-symplectic formulation of a partial differential equation may lead to numerical schemes which render well the local properties of the equation. To derive multi-symplectic integrators, we follow the approach given in [2] (see also [3]) and write the partial differential equation as a system of first order equations (43) and then discretise it. For an alternative construction of multi-symplectic integrators see for example [31].

The main philosophy behind the use of symplectic integrators for Hamiltonian differential equation is that the schemes are designed to preserve the symplectic form of the equation at each time step. For multi-symplectic partial differential equations, the idea of Bridges and Reich [3] was to develop integrators which satisfy a discretised version of the multi-symplectic conservation law (44). For this purpose, they considered a direct discretisation of (43), replacing the derivatives with divided differences, and the continuous function $z(x,t)$ by a discrete version $z^{n,i} \approx z(x_n, t_i)$ on a uniform rectangular grid. We set $\Delta x = x_{n+1} - x_n, n \in \mathbb{Z}$, and $\Delta t = t_{i+1} - t_i$, $i \geq 0$ as in Section 3.

Following their notation, we write

$$M\partial_t^{n,i} z^{n,i} + K\partial_x^{n,i} z^{n,i} = \nabla_z S(z^{n,i}), \tag{45}$$

where $\partial_t^{n,i}$ and $\partial_x^{n,i}$ are discretisations of the partial derivatives ∂_t and ∂_x, respectively. A natural way of inferring multi-symplecticity on the discrete level is to demand that for any pairs $(U^{n,i}, V^{n,i})$ of solutions to the corresponding variational equation of (45), one has

$$\partial_t^{n,i}\omega_{n,i} + \partial_x^{n,i}\kappa_{n,i} = 0,$$

where

$$\omega_{n,i}(U^{n,i}, V^{n,i}) = \langle MU^{n,i}, V^{n,i}\rangle, \qquad \kappa_{n,i}(U^{n,i}, V^{n,i}) = \langle KU^{n,i}, V^{n,i}\rangle,$$

with the Euclidean scalar product $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^d$.

As for the Camassa-Holm equation, see [10], setting $z = [u, \phi, w, v, \nu]^T$, we derive the following multi-symplectic formulation (43) for the generalised hyperelastic-rod wave (1):

$$\begin{pmatrix} 0 & 1/2 & 0 & 0 & -1/2 \\ -1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 \end{pmatrix} z_t + \begin{pmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} z_x = \begin{pmatrix} -w - \frac{1}{2}g(u) - \gamma\frac{\nu^2}{2} \\ 0 \\ -u \\ \nu \\ -\gamma u\nu + v \end{pmatrix}, \tag{46}$$

with the scalar function $S(z) = -wu - \frac{1}{2}G(u) - \gamma u\frac{\nu^2}{2} + \upsilon\nu$, recalling $G(u) := \int g(u)$.

We now turn to the calculation of the global invariants (energy and momentum) defined above. For the hyperelastic-rod wave, an integration of the conservation law $\partial_t\widetilde{I}(z) + \partial_x\widetilde{G}(z) = 0$ leads to:

$$\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int(-u_x\phi + u_x^2 + u^2 - uu_{xx})\mathrm{d}x + \big[\widetilde{G}(z)\big] = 0,$$

where the brackets stand for the difference of the function evaluated at the upper and lower limit of the integral. As in [10], after an integration by parts on the first and last term, using periodic (or vanishing at infinity) boundary conditions of u (i.e. $[u] = [u_x] = [u_{xx}] = [\phi_t] = 0$), we obtain the following global invariant for the hyperelastic-rod wave:

$$\mathcal{I} = \frac{1}{2}\int(u^2 + u_x^2)\mathrm{d}x.$$

Similarly, the second conservation law $\partial_t\widetilde{E}(z) + \partial_x\widetilde{F}(z) = 0$ leads to

$$\mathcal{E} = -\frac{1}{2}\int(u^3 + \gamma u u_x^2)\mathrm{d}x.$$

We remark that these two conserved quantities are (up to a multiplicative constant) the Hamiltonian functionals given in (9)-(10).

The Euler box scheme. By taking the splitting $M = M_+ + M_-$ with $M_+ = M_- = \frac{1}{2}M$ (and similarly for K) we obtain the Euler-box scheme, a multi-symplectic integrator for the generalised hyperelastic-rod wave, expressed in terms of u (see [10] and [34]):

$$\begin{aligned}
- \quad & 4\Delta x^2 u^{n,i+1} + \big(u^{n+2,i+1} - 2u^{n,i+1} + u^{n-2,i+1}\big) = \\
8 \quad & \Delta x^2\Delta t\Big\{-\frac{1}{2\Delta t}u^{n,i-1} - \frac{1}{2\Delta x}\Big(-\frac{1}{2}g(u^{n+1,i}) + \frac{1}{2}g(u^{n-1,i}) - \frac{\gamma}{8\Delta x^2}(u^{n+2,i} - u^{n,i})^2 \\
+ \quad & \frac{\gamma}{8\Delta x^2}(u^{n,i} - u^{n-2,i})^2 + \frac{1}{4\Delta x\Delta t}(-u^{n+2,i-1} + 2u^{n,i-1} - u^{n-2,i-1}) \\
+ \quad & \frac{\gamma}{4\Delta x^2}(u^{n+2,i}(u^{n+3,i} - u^{n+1,i}) - 2u^{n,i}(u^{n+1,i} - u^{n-1,i}) + u^{n-2,i}(u^{n-1,i} - u^{n-3,i}))\Big)\Big\}.
\end{aligned}$$

Equation (1) can be rewritten in the form

$$u_t - u_{xxt} + \big(\tfrac{1}{2}g(u) + \tfrac{\gamma}{2}u_x^2\big)_x - \gamma(uu_x)_{xx} = 0 \tag{47}$$

and the corresponding Euler-box scheme is given in a more compact form by

$$\delta_t u^{n,i} - \delta_x^2\delta_t u^{n,i} + \delta_x\big(\tfrac{1}{2}g(u^{n,i}) + \tfrac{\gamma}{2}(\delta_x u^{n,i})^2\big) - \gamma\delta_x^2(u^{n,i}\delta_x u^{n,i}) = 0, \tag{48}$$

recalling from Section 3 the definitions of the centered differences $\delta_x = \frac{1}{2}(\delta_x^+ + \delta_x^-)$ and, similarly in time, $\delta_t = \frac{1}{2}(\delta_t^+ + \delta_t^-)$. Note that this scheme is only linearly implicit.

5 Numerical experiments

In this section, we present some numerical experiments for the hyperelastic-rod wave equation (3). We consider two types of initials conditions: A smooth traveling wave and a single peakon. They are obtained in the following way (see [19, 29] for a derivation of all the traveling wave of (3)). Looking at the "Hamiltonian-like" formulation of (3) with (9), we define

$$v = u - \gamma u_{xx}$$

so that $m = \frac{\gamma - 1}{\gamma} u + \frac{v}{\gamma}$ and the partial differential equation becomes

$$\frac{1}{\gamma}((\gamma - 1)u_t + v_t) + (vu)_x + vu_x = 0. \tag{49}$$

For a traveling wave with speed c, we have

$$u(t, x) = U(x - ct) \text{ and } v(t, x) = V(x - ct)$$

and (49) yields

$$-\frac{c}{\gamma}((\gamma - 1)U' + V') + V'U + 2VU' = 0.$$

Thus,

$$(U - \frac{c}{\gamma})V' + 2U'(V - \frac{c}{2\gamma}(\gamma - 1)) = 0. \tag{50}$$

After multiplying both sides of (50) by $(u - \frac{c}{\gamma})$, we get

$$(U - \frac{c}{\gamma})^2 V' + 2U'(U - \frac{c}{\gamma})(V - \frac{c}{2\gamma}(\gamma - 1)) = 0$$

which can be integrated and gives

$$(V - \frac{c}{2\gamma}(\gamma - 1))(U - \frac{c}{\gamma})^2 = \alpha \tag{51}$$

for some constant α. Using the fact that $V = U - \gamma U''$, we can rewrite (51) and obtain

$$U'' = -\frac{c(\gamma - 1)}{2\gamma^2} + \frac{1}{\gamma} U - \frac{\alpha\gamma}{(\gamma U - c)^2} \tag{52}$$

which is a second order equation for the traveling wave U. After multiplying (52) by U_x and integrating one more time we recover the equations given in [19, 29]. However, (52) may be easier to implement numerically. We do not obtain smooth traveling waves for all the values of the parameters α, c and γ. For $c = \alpha = 3$, we solve numerically (52) with initial data $U(0) = 1$ and $U_x(0) = 0$. The results are presented in Figure 1 for different values of γ.

Taking $\alpha = 0$ in (52), we obtain the peakons. Indeed, on the line, the general solution of this second order differential equation is given by

$$U(\zeta) = \frac{c(\gamma - 1)}{2\gamma} + A\mathrm{e}^{-\zeta/\sqrt{\gamma}} + B\mathrm{e}^{\zeta/\sqrt{\gamma}},$$

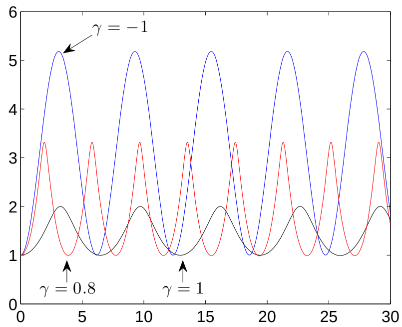

Figure 1: Smooth traveling waves for the hyperelastic rod equation for different values of γ.

for some constants A and B. As it is noted in [29], a traveling wave can only have a point of discontinuity ζ_0 when U reaches the value $\frac{c}{\gamma}$, that is, $U(\zeta_0) = \frac{c}{\gamma}$. For a single peakon, there is only one point of discontinuity ζ_0 (the top of the peak) and we impose $\zeta_0 = 0$. To obtain vanishing at infinity boundary conditions, we must have $A = B$ and thus

$$U(\zeta) = \frac{c}{\gamma}\big(\frac{\gamma-1}{2} + (1 - \frac{\gamma-1}{2})\mathrm{e}^{-|\zeta|/\sqrt{\gamma}}\big)$$

so that, on the line, the peakon-solution of the hyperelastic-rod wave is then given by

$$u(x,t) = \frac{c}{2\gamma}\big(\gamma - 1 + (3-\gamma)\mathrm{e}^{-|x-ct|/\sqrt{\gamma}}\big).$$

Still for $\alpha = 0$, by choosing the points of discontinuity at $-T/2$ and $T/2$, we obtain the periodic peakon. On the interval $[-T/2, T/2]$, this gives

$$U(\zeta) = \frac{c}{2\gamma}\big(\gamma - 1 + \frac{(3-\gamma)}{\cosh(T/(2\sqrt{\gamma}))}\cosh(\frac{\zeta}{\sqrt{\gamma}})\big),$$

so that the periodic peakon is

$$u(x,t) = \frac{c}{2\gamma}\Big(\gamma - 1 + \frac{(3-\gamma)}{\cosh(T/(2\sqrt{\gamma}))}\cosh\Big(\frac{d(x-ct)}{\sqrt{\gamma}}\Big)\Big), \tag{53}$$

for $d(x) = \bar{x} - \frac{T}{2}$ where $\bar{x}$ is the unique element of $[0, T)$ for which there exists $k \in \mathbb{Z}$ such that $\bar{x} = x + \frac{T}{2} + kT$.

Before we proceed with the numerical experiments, let us give some remarks concerning implementation issues. For the multi-symplectic scheme (48) applied to equation (3), the first needed step for the iteration will be computed along the exact solution of the problem. The integrals in the Hamiltonians given in (9) and (10) will be discretised by the trapezoidal rule and the derivatives appearing in these quantities by the symmetric derivative δ_x. Thus, these discretisations will give us the conserved quantities (24), resp. (25) from the energy-preserving schemes (40), (41) and (42).

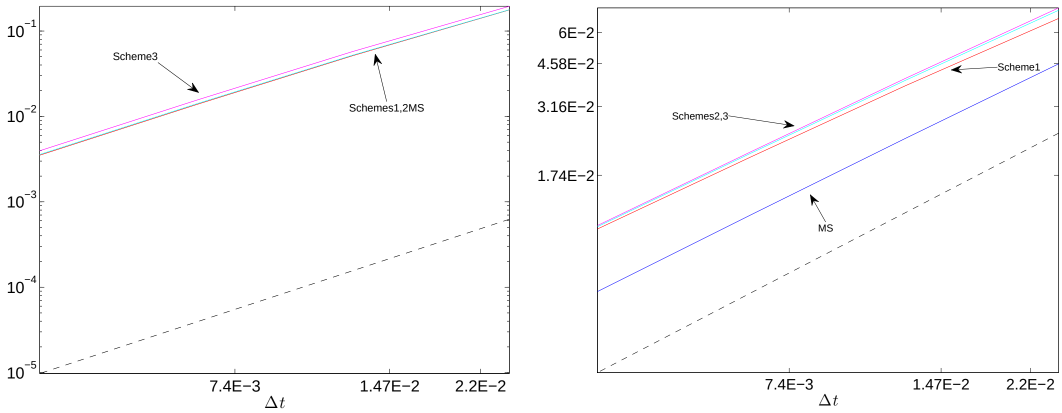

Figure 2: L^1-error of the schemes (40), (41), (42) and (48) at time $Tend = 2$ for the smooth solution (left) and for the peakon (right). The dashed lines have slopes two, resp. one.

All the numerical experiments will be done for the hyperelastic-rode wave with the constant $\gamma = 0.8$. The smooth traveling wave considered will be the solution of (52) with $c = \alpha = 3$. In this case we obtain a period $T \approx 3.8609$ for the traveling wave. For the single peakon (53), we take $T = 40$ and $c = 1$.

We first consider the temporal rate of convergence of our schemes. We vary the time step Δt and set the space step to $\Delta x = 0.9\ \Delta x/c$. One can see from Figure 2 that the order of convergence is two for the smooth solution and one for non-smooth one, and this holds for all the schemes.

Similar behaviours are also observed for the spatial rate of convergence of the numerical methods: order one for the non-smooth solution and order two for the smooth one. The results are however not displayed.

We next plot the discretisations (24) and (25) of the Hamiltonian functionals of our problem. For the smooth solution, the grid parameters are $\Delta x = 0.04$ and $\Delta t = 0.01$. For the single peakon solution, they are given by $\Delta x = 0.27$ and $\Delta t = 0.01$. The integrations are done over the time interval $[0, 5]$. Figures 3 and 4 display the results for the discretisation of the Hamiltonians given by (24) and (25), respectively. We have noted that, when taking smaller time step size, the results given by the multi-symplectic scheme tends to those of the energy-preserving schemes (41) and (42).

Let us conclude this paper by two remarks. First, from our numerical experiments, we can see that all the schemes perform in a comparable manner, and in particular it is not clear if one can take advantage of the global or local nature of the schemes (global for the schemes (40), (41), (42) as they preserve one of the Hamiltonians, or local for the multi-symplectic scheme (48)). The second remark is about the degree of freedom we have when deriving the schemes that have been presented. We already saw that the discrete gradient

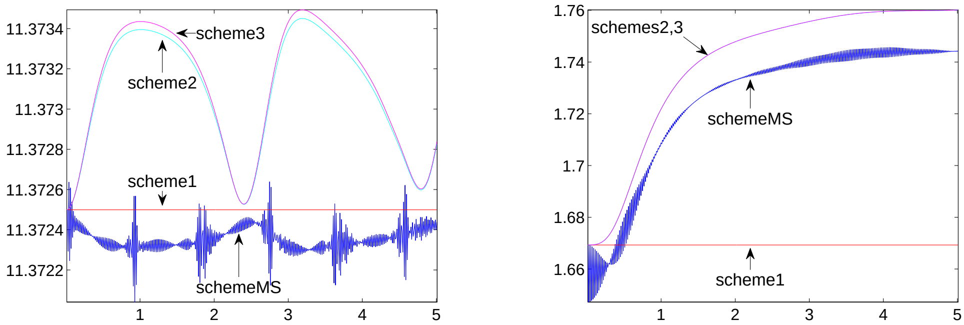

Figure 3: The Hamiltonian (24) along the numerical solutions given by the schemes (40), (41), (42) and (48) for the smooth (left) and non-smooth (right) solution.

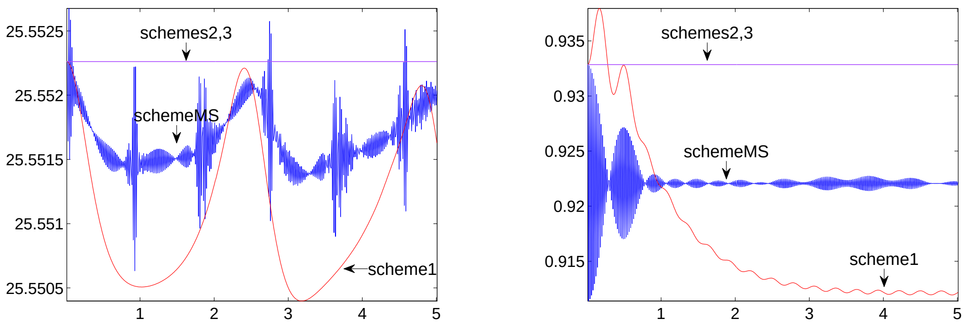

Figure 4: The Hamiltonian (25) along the numerical solutions given by the schemes (40), (41), (42) and (48) for the smooth (left) and non-smooth (right) solution.

of a function is not unique and presented two ways of computing it. In addition, when discretising the Hamiltonian functionals $\mathcal{H}_1$ and $\mathcal{H}_2$ and the antisymmetric operators $\mathcal{D}_1$ and $\mathcal{D}_2$, we used systematically the symmetric discrete derivative δ. We could have used instead left and right discrete derivatives and obtain schemes with the same preserving property. For example, instead of (35), we can take

$$D_1(m)(v) = -((u - \gamma\delta_x^2 u)\delta_x^- v) - \delta_x^+((u - \gamma\delta_x^2 u)v). \tag{54}$$

By using the discrete summation by part rule (23), we can check that this operator is antisymmetric and, in the same way as we derived from (35) the numerical scheme (40), we can obtain from (54) a numerical scheme that *exactly* preserves the discrete Hamiltonian H_1. We have implemented this particular scheme and observed that it may be very unstable, for example in the case of a smooth wave (traveling from left to right) as initial data. This bad behaviour is due to the discrete difference operator δ_x^+ in (54), which models the transport of the momentum $u - \gamma u_{xx}$ at a speed u. In the case we are looking at, the "information" is traveling in the same direction as the wave, from left to right, but the right discrete derivative δ_x^+ compute the difference by taking values from the opposite direction, from the right. We can observe that, if we consider as initial data a wave now traveling from right to left, the same scheme performs well. This confirms the stabilizing effect of the symmetric discrete derivative and justifies its use. It also shows that the preservation of energy alone does not guarantee the well-behaviour of a scheme.

6 Acknowledgement

We would like to thank Ernst Hairer, Brynjulf Owren for interesting discussions.

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