C^*-DYNAMICS AND CROSSED PRODUCTS
Third exercise sheet

EUSEBIO GARDELLA

You must hand in a minimum of three exercises by December 12th.

Exercise 1. Let $\alpha\colon \{-1,1\} \to \mathrm{Aut}(C([-1,1]))$ be the action induced by multiplication by -1 on $[-1,1]$.

(1) Describe the spectral subspaces of α.
(2) Compute $\mathrm{Sp}(\alpha)$, $\Gamma(\alpha)$, $\widetilde{\mathrm{Sp}}(\alpha)$ and $\widetilde{\Gamma}(\alpha)$ directly from their definitions.

Exercise 2. Let G be a compact abelian group, let $\mathcal{H}$ be a Hilbert space, and let $u\colon G \to \mathcal{U}(\mathcal{H})$ be a unitary representation. We set

$$\mathrm{Sp}(u) = \{\chi \in \widehat{G}\colon \text{there exists } \xi \in \mathcal{H} \setminus \{0\} \text{ with } u_g(\xi) = \chi(g)\xi \text{ for all } g \in G\}.$$

(1) Describe the eigenspaces of the action $\mathrm{Ad}(u)\colon G \to \mathrm{Aut}(\mathcal{K}(\mathcal{H}))$ in terms of the eigenspaces of u.
(2) Prove that $\widetilde{\mathrm{Sp}}(\mathrm{Ad}(u)) = \widehat{G}$ if and only if $\mathrm{Sp}(u) = \widehat{G}$.

Exercise 3.

(1) Construct an example of a compact abelian group action $\alpha\colon G \to \mathrm{Aut}(A)$ for which there exists $B \in \mathrm{Her}_B(A)$ such that $\mathrm{Sp}(\alpha) \neq \mathrm{Sp}(\alpha|_B)$. Deduce that $\mathrm{Sp}(\alpha)$ is generally different from $\widetilde{\mathrm{Sp}}(\alpha)$.
(2) Construct an example of a compact abelian group action $\alpha\colon G \to \mathrm{Aut}(A)$ for which there exists $B \in \mathrm{Her}_B(A)$ such that $\Gamma(\alpha) \neq \Gamma(\alpha|_B)$. Deduce that $\Gamma(\alpha)$ is generally different from $\widetilde{\Gamma}(\alpha)$.

Exercise 4. Let G be a locally compact abelian group, acting trivially on $\mathbb{C}$. Show that the dual action $\widehat{G} \to \mathrm{Aut}(C^*(G))$ can be identified with the left translation action of $\widehat{G}$ on itself.

Exercise 5. Give a complete proof of Takai duality for $G = \mathbb{Z}_2$ and A unital. In this case, the isomorphism can be described very explicitly.

Exercise 6. Let Γ be a discrete abelian group, and set $G = \widehat{\Gamma}$. Let $\beta\colon \Gamma \to \mathrm{Aut}(B)$ be an action, and set

$$A = B \rtimes_\beta \Gamma \quad \text{and} \quad \alpha = \widehat{\beta}\colon G \to \mathrm{Aut}(A).$$

Show that $A(\chi) = Bu_\chi$ for all $\chi \in \widehat{G} \cong \Gamma$. In particular, this shows that the fixed point algebra of the dual action $\widehat{\beta}$ is B.

Exercise 7. Let $\theta \in \mathcal{R} \setminus \mathbb{Q}$ and consider the irrational rotation algebra A_θ. Let $\gamma\colon S^1 \to \mathrm{Aut}(A_\theta)$ be the gauge action, which is given by

$$\gamma_z(u) = u \quad \text{and} \quad \gamma_z(v) = zv$$

for all $z \in S^1$.

Date: November 23, 2018.

(1) Compute the spectral subspaces of A_θ with respect to γ.
(2) Compute $\widetilde{\Gamma}(\gamma)$.
(3) Show, using Takai duality and the characterization of ideals in the crossed product in the case of full strong Connes spectrum, that A_θ is simple.

Exercise 8. Let G be a locally compact abelian group, let A be a C^*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. Given a $\widehat{G}$-invariant ideal J in $A \rtimes_\alpha G$, let I_J be the unique G-invariant ideal in A such that $J \rtimes_{\widehat{\alpha}} \widehat{G} = I_J \otimes \mathcal{K}(L^2(G))$. Show that $J = I_J \rtimes_\alpha G$.

Exercise 9. Let V be an abelian semigroup. If G is an abelian group and $\varphi\colon V \to G$ is a semigroup homomorphism, show that there exists a unique group homomorphism $\psi\colon G(V) \to G$ satisfying $\psi \circ \iota_V = \varphi$. In other words, the following diagram commutes

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\iota_V"'] & G \\ G(V). \arrow[ur, dashed, "\exists!\psi"'] & \end{tikzcd}$$

Morever, if $\mathcal{G}(V)$ is another abelian group and $j_V\colon V \to \mathcal{G}(V)$ is a semigroup homomorphism satisfying the same property as above, show that there exists an isomorphism $\theta\colon G(V) \to \mathcal{G}(V)$ satisfying $\theta \circ \iota_V = j_V$.

Eusebio Gardella Mathematisches Institut, Fachbereich Mathematik und Informatik der Universität Münster, Einsteinstrasse 62, 48149 Münster, Germany.
E-mail address: gardella@uni-muenster.de
URL: www.math.uni-muenster.de/u/gardella/