Weak convergence analysis of numerical schemes for stochastic PDEs

Adam Andersson

Matematiska vetenskaper
Chalmers tekniska högskola

Svenska matematikersamfundets höstmöte
Luleå
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Outline

- A first look at stochastic PDEs

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- Malliavin calculus

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- Numerical approximation

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- Weak error analysis

Applications of stochastic PDEs

- Filtering theory (The Zakai equation)

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- Spatial statistics (Ozon data / Desert spread)

A first look at stochastic PDEs

A semi-linear evolution equation

$$\dot{X}(t) + AX(t) = F(X(t)), \quad t \in (0, T], \; X(0) = X_0.$$

on H, a separable Hilbert space.

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Has a mild solution $X \in C([0, T], H)$ satisfying the fixed point equation

$$X(t) = E(t)X_0 + \int_0^t E(t-s)F(X(s))\,\mathrm{d}s, \quad t \in (0, T].$$

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A stochastic semi-linear heat equation in a preliminary form

$$\mathrm{d}X(t) + AX(t)\,\mathrm{d}t = F(X(t))\,\mathrm{d}t + \sum_{i\in\mathbf{N}} g_i(X(t))\,\mathrm{d}\beta_i(t),$$

where $g_i \in \mathrm{Lip}(H, H)$, $i \in \mathbf{N}$, with $\sum_{i\in\mathbf{N}} \|g_i(x)\|^2 < \infty$, $\forall x \in H$.

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Let $(h_i)_{i\in\mathbf{N}} \subset H$ be an ON-basis. We write

$$\int_0^t \sum_{i\in\mathbf{N}} g_i(X(t))\,\mathrm{d}\beta_i(t) = \int_0^t \underbrace{\Big(\sum_{i\in\mathbf{N}} g_i(X(t)) \otimes h_i\Big)}_{:=G(X(t))}\,\mathrm{d}\underbrace{\Big(\sum_{j\in\mathbf{N}} h_j \otimes \beta_j(t)\Big)}_{:=W(t)}$$

$$=: \int_0^t G(X(t))\,\mathrm{d}W(t).$$

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$G \in \mathrm{Lip}(H, \mathrm{HS})$. $W(t)$ can be defined properly.

A first look at stochastic PDEs

We can now write

$$\begin{aligned} &\mathrm{d}X(t) + AX(t)\,\mathrm{d}t = F(X(t))\,\mathrm{d}t + G(X(t))\,\mathrm{d}W(t), \quad t \in (0, T], \\ &X(0) = X_0. \end{aligned}$$

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The equation has a unique mild solution $X \in C([0, T], L_2(\Omega, H))$

$$\begin{aligned}X(t) = E(t)X_0 &+ \int_0^t E(t-s)F(X(s))\,\mathrm{d}s\\ &+ \int_0^t E(t-s)G(X(s))\,\mathrm{d}W(s), \quad t \in (0, T].\end{aligned}$$

Malliavin calculus

Define the map $I\colon L_2([0,T],H) \to L_2(\Omega)$ by

$$I(\phi) = \int_0^T \langle \phi(t), \mathrm{d}W(t)\rangle, \quad \phi \in L_2([0,T],H).$$

Let $C_{\mathrm{p}}^\infty(\mathbf{R}^n)$ denote the space of all C^∞-functions over $\mathbf{R}^n$ with polynomial growth. Define

$$\mathcal{S} = \{X = f(I(\phi_1),\dots,I(\phi_n))\colon f \in C_{\mathrm{p}}^\infty(\mathbf{R}^n), \\ \phi_1,\dots,\phi_n \in L_2([0,T],H),\ n \geq 1\}$$

and

$$\mathcal{S}(H) = \Big\{F = \sum_{k=1}^m X_k \otimes h_k\colon X_1,\dots,X_m \in \mathcal{S},\ h_1,\dots,h_m \in H,\ m \geq 1\Big\}.$$

Malliavin calculus

For $F \in \mathcal{S}(H)$ with representation

$$F = \sum_{k=1}^{m} f_k(I(\phi_1), \ldots, I(\phi_n)) \otimes h_k,$$

We define the Malliavin derivative $DF \in L_2([0, T] \times \Omega, \mathrm{HS})$ as the process

$$D_t F = \sum_{k=1}^{m} \sum_{i=1}^{n} \partial_i f_k(I(\phi_1), \ldots, I(\phi_n)) \otimes (h_k \otimes \phi_i(t)),$$

and let, for $h \in H$, the directional derivative be given by

$$D_t^h F = D_t F h = \sum_{k=1}^{m} \sum_{i=1}^{n} \partial_i f_k(I(\phi_1), \ldots, I(\phi_n)) \otimes \langle \phi_i(t), h \rangle \otimes h_k.$$

Malliavin calculus: integration by parts

For all $F \in \mathcal{S}(H)$ and $\Phi \in L_2([0,T],\mathrm{HS})$,

$$\mathbf{E}\langle DF, \Phi\rangle_{L_2([0,T],\mathrm{HS})} = \mathbf{E}\Big\langle F, \int_0^T \Phi(t)\,\mathrm{d}W(t)\Big\rangle_H. \qquad (1)$$

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Let $\mathbf{D}^{1,2}(H)$ be the closure of $\mathcal{S}(H)$ with respect to the norm

$$\|F\|_{\mathbf{D}^{1,2}(H)} = \Big(\mathbf{E}[\|F\|_H^2] + \mathbf{E}\Big[\int_0^T \|D_tF\|_{\mathrm{HS}}^2\,\mathrm{d}t\Big]\Big)^{\frac{1}{2}}.$$

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For $\Phi \in \mathrm{dom}(\delta) \subset L_2([0,t] \times \Omega, H)$, i.e., for Φ such that the left hand side of (1) defines a bounded linear functional on $\mathbf{D}^{1,2}(H)$, we define the adjoint operator δ by

$$\mathbf{E}\langle DF, \Phi\rangle_{L_2([0,T],\mathrm{HS})} = \mathbf{E}\langle F, \delta(\Phi)\rangle_H, \quad \forall F \in \mathbf{D}^{1,2}(H).$$

The Itô integral

Let $\mathcal{F}_t = \sigma\{\beta_i(s)\colon 0 \le s \le t, i \in \mathbf{N}\}$. One can prove that for an $\mathcal{F}_t$-adapted process $\Phi \in L_2([0, T] \times \Omega; \mathrm{HS})$, $\delta(\Phi)$ coincides with the Itô integral of Φ, as defined by Da Prato and Zabczyk (1992). We write

$$\delta(\Phi) = \int_0^T \Phi(t)\,\mathrm{d}W(t).$$

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Itô's formula

$$G(X(t)) = \int_0^t \langle DG(X(s)), \phi(s)\rangle\,\mathrm{d}s + \int_0^t \langle DG(X(s)), \Phi(s)\,\mathrm{d}W(s)\rangle + \frac{1}{2}\int_0^t \mathrm{Tr}(D^2 G(X(s))\Phi(s)\Phi(s)^*)\,\mathrm{d}s,$$

where

$$X(t) = \int_0^t \phi(s)\,\mathrm{d}s + \int_0^t \Phi(s)\,\mathrm{d}W(s).$$

A non-linear stochastic heat equation

We consider the following stochastic fixed point equation

$$X(t) = E(t)X_0 + \int_0^t E(t-s)F(X(s))\,\mathrm{d}s + \int_0^t E(t-s)G(X(s))\,\mathrm{d}W(s), \quad t \in (0, T],$$

where

- $H = L_2([0,1])$. $A = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$, $\mathcal{D}(A) = H_0^1([0,1]) \cap H^2([0,1])$, in this case $(E(t))_{t\geq 0}$ is an analytic semigroup,

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- $X_0 \in H$,
- $F \in C_b^2(H,H)$,
- $G(x) = B + Cx + \tilde{G}(x)$, where $B \in \mathcal{L}(H,H)$, $C \in \mathcal{L}(H,\mathcal{L}(H,H))$ and $\tilde{G} \in C_b^2(\mathcal{D}(A^{-\frac{1}{4}}), \mathcal{L}(H,H))$.

Approximation by the finite element method

A discretized equation:

$$\begin{cases} \mathrm{d}X_h(t) + [A_h X_h(t) - P_h F(X_h(t))]\,\mathrm{d}t = P_h G(X_h(t))\,\mathrm{d}W(t), & t \in (0, T] \\ X_h(0) = P_h X_0. \end{cases}$$

Finite element spaces $\{S_h\}_{h\in(0,1]}$ of continuous piecewise linear functions corresponding to a quasi-uniform family of partitions of the form $0 = \xi_1 < \xi_2 < \cdots < \xi_{N_h} = 1$ with $h = \max_{1\le n\le N}(\xi_n - \xi_{n-1})$.

A_h is the discrete Laplacian defined by

$$\langle A_h \psi, \chi \rangle_H = \langle \nabla\psi, \nabla\chi \rangle_H, \quad \forall \psi, \chi \in S_h.$$

$P_h \colon H \to S_h$ orthogonal projection w.r.t. $\langle \cdot, \cdot \rangle_H$.

Mild solution of spatially discretized equation

Let $\{E_h(t)\}_{t\geq 0}$ be the analytic semigroup generated by $-A_h$.

For every $h \in (0,1]$ $\exists!$ solution $X_h \in C([0,T], L_2(\Omega, S_h))$ to the mild equation

$$X_h(t) = E_h(t)P_hX_0 + \int_0^t E_h(t-s)P_hF(X_h(s))\,\mathrm{d}s + \int_0^t E_h(t-s)P_hG(X_h(s))\,\mathrm{d}W(s), \quad t \in (0,T].$$

Main result

Theorem

For every test function $\Phi \in C_b^2(H, \mathbf{R})$ *and* $\epsilon > 0$ *there exists a* $C > 0$ *such that*

$$\begin{aligned} \mathbf{E}[\Phi(X(T)) - \Phi(X_h(T))] &= \int_H \Phi(x)\, d\mu(x) - \int_H \Phi(x)\, d\mu_h(x) \\ &\leq Ch^{1-\epsilon}, \end{aligned}$$

where $\mu = \mathcal{L}(X(T)) = \mathbf{P} \circ X(T)^{-1}$ *and* $\mu_h = \mathcal{L}(X_h(T)) = \mathbf{P} \circ X_h(T)^{-1}$.

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Sketch of proof:

Method by A. Debussche (2011).

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Method by A. Debussche (2011).

For the presentation, let $F = 0$ and $X_0 = 0$.

Let $u(t,x) = \mathbf{E}\Phi(X^x(t))$, $t \in [0,T]$ and $x \in H$ where $X^x(0) = x$.
Satisfies the Kolmogorov equation

$$
\begin{aligned}
&u_t(t,x) + Lu(t,x) = 0, \quad t \in [0,T), \quad x \in \mathcal{D}(A),\\
&u(0,x) = \Phi(x), \quad x \in H.
\end{aligned}
$$

Here

$$
Lu(t,x) = \langle Ax, u_x(t,x)\rangle - \frac{1}{2}\,\mathrm{Tr}\{g(x)g^*(x)u_{xx}(t,x)\}.
$$

By Itô's formula and the Kolmogorov equation

$$
\begin{aligned}
&\mathbf{E}[\Phi(X(T)) - \Phi(X_h(T))] \\
&\quad = \mathbf{E}[u(T, X_0) - u(0, X_h(T))] \\
&\quad = \mathbf{E}\Big[\int_0^T u_t(T-s, X_h(s)) + L_h u(T-s, X_h(s))\,\mathrm{d}s\Big] \\
&\quad = \mathbf{E}\Big[\int_0^T (L_h - L)u(T-s, X_h(s))\,\mathrm{d}s\Big] \\
&\quad = \mathbf{E}\Big[\int_0^T \langle (A_h - A)X_h(s), u_x(T-s, X_h(s))\rangle\,\mathrm{d}s \\
&\qquad -\frac{1}{2}\int_0^T \mathrm{Tr}\{(P_h g(X_h(s))g^*(X_h(s))P_h \\
&\qquad\qquad -g(X_h(s))g^*(X_h(s)))u_{xx}(t, X_h(s))\}\,\mathrm{d}s\Big] \\
&\quad = I + J
\end{aligned}
$$

$$I \leq \left| \mathbf{E} \int_0^T \left\langle A_h P_h (I - R_h) u_x(T-s, X_h(s)), X_h(s) \right\rangle \mathrm{d}s \right| \qquad (R_h = A_h^{-1} P_h A)$$

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Recall that

$$X_h(s) = \int_0^s E_h(s-r) P_h g(X_h(r)) \, \mathrm{d}W(r).$$

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Recall that

$$X_h(s) = \int_0^s E_h(s-r) P_h g(X_h(r)) \, \mathrm{d}W(r).$$

Then by substitution and integration by parts

$$\begin{aligned} I \leq \Big| \mathbf{E} \int_0^T \Big\langle & u_x(T-s, X_h(s)), \\ & \int_0^s (A_h P_h (I - R_h))^* E_h(s-r) P_h g(X_h(r)) \, \mathrm{d}W(r) \Big\rangle \, \mathrm{d}s \Big| \\ = \Big| \mathbf{E} \int_0^T \int_0^s & \operatorname{Tr} \Big\{ g^*(X_h(r)) P_h E_h(s-r) A_h P_h (I - R_h) \\ & u_{xx}(T-s, X_h(s)) D_r X_h(s) \Big\} \, \mathrm{d}r \, \mathrm{d}s \Big|. \end{aligned}$$

Distributing powers of A and A_h yields

$$
\begin{aligned}
I \leq \Big| \mathbf{E} \int_0^T \int_0^s \operatorname{Tr} \Big\{ & g^*(X_h(r)) P_h E_h(s-r) A_h P_h (I - R_h) \\
& u_{xx}(T-s, X_h(s)) D_r X_h(s) \Big\} \, \mathrm{d}r \, \mathrm{d}s \Big| \\
\leq \mathbf{E} \int_0^T \int_0^s & \| g^*(X_h(s)) \|_{\mathcal{L}(H)} \| A_h^{1-3\epsilon} E_h(s-r) P_h \|_{\mathcal{L}(H)} \\
& \times \| A_h^{3\epsilon} P_h (I - R_h) A^{-\frac{1}{2}+\epsilon} \|_{\mathcal{L}(H)} \| A^{\frac{1}{2}-\epsilon} u_{xx}(T-s, X_h(s)) A^{\frac{1}{2}-\epsilon} \|_{\mathcal{L}(H)} \\
& \times \| A^{-\frac{1}{2}+\epsilon} A^{-2\epsilon} \|_{\mathcal{L}_1(H)} \| A^{2\epsilon} D_r X_h(s) \|_{\mathcal{L}(H)} \, \mathrm{d}r \, \mathrm{d}s \\
\leq C h^{1-8\epsilon} \int_0^T \int_0^s & (T-s)^{-1+2\epsilon} (s-r)^{-1+\epsilon} \, \mathrm{d}r \, \mathrm{d}s.
\end{aligned}
$$

Estimates

Let $0 \leq s \leq r \leq 1$. Then

$$\|A_h^{\frac{s}{2}} P_h (I - R_h) A^{-\frac{r}{2}}\|_{\mathcal{L}(H)} \leq C h^{r-s}, \quad h \in (0, 1].$$

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Let $\gamma \geq 0$. Then

$$\|A^{\gamma} E(t)\|_{\mathcal{L}(H)} + \|A_h^{\gamma} E_h(t) P_h\|_{\mathcal{L}(H)} \leq C t^{-\gamma}, \quad \forall t > 0.$$

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Let $\lambda \in [0, \frac{1}{2})$. Then

$$\|A^{\lambda} u_x(t,x)\|_{\mathcal{L}(H)} \le C t^{-\lambda} |\Phi|_{C_{\mathrm{b}}^1}, \quad \forall t \in (0,T], \quad \forall x \in H.$$

Estimates

Let $0 \le s \le r \le 1$. Then

$$\|A_h^{\frac{s}{2}} P_h (I - R_h) A^{-\frac{r}{2}}\|_{\mathcal{L}(H)} \le C h^{r-s}, \quad h \in (0,1].$$

Let $\gamma \ge 0$. Then

$$\|A^\gamma E(t)\|_{\mathcal{L}(H)} + \|A_h^\gamma E_h(t) P_h\|_{\mathcal{L}(H)} \le C t^{-\gamma}, \quad \forall t > 0.$$

Let $\lambda \in [0, \frac{1}{2})$. Then

$$\|A^\lambda u_x(t,x)\|_{\mathcal{L}(H)} \le C t^{-\lambda} |\Phi|_{C_{\mathrm{b}}^1}, \quad \forall t \in (0,T], \quad \forall x \in H.$$

Let $\lambda, \rho \in [0, \frac{1}{2})$. Then

$$\|A^\rho u_{xx}(t,x) A^\lambda\|_{\mathcal{L}(H)} \le C t^{-(\rho+\lambda)} |\Phi|_{C_{\mathrm{b}}^2}, \quad \forall t \in (0,T], \quad \forall x \in H.$$

Estimates

Let $0 \le s \le r \le 1$. Then

$$\|A_h^{\frac{s}{2}} P_h (I - R_h) A^{-\frac{r}{2}}\|_{\mathcal{L}(H)} \le C h^{r-s}, \quad h \in (0,1].$$

Let $\gamma \ge 0$. Then

$$\|A^{\gamma} E(t)\|_{\mathcal{L}(H)} + \|A_h^{\gamma} E_h(t) P_h\|_{\mathcal{L}(H)} \le C t^{-\gamma}, \quad \forall t > 0.$$

Let $\lambda \in [0, \frac{1}{2})$. Then

$$\|A^{\lambda} u_x(t,x)\|_{\mathcal{L}(H)} \le C t^{-\lambda} |\Phi|_{C_{\mathrm{b}}^1}, \quad \forall t \in (0,T], \quad \forall x \in H.$$

Let $\lambda, \rho \in [0, \frac{1}{2})$. Then

$$\|A^{\rho} u_{xx}(t,x) A^{\lambda}\|_{\mathcal{L}(H)} \le C t^{-(\rho+\lambda)} |\Phi|_{C_{\mathrm{b}}^2}, \quad \forall t \in (0,T], \quad \forall x \in H.$$

Let $\gamma \in [0, \frac{1}{2})$. Then

$$\mathbf{E}[\|A_h^{\gamma} D_s X_h(t)\|_{\mathcal{L}(H)}^2] \le C(t-s)^{-2\gamma}, \quad \forall t \in [0,T], \quad \forall s \in [0,t).$$

Thank you for your attention!