AN INTRODUCTION TO L^p-OPERATOR ALGEBRAS

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Abstract. The study of operator algebras on Hilbert spaces, and C^*-algebras in particular, is one of the most active areas within Functional Analysis. A natural generalization of these is to replace Hilbert spaces (which are L^2-spaces) with L^p-spaces, for $p \in [1, \infty)$. The study of such algebras of operators is notoriously more complicated, due to the very complicated geometry of L^p-spaces (including the fact that they are not self-dual unless $p = 2$).

These notes are based on a postgraduate course given in July-August of 2019 at the *Instituto de Matemática y Estadística Rafael Laguardia* of the *Universidad de la República* in Montevideo, Uruguay. This course provided an introduction to L^p-operator algebras, with special emphasis on group algebras and Cuntz algebras.

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1. Introduction

Given $p \in [1, \infty)$, we say that a Banach algebra A is an *L^p-operator algebra* if it admits an isometric representation on an L^p-space. L^p-operator algebras have been historically studied by example, starting with Herz's influential works [21] on harmonic analysis on L^p-spaces. Given a locally compact group G, Herz studied the Banach algebra $PF_p(G) \subseteq \mathcal{B}(L^p(G))$ generated by the left regular representation, as well as its weak-$*$ closure $PM_p(G)$ and its double commutant $CV_p(G)$. The study of the structure of these algebras has attracted the attention of a number of mathematicians in the last decades (see, for example, [5], [24], [9], [8], and [10]), particularly in what refers to the so-called "convolvers and pseudomeasures" problem, which asks whether $CV_p(G) = PM_p(G)$ for all groups G and for all $p \in [1, \infty)$. We refer the interested reader to the recent paper [7] for an excellent

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survey on the problem as well as for a proof that $CV_p(G) = PM_p(G)$ when G has the approximation property.

L^p-operator algebras have recently seen renewed interest, thanks to the infusion of ideas and techniques from operator algebras, particularly in the works of Phillips [25, 26]. There, Phillips introduced and studied the L^p-analogs $\mathcal{O}_n^p$ of the Cuntz algebras $\mathcal{O}_n$ from [6] (which are the case $p = 2$), and of UHF-algebras. The work of Phillips motivated other authors to study L^p-analogs of other well-studied families of C^*-algebras. These classes include group algebras [26, 15, 19]; groupoid algebras [13]; crossed products by topological systems [26]; AF-algebras [27, 12]; and graph algebras [4]. In these works, an L^p-operator algebra is obtained from combinatorial or dynamical data, and properties of the underlying data are related to properties of the algebra. Quite surprisingly, the lack of symmetry of the unit ball of an L^p-space for $p \neq 2$ allows one to prove isomorphism results that show a stark contrast with the case $p = 2$.

More recent works have approached the study of L^p-operator algebras in a more abstract and systematic way [14, 17, 2], showing that there is an interesting theory waiting to be unveiled, of which only very little is currently known.

These notes are an introduction to L^p-operator algebras, beginning in Section 2 with what is arguably the most fundamental result in the area: Lamperti's theorem (see Theorem 2.13), which characterizes the invertible isometries of an L^p-space for $p \in [1, \infty) \setminus \{2\}$. In Section 3 we define L^p-operator algebras, prove some elementary facts about them, and give some basic examples. The next four sections are devoted to the study of three very prominent classes of examples: group algebras (Sections 4 and 5); Cuntz and graph algebras (Section 6); and crossed products (Section 7). Finally, in Section 8 we discuss a recent result obtained in [3]: $\mathcal{O}_2^p \otimes \mathcal{O}_2^p$ is not isomorphic to $\mathcal{O}_2^p$ for $p \in [1, \infty) \setminus \{2\}$ (while it is well-known that an isomorphism exists for $p = 2$; see [28]). This answers a question of Phillips.

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2. Lamperti's theorem

In [22], Lamperti gave a description of the linear isometries of the L^p-space of a σ-finite measure space, for $p \in [1, \infty]$ with $p \neq 2$. This result had been earlier announced (without proof) by Banach for the unit interval with the Lebesgue measure, and for this reason it is also sometimes referred to as the "Banach-Lamperti Theorem". In this section, which is based on Sections 2 and 3 of [18], we generalize their result by characterizing the surjective, linear isometries on the L^p-space of a localizable measure algebra; see Theorem 2.13. The generalization from σ-finite spaces to localizable ones will allow us in the next sections to deal with locally compact groups that are not σ-compact.

Definition 2.1. A *Boolean algebra* is a set $\mathcal{A}$ containing two distinguished elements $\emptyset$ and I, and with commutative, associative operations $\vee$ (disjoint union/orthogonal sum) and $\wedge$ (intersection/multiplication), and a notion of complementation $E \mapsto E^c$, satisfying the following properties:

(1) Idempotency: $E \vee E = E \wedge E = E$ for all $E \in \mathcal{A}$;
(2) Absorption: $E \vee (E \wedge F) = E \wedge (E \wedge F) = E$ for all $E, F \in \mathcal{A}$;
(3) Universality: for all $E \in \mathcal{A}$, we have

$$E \vee \emptyset = E = E \wedge I, \quad E \wedge \emptyset = \emptyset \quad \text{and} \quad E \vee I = I;$$

(4) Complementation: $E \vee E^c = I$ and $E \wedge E^c = \emptyset$ for all $E \in \mathcal{A}$.

A *homomorphism* between Boolean algebras is a function preserving all the operations and the distinguished sets $\emptyset$ and I.

Given $E, F \in \mathcal{A}$, we write $E \leq F$ if $E \wedge F = E$, and we write $E \perp F$ if $E \wedge F = \emptyset$.

We say that $\mathcal{A}$ is *(σ)-complete* if every nonempty countable subset of $\mathcal{A}$ has a supremum, and every nonempty (countable) subset of $\mathcal{A}$ has an infimum.

The reader is referred to [11] for a thorough treatment of Boolean algebras. The most important example for the purposes of these notes is the following.

Example 2.2. Let (X, Σ, μ) be a measure space. Set $\mathcal{N} = \{E \in \Sigma \colon \mu(E) = 0\}$, and let $\mathcal{A}$ denote the quotient $\Sigma/\mathcal{N}$. Then $\mathcal{A}$ is a σ-complete Boolean algebra, with countable suprema given by union, and countable infima given by intersection.

There is a natural notion of measure on a Boolean algebra.

Definition 2.3. Let $\mathcal{A}$ be a Boolean algebra. A map $\mu \colon \mathcal{A} \to [0, \infty]$ is said to be a *measure* if it satisfies $\mu(\emptyset) = 0$ and $\mu(\bigvee_{n\in\mathbb{N}} E_n) = \sum_{n\in\mathbb{N}} \mu(E_n)$ whenever the E_n are pairwise orthogonal. We call a measure μ *semi-finite* if for every $E \in \mathcal{A}$ there exists $F \leq E$ with $0 < \mu(F) < \infty$.

Finally, we say that the measured algebra $(\mathcal{A}, \mu)$ is *localizable* if $\mathcal{A}$ is σ-complete and μ is semi-finite.

Example 2.4. In Example 2.2, the map $\mathcal{A} \to [0, \infty]$ which μ naturally induces is a measure. Localizability can be easily characterized in terms of the measure space (X, Σ, μ): for every $E \in \Sigma$ with $0 < \mu(E)$, there exists $F \in \Sigma$ with $F \subseteq E$ such that $0 < \mu(F) < \infty$.

Our next goal is to define L^p-spaces associated to a measured algebra. We denote by $\mathcal{B}(\mathbb{R})$ the Boolean algebra of all Borel-measurable subsets of $\mathbb{R}$.

Definition 2.5. Let $\mathcal{A}$ be a Boolean algebra. A *measurable real valued function* is a Boolean homomorphism $f \colon \mathcal{B}(\mathbb{R}) \to \mathcal{A}$ which preserves the suprema of countable sets. For $t \in \mathbb{R}$, we write $\{f > t\}$ for the set $f((t, \infty))$.

Note that two functions $f, g \colon \mathcal{B}(\mathbb{R}) \to \mathcal{A}$ are equal if and only if $\{f > t\} = \{g > t\}$ for all $t \in \mathbb{R}$.

Example 2.6. In the context of Example 2.2, a measurable function $f \colon X \to \mathbb{R}$ is identified with the homomorphism $\widetilde{f} \colon \mathcal{B}(\mathbb{R}) \to \mathcal{A}$ given by $\widetilde{f}(E) = f^{-1}(E) + \mathcal{N} \in \mathcal{A}$.

The set of all measurable functions on $\mathcal{A}$ is denoted $L^0_{\mathbb{R}}(\mathcal{A})$. We set $L^0(\mathcal{A}) = L^0_{\mathbb{R}}(\mathcal{A}) + iL^0_{\mathbb{R}}(\mathcal{A})$. For $f \in L^0_{\mathbb{R}}(\mathcal{A})$, we define its integral by

$$\int f \, d\mu = \int_0^\infty \mu(\{f > t\}) \, dt.$$

For $f \in L^0(\mathcal{A})$ and $p \in [1, \infty)$, we set $\|f\|_p^p = \int |f|^{1/p} d\mu$.

The context of measured Boolean algebras seems to be the most appropriate one to do measure theory. The notion of σ-finiteness for measure spaces is technically very useful, but virtually every result for σ-finite measure spaces can be proved in the more general context of localizable measures. One instance of this is the Radon-Nikodym theorem; in fact, localizability is *characterized* by the validity of the Radon-Nikodym theorem.

Theorem 2.7. Let $\mathcal{A}$ be a σ-complete Boolean algebra, and let μ and ν be measures on $\mathcal{A}$ with μ localizable. Then there exists a unique function $\frac{d\nu}{d\mu} \in L^0_{\mathbb{R}}(\mathcal{A})$, called the *Radon-Nikodym derivative* of ν with respect to μ, satisfying

$$\int f \; d\nu = \int f \frac{d\nu}{d\mu} \; d\mu$$

for all $f \in L^1(\nu)$.

In the context of the theorem above, the function $\frac{d\nu}{d\mu}$ is characterized by the fact that, for $t \in (0, \infty)$, the element $\{\frac{d\nu}{d\mu} > t\} \in \mathcal{A}$ is the supremum of all the elements $E \in \mathcal{A}$ satisfying $\nu(E) > t\mu(E)$.

Exercise 2.8. Let $(\mathcal{A}, \mu)$ be a localizable measured Boolean algebra, and let $\varphi \in \mathrm{Aut}(\mathcal{A})$. Show that

$$\int f \; d\mu = \int (\varphi \circ f) \frac{d(\mu \circ \varphi^{-1})}{d\mu} \; d\mu$$

for all $f \in L^0_{\mathbb{R}}(\mathcal{A})$. This identity is known as the "change of variables formula".

The problem we will address in the rest of this section is to describe all isometric isomorphisms (surjective isometries) between L^p-spaces, for $p \in [1, \infty)$. For $\ell^p(\{0, 1\})$, this is easy to answer.

Example 2.9. Endow $\{0, 1\}$ with the counting measure. The geometric description of the unit ball of $\ell^p(\{0, 1\})$, for $p \in [1, \infty]$, reveals that the case $p = 2$ has *many* more symmetries than the other ones. In particular, for $p \neq 2$, it is clear that δ_0 must be mapped either to a complex multiple of δ_0 or to a complex multiple of δ_1, and similarly for δ_1. In other words, an invertible isometry in this case has one of the following forms:

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & \lambda_1 \\ \lambda_2 & 0 \end{bmatrix}$$

for $\lambda_1, \lambda_2 \in S^1$. For $p = 2$, rotations by angles other than multiples of $\pi/2$ also give rise to invertible isometries (also known as unitary matrices), which are not isometric when regarded as maps on $\ell^p(\{0, 1\})$ for $p \neq 2$.

Proposition 2.10. Let $(\mathcal{A}, \mu)$ be a localizable measured Boolean algebra and let $p \in [1, \infty)$.

(1) Set

$$\mathcal{U}(L^\infty(\mu)) = \{f \in L^0(\mathcal{A}) \colon \{|f| > 1\} = \{|f| < 1\} = \emptyset\},$$

which is a group under multiplication. Then there is a group homomorphism

$$m \colon \mathcal{U}(L^\infty(\mu)) \to \mathrm{Isom}(L^p(\mu))$$

given by $m_f(\xi) = f\xi$ for all $f \in \mathcal{U}(L^\infty(\mu))$ and all $\xi \in L^p(\mu)$.

(2) There is a group homomorphism

$$u\colon \mathrm{Aut}(\mathcal{A}) \to \mathrm{Isom}(L^p(\mu))$$

given by

$$u_\varphi(\xi) = \varphi \circ \xi \left(\frac{d(\mu \circ \varphi^{-1})}{d\mu} \right)^{1/p}$$

for all $\varphi \in \mathrm{Aut}(\mathcal{A})$ and all $\xi \in L^p(\mu)$.

(3) Let $\varphi \in \mathrm{Aut}(\mathcal{A})$ and let $f \in \mathcal{U}(L^\infty(\mu))$. Then

$$u_\varphi m_f u_\varphi^{-1} = m_{\varphi \circ f}.$$

In particular, there exists a group homomorphism

$$\mathcal{U}(L^\infty(\mu)) \rtimes \mathrm{Aut}(\mathcal{A}) \to \mathrm{Isom}(L^p(\mu)).$$

(4) Given $f, g \in \mathcal{U}(L^\infty(\mu))$ and $\varphi, \psi \in \mathrm{Aut}(\mathcal{A})$, we have

$$\|m_f u_\varphi - m_g u_\psi\| = \max\{\|f - g\|_\infty, 2 - 2\delta_{\varphi,\psi}\}.$$

Proof. Most of the proposition is routine. We will prove in (2) that u_φ is isometric. Given $\xi \in L^p(\mu)$, we use Exercise 2.8 at the third step to get

$$\begin{aligned}
\|u_\varphi(\xi)\|_p^p &= \int \left| (\varphi \circ \xi) \left(\frac{d(\mu \circ \varphi^{-1})}{d\mu} \right)^{1/p} \right|^p d\mu \\
&= \int |\varphi \circ \xi|^p \frac{d(\mu \circ \varphi^{-1})}{d\mu} \, d\mu \\
&= \int |\xi|^p d\mu \\
&= \|\xi\|_p^p.
\end{aligned}$$

□

Exercise 2.11. Complete the proof of Proposition 2.10.

The main result of this section, Theorem 2.13, asserts that for $p \neq 2$, the only isometries of $L^p(\mu)$ for localizable μ are the ones described in Proposition 2.10.

We need a preparatory lemma.

Lemma 2.12. Let $(\mathcal{A}, \mu)$ be a measured Boolean algebra, let $p \in [1, \infty)$, and let $\xi, \eta \in L^p(\mu)$.

(1) For $2 \leq p$, we have

$$\|\xi + \eta\|_p^p + \|\xi - \eta\|_p^p \geq 2 \left(\|\xi\|_p^p + \|\eta\|_p^p \right),$$

and equality holds for $p \neq 2$ if and only if $\xi\eta = 0$.

(2) For $p \leq 2$, we have

$$\|\xi + \eta\|_p^p + \|\xi - \eta\|_p^p \leq 2 \left(\|\xi\|_p^p + \|\eta\|_p^p \right),$$

and equality holds for $p \neq 2$ if and only if $\xi\eta = 0$.

(3) If $p \neq 2$ and $T\colon L^p(\mu) \to L^p(\mu)$ is isometric, then $T(\xi)T(\eta) = 0$ whenever $\xi\eta = 0$. (In other words, T is "disjointness preserving".)

Proof. (1). Set $\phi(t) = t^p$ for $t \in \mathbb{R}$. Then $\phi(\sqrt(t))$ is convex, and by standard results, we have

$$\phi^{-1}\left(\frac{\phi(|z+w|)+\phi(|z-w|)}{2}\right) \geq (|z|^2+|w|^2)^{1/2} \geq \phi^{-1}(\phi(|z|)+\phi(|w|)),$$

for all $z, w \in \mathbb{C}$. Moreover, when ϕ is strictly convex, equality holds if and only if $zw = 0$. Since ϕ^{-1} is increasing, the result follows by integration.

(2). This is entirely analogous to (1).

(3). Suppose $p \neq 2$. If $\xi\eta = 0$, then

$$\begin{aligned} \|T(\xi)+T(\eta)\|_p^p + \|T(\xi)-T(\eta)\|_p^p &= \|\xi+\eta\|_p^p + \|\xi-\eta\|_p^p \\ &= 2\left(\|\xi\|_p^p + \|\eta\|_p^p\right) \\ &= 2\left(\|T(\xi)\|_p^p + \|T(\eta)\|_p^p\right). \end{aligned}$$

It follows that $T(\xi)T(\eta) = 0$, as desired. □

We have now arrived at Lamperti's theorem.

Theorem 2.13. Let $(\mathcal{A}, \mu)$ be a localizable measured Boolean algebra, let $p \in [1, \infty) \setminus \{2\}$ and let $T \colon L^p(\mu) \to L^p(\mu)$ be an invertible isometry. Then there exist unique $\varphi \in \mathrm{Aut}(\mathcal{A})$ and $f \in \mathcal{U}(L^\infty(\mu))$ such that $T = m_f u_\varphi$.

Proof. To make the proof more transparent, we will assume that $(\mathcal{A}, \mu)$ arises as in Example 2.2 from a *finite* measure space (X, Σ, μ). Given $E \in \mathcal{A} = \Sigma/\mathcal{N}$, set

$$\varphi(E) = \{|T(\chi_E)| > 0\}.$$

If $E, F \in \mathcal{A}$ are disjoint, then so are $\varphi(E)$ and $\varphi(F)$ by part (3) of Lemma 2.12. In particular, $\varphi(X - E) = \varphi(X) - \varphi(E)$, so φ preserves complements and is thus a homomorphism from $\mathcal{A}$ to itself. Since T is invertible, so is φ.

Using that $\mu(X) < \infty$, set $h = T(\chi_X)$.

We claim that $T(\xi) = (\xi \circ \varphi)h$ for all $\xi \in L^p(\mu)$. To check this, suppose first that $\xi = \chi_E$ for $E \in \mathcal{A}$. Note that

$$h = T(\chi_E) + T(\chi_{E^c}),$$

and that the supports of the functions on the right-hand side are disjoint. In particular, h agrees with $T(\chi_E)$ on the support of $T(\chi_E)$, which is $\varphi(E)$. Thus,

$$T(\chi_E) = h\chi_{\varphi(E)} = h(\chi_E \circ \varphi).$$

This verifies the claim for indicator functions, and thus for step functions. Since these are dense in $L^p(\mu)$, the claim follows.

It remains to identify h. For $E \in \mathcal{A}$, we have

$$\mu(E) = \|\chi_E\|_p^p = \|T(\chi_E)\|_p^p = \int |h|^p \chi_{\varphi(E)}^p \, d\mu = \int_{\varphi(E)} |h|^p d\mu.$$

On the other hand,

$$\mu(E) = (\mu \circ \varphi^{-1})(\varphi(E)) = \int_{\varphi(E)} \frac{d(\mu \circ \varphi^{-1})}{d\mu} \, d\mu.$$

It follows that $|h|^p = \frac{d(\mu\circ\varphi^{-1}}{d\mu}$, so there exists $f \in \mathcal{U}(L^\infty(\mu))$ such that $h = f\left(\frac{d(\mu\circ\varphi^{-1}}{d\mu}\right)^{1/p}$. This finishes the proof. □

Remark 2.14. Suppose that $(\mathcal{A}, \mu)$ arises from a measure space (X, Σ, μ) as in Example 2.2. In some cases, it is possible to "lift" the automorphism φ to a bi-measurable bijective transformation $T\colon X \to X$ satisfying

$$\mu(E) = 0 \Leftrightarrow \mu(T(E)) = 0 \Leftrightarrow \mu(T^{-1}(E)) = 0.$$

This is always the case, for example, when μ is an atomic measure, in which case one even has $\mu(T(E)) = \mu(E)$ for all $E \in \Sigma$.

Exercise 2.15. Let $(\mathcal{A}, \mu)$ be a localizable measure algebra, let $p, q \in [1, \infty)$ with $p \neq q$, and let $T\colon L^p(\mu) \cap L^q(\mu) \to L^p(\mu) \cap L^q(\mu)$ be a linear map that extends to isometric surjections $L^p(\mu) \to L^p(\mu)$ and $L^q(\mu) \to L^q(\mu)$. Prove that there exist $f \in \mathcal{U}(L^\infty(\mu))$ and $\varphi \in \mathrm{Aut}(\mathcal{A})$ with $\mu \circ \varphi = \mu$ and

$$T(\xi) = f(\varphi \circ \xi).$$

3. L^p-operator algebras: basic examples

Recall that a *Banach algebra* is a complex algebra A with a Banach space structure, satisfying $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in A$. When A has a multiplicative unit, we moreover demand that $\|1\| = 1$.

Definition 3.1. Let A be a Banach algebra. We say that A is an *L^p-operator algebra* if there exist an L^p-space E and an isometric homomorphism $A \to \mathcal{B}(E)$.

A *representation* of A (on an L^p-space E) is a contractive homomorphism $\varphi\colon A \to \mathcal{B}(E)$. We say that φ is *non-degenerate* if $\mathrm{span}\{\varphi(a)\xi\colon a \in A, \xi \in E\}$ is dense in E.

We make some comments on why we restrict to $p \in [1, \infty)$. First, for $p < 1$, the vector space $L^p(\mu)$ is not normed (and its dual space is in fact trivial). On the other hand, for $p = \infty$ we do not have a Lamperti-type theorem that allows us to represent the invertible isometries spatially (neither do we for $p = 2$, but in this case there are adjoints). For $p \in (1, \infty)$, the fact that an L^p-space is reflexive is sometimes quite useful and produces legitimate differences with the case $p = 1$; see, for example, Theorem 4.10.

Remark 3.2. For $p = 2$, an L^2-operator algebra is a not necessarily self-adjoint operator algebra.

Example 3.3. If E is an L^p-space, then $\mathcal{B}(E)$ is trivially an L^p-operator algebra. When $E = \ell^p(\{1, \ldots, n\})$, then $\mathcal{B}(E)$ is algebraically isomorphic to M_n, and we denote the resulting Banach algebra by M_n^p.

The norm described above is not the only L^p-operator norm on M_n (even for $p = 2$):

Example 3.4. If $s \in M_n^p$ is an invertible operator, one can define a new L^p-operator norm $\|\cdot\|_s$ on M_n by setting $\|x\|_s = \|sxs^{-1}\|$. This norm is in general different from the one on M_n^p.

Example 3.5. Let X be a locally compact topological space. Then $C_0(X)$ is an L^p-operator algebra. In the case that there exists a regular Borel measure μ on X, one can represent $C_0(X)$ isometrically on $L^p(\mu)$ via multiplication operators. (Such a measure does not always exist, but one can always find a "separating" family of such measures and take the direct sum of the resulting representations by multiplication.)

It is very convenient to work with non-degenerate representations. However, such representations don't always exist.

Example 3.6. Let $\mathbb{C}_0$ be the Banach algebra whose underlying Banach space is $\mathbb{C}$, endowed with the trivial product: $ab = 0$ for all $a, b \in \mathbb{C}$. Then $\mathbb{C}_0$ is an L^p-operator algebra for all $p \in [1, \infty)$, since it is isometrically isomorphic to the upper-triangular matrices in M_2^p. However, $\mathbb{C}_0$ does not admit non-degenerate representations.

For L^p-operator algebras with contractive approximate identities, one can always "cut-down" a given representation to obtain a non-degenerate one. This is much more subtle than in the Hilbert space case (where one just co-restricts to the essential range, which is automatically a Hilbert space), since there are subspaces of an L^p-space which are not themselves L^p-spaces. The case of unital algebras is much easier to prove.

Proposition 3.7. Let A be a unital L^p-operator algebra. Then A admits an isometric, unital representation $\varphi\colon A \to \mathcal{B}(E)$ on an L^p-space E.

Proof. Let F be an L^p-space and let $\psi\colon A \to \mathcal{B}(F)$ be an isometric representation. Set $e = \varphi(1)$, which is a contractive idempotent. By the main result of [29], the image $E = e(F)$ of e is an L^p-space. Define $\varphi\colon A \to \mathcal{B}(E)$ by $\varphi(a)(\xi) = \psi(a)(\xi)$ for all $a \in A$ and all $\xi \in E \subseteq F$. Then φ is an isometric representation. □

For general algebras with a contractive approximate identity, one shows that there is a contractive idempotent from the ambient L^p-space to the essential range, which implies that the essential range is an L^p-space; see [17].

The study of L^p-operator algebras is generally much more complicated than that of (L^2-)operator algebras, largely due to the complicated geometry of L^p-spaces. Even for algebras that "look like" C^*-algebras, many of the most fundamental facts about C^*-algebras fail. To mention a few:

- There is no abstract characterization of L^p-operator algebras among all Banach algebras, or canonical way of obtaining a representation on an L^p-space for a given L^p-operator algebra;
- L^p-operator norms are not unique; in particular, a homomorphism between L^p-operator algebras does not necessarily have closed range, and an injective homomorphism is not necessarily isometric.
- For $p \neq 2$, not every quotient of an L^p-operator algebra can be represented on an L^p-space; see [16]. There is also no known characterization of which ideals give L^p-operator quotients.

There is so far no general theory, and it has been very productive to study concrete families of L^p-operator algebras, typically (but not always) constructed from some topological/algebraic data. In the following sections, we will introduce some of the most studied classes of examples.

4. Group algebras acting on L^p-spaces

Let G be a locally compact group, and let μ denote its Haar measure. For functions f and g defined on G, their *convolution* is defined as

$$(f * g)(s) = \int_G f(t)g(t^{-1}s)\; d\mu(t)$$

for all $s \in G$. When $f \in L^1(G)$ and $g \in L^p(G)$, the convolution $f * g$ belongs to $L^p(G)$, and $\|f * g\|_p \leq \|f\|_1 \|g\|_p$. In particular, $L^1(G)$ is a Banach algebra under convolution, and we denote by $\lambda_p \colon L^1(G) \to \mathcal{B}(L^p(G))$ the action by left convolution (also called the *left regular representation*).

Exercise 4.1. Let G be a locally compact group. Show that $L^1(G)$ is unital if and only if G is discrete.

The following definitions are due to Herz [21].

Definition 4.2. Let G be a locally compact group.

(1) We define the *algebra of p-pseudofunctions* $PF_p(G)$ of G, usually also denoted $F^p_\lambda(G)$ for consistency with other notations, to be
$$PF_p(G) = \overline{\lambda_p(L^1(G))}^{\|\cdot\|} \subseteq \mathcal{B}(L^p(G)).$$
(This algebra is sometimes also denoted $F^p_\lambda(G)$ and called the *reduced group L^p-operator algebra* of G.)

(2) We define the *algebra of p-pseudomeasure* $PM_p(G)$ of G to be
$$PM_p(G) = \overline{\lambda_p(L^1(G))}^{\text{weak}} \subseteq \mathcal{B}(L^p(G)).$$

(3) We define the *algebra of p-convolvers* $CV_p(G)$ of G to be
$$CV_p(G) = \lambda_p(L^1(G))'' \subseteq \mathcal{B}(L^p(G)).$$

In general, we have $PF_p(G) \subseteq PM_p(G) \subseteq CV_p(G)$.

Notation 4.3. In these notes, we will usually denote the algebra of p-pseudofunctions on G by $F^p_\lambda(G)$. This algebra is also sometimes called the "reduced group L^p-operator algebra".

There is another very important L^p-operator algebra associated to a locally compact group, due to Phillips [26].

Definition 4.4. Let G be a locally compact group and let $p \in [1, \infty)$. We define its *full group L^p-operator algebra* $F^p(G)$ to be the completion of $L^1(G)$ in the norm
$$\|f\|_{F^p(G)} = \sup\{\|\varphi(f)\| \colon \varphi \colon L^1(G) \to \mathcal{B}(E) \text{ contractive representation}\},$$
for $f \in L^1(G)$. (Where E ranges over all possible L^p-spaces.)

Representations of $L^1(G)$ as above are in one-to-one correspondence with isometric representations of G, via the integrated form. The case of discrete groups is particularly easy to prove:

Exercise 4.5. Let G be a discrete group, and let E be any Banach space. Given a unital, contractive homomorphism $\varphi \colon \ell^1(G) \to \mathcal{B}(E)$, let $u_\varphi \colon G \to \mathrm{Isom}(E)$ be given by $u_\varphi(g) = \varphi(\delta_g)$ for $g \in G$. Conversely, given $u \colon G \to \mathrm{Isom}(E)$, let $\varphi_u \colon \ell^1(G) \to \mathcal{B}(E)$ be the bounded linear map determined by $\varphi_u(\delta_g) = u_g$ for all $g \in G$.

(1) Prove that u_φ is an isometric representation of G on E (that is, show that $u_\varphi(g)$ is an invertible isometry of E for all $g \in G$, and that $u_\varphi(gh) = u_\varphi(g)u_\varphi(h)$ for all $g, h \in G$.)

(2) Prove that φ_u extends to a well-defined algebra homomorphism, and that it is unital and contractive.

(3) Prove that $\varphi_{u_\varphi} = \varphi$ and $u_{\varphi_u} = u$.

(4) Suppose that $E = \ell^p(G)$, and let $\mathtt{Lt}\colon G \to \ell^p(G)$ be given by $\mathtt{Lt}_g(\xi)(h) = \xi(g^{-1}h)$ for all $g, h \in G$ and all $\xi \in E$. Prove that $\varphi_{\mathtt{Lt}} = \lambda_p$.

It is not entirely obvious from the definition that $F^p(G)$ is indeed an L^p-operator algebra (unlike for the algebras defined in Definition 4.2, which is explicitly constructed as a Banach subalgebras of $\mathcal{B}(L^p(G))$). For this, one needs to produce an isometric representation on some L^p-space.

Proposition 4.6. Let G be a locally compact group and let $p \in [1, \infty)$. Then $F^p(G)$ is an L^p-operator algebra.

Proof. For $f \in L^1(G)$ and $n \in \mathbb{N}$, let $\varphi_{f,n}\colon L^1(G) \to \mathcal{B}(E_{f,n})$ be a contractive representation satisfying

$$\|\varphi_{f,n}\| \geq \|f\|_{F^p(G)} - \frac{1}{n}.$$

Set $E = \bigoplus\limits_{f \in L^1(G)} \bigoplus\limits_{n \in \mathbb{N}} E_{f,n}$ and let $\varphi\colon L^1(G) \to \mathcal{B}(E)$ denote the "diagonal" representation. Then E is an L^p-space. Moreover, for $f \in L^1(G)$ and $n \in \mathbb{N}$, one has

$$\|\varphi(f)\| = \sup_{m \in \mathbb{N}} \sup_{g \in L^1(G)} \|\varphi_{g,m}(f)\| \geq \|\varphi_{f,n}(f)\| \geq \|f\|_{F^p(G)} - \frac{1}{n}.$$

In particular, $\|\varphi(f)\| \geq \|f\|_{F^p(G)}$. Since φ is a contractive representation of $L^1(G)$ on some L^p-space, we also have $\|\varphi(f)\| \leq \|f\|_{F^p(G)}$, and hence $\|\varphi(f)\| = \|f\|_{F^p(G)}$. It follows that the norm-closure of $\varphi(L^1(G))$ in $\mathcal{B}(E)$ is isometrically isomorphic to $F^p(G)$, and thus $F^p(G)$ is an L^p-operator algebra. $\square$

Remark 4.7. When G is discrete, $F^p(G)$ is the universal L^p-operator algebra generated by invertible isometries u_s, for $s \in G$, satisfying $u_s u_t = u_{st}$.

Since $\lambda_p\colon L^1(G) \to \mathcal{B}(L^p(G))$ is a contractive representation, it follows that $\|\cdot\|_{F^p_\lambda(G)} \leq \|\cdot\|_{F^p(G)}$. In other words, the identity map

$$\mathrm{id}\colon (L^1(G), \|\cdot\|_{F^p_\lambda(G)}) \to (L^1(G), \|\cdot\|_{F^p(G)})$$

is contractive, and it extends to a contractive map $\kappa_p\colon F^p(G) \to F^p_\lambda(G)$ between their completions, which has dense range since it contains $L^1(G)$.

The cases $p = 1$ and $p = 2$ of the algebras in Definition 4.2 and Definition 4.4 are easy to describe. For the identification of $F^2(G)$, we will need the following exercise:

Exercise 4.8. Let $\mathcal{H}$ be a Hilbert space and let $u \in \mathcal{B}(\mathcal{H})$. Show that u is a unitary if and only if u is invertible and $\|u\| = \|u^{-1}\| = 1$.

Proposition 4.9. Let G be a locally compact group.

- When $p = 1$, we get $F^1_\lambda(G) = F^1(G) = L^1(G)$ and $PM_1(G) = CV_1(G) = M(G)$.
- When $p = 2$, we get $F^2_\lambda(G) = C^*_\lambda(G)$, $F^2(G) = C^*(G)$ and $PM_2(G) = CV_2(G) = W^*(G)$.

Proof. (1). Recall that $L^1(G)$ has a contractive approximate identity $(f_n)_{n \in \mathbb{N}}$. Given $f \in L^1$ and $\varepsilon > 0$, find $n \in \mathbb{N}$ such that $\|f * f_n\|_1 \geq \|f\|_1 - \varepsilon$. Then

$$\|f\|_1 \geq \|\lambda_1(f)\|_{\mathcal{B}(L^1(G))} \geq \frac{\|f * f_n\|_1}{\|f_n\|_1} \geq \|f\|_1 - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that $\|\lambda_1(f)\|_{\mathcal{B}(L^1(G))} = \|f\|_1$ and thus $F^1_\lambda(G) = L^1(G)$.

Since $\|\cdot\|_{F^1_\lambda(G)} \leq \|\cdot\|_{F^1(G)} \leq \|\cdot\|_1$, it follows that $\|\cdot\|_{F^1(G)} = \|\cdot\|_1$ and hence $F^1(G) = L^1(G)$ as well. We omit the proofs of the identities $PM_1(G) = CV_1(G) = M(G)$, which are analogous.

(2). The identities $F^2_\lambda(G) = C^*_\lambda(G)$ and $PM_p(G) = W^*(G)$ are true by definition, and $CV_2(G) = W^*(G)$ by the double-commutant theorem. The identity $F^2(G) = C^*(G)$ follows from Exercise 4.5 and Exercise 4.8. □

In view of the previous proposition, we often regard the group algebras from Definition 4.2, for different values of p, as a continuously varying family of Banach algebras that deform $L^1(G)$ or $M(G)$ into $C^*_\lambda(G)$, $C^*(G)$, or $W^*(G)$.

The fact that $F^1_\lambda(G)$ and $F^1(G)$ agree is misleading, since, for other values of p, this happens if and only if G is amenable. (Recall that a group G is *amenable* if for every $\varepsilon > 0$ and for every compact subset $K \subseteq G$, there exists a compact subset $F \subseteq G$ such that $\mu(FK \triangle F) < \varepsilon\mu(F)$.) The following is Theorem 3.20 in [15], and it was also independently proved by Phillips.

Theorem 4.10. Let G be a locally compact group and let $p > 1$. Then the canonical map $\kappa_p \colon F^p(G) \to F^p_\lambda(G)$ is an isometric isomorphism if and only if G is amenable.

This implies, among others, that for G amenable the reduced group algebra $F^p_\lambda(G)$ admits a characterization in terms of generators and relations. For $G = \mathbb{Z}$, this description is particularly nice:

Corollary 4.11. $F^p_\lambda(\mathbb{Z})$ is the Banach subalgebra of $\mathcal{B}(\ell^p(\mathbb{Z}))$ generated by the forward and backward shifts. For $p = 2$, this algebra is isometrically isomorphic to $C(S^1)$, but in general the norm is larger.

A natural question that arises from looking at the cases $p = 1, 2$ is whether the equality $PM_p(G) = CV_p(G)$ always holds. This is arguably the most important open problem in the area, dating back to Herz's work in the 70's, and is known as the "convolvers and pseudomeasures" problem.

Question 4.12. Let G be a locally compact group and let $p \in [1, \infty)$. Is it true that $PM_p(G) = CV_p(G)$?

The question above asks whether a specific case of the double-commutant theorem holds for operators on L^p-spaces. It is known that Question 4.12 has a positive answer, for all $p \in [1, \infty)$, whenever G has the so-called approximation property. This is in particular the case when G is amenable.

We finish this subsection by discussing the smallest non-trivial group algebra.

Example 4.13. $F^p_\lambda(\mathbb{Z}_2)$ is the Banach subalgebra of $\mathcal{B}(\ell^p(\{0,1\}))$ generated by the rotation matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (and its inverse, which is itself). This algebra can be identified with $\mathbb{C}^2$, but its norm is not the maximum norm. The norm of $(a, b) \in F^p_\lambda(\mathbb{Z}_2)$ is the L^p-operator norm of the matrix $\frac{1}{2}\begin{bmatrix} a+b & a-b \\ a-b & a+b \end{bmatrix}$.

One can verify with elementary methods that the algebras $F^p_\lambda(\mathbb{Z}_2)$, for different values of p, are pairwise not isometrically isomorphic (unless $\frac{1}{p} + \frac{1}{q} = 1$ or $p = q$).

Exercise 4.14. Let $p \in [1,\infty)$. Prove that

$$\|(1,i)\|_{F^p_\lambda(\mathbb{Z}_2)} = 2^{\left|\frac{1}{p}-\frac{1}{2}\right|}.$$

Deduce that $F^p_\lambda(\mathbb{Z}_2)$ is not isometrically isomorphic to $F^q_\lambda(\mathbb{Z}_2)$ unless $\frac{1}{p}+\frac{1}{q}=1$. *(Hint: for the upper bound, use the Riesz-Thorin interpolation theorem.)*

The previous exercise can be used to deduce a similar result for $F^p_\lambda(\mathbb{Z})$; see Theorem 3.5 in [19].

4.1. **Subgroups and quotients.** In this short subsection, we study some elementary functoriality properties of the L^p-operator group algebras.

Remark 4.15. Let G be a locally compact group, let $H \subseteq G$ be an open subgroup, and let $p \in [1,\infty)$. Regard $L^1(H)$ as a subalgebra of $L^1(G)$ canonically (by extending a function on H as zero on its complement). Then $\lambda_p^G|_{L^1(H)}$ is isometrically conjugate to the representation

$$\lambda_p^H \otimes \mathrm{id}_{\ell^p(G/H)} \colon L^1(H) \to \mathcal{B}\left(L^p(H)\otimes \ell^p(G/H)\right).$$

Exercise 4.16. Prove the claim in Remark 4.15 in the case that G is countable and discrete.

Proposition 4.17. Let G be a locally compact group, let $H \subseteq G$ be an open subgroup, and let $p \in [1,\infty)$. Denote by $\iota\colon L^1(H) \to L^1(G)$ the canonical isometric inclusion described in Remark 4.15. Then there are canonical injective contractive homomorphisms

$$\iota^p_\lambda \colon F^p_\lambda(H) \to F^p_\lambda(G) \quad \text{and} \quad \iota^p \colon F^p(H) \to F^p(G).$$

Moreover, ι^p_λ is isometric.

Proof. For the map ι^p, one needs to show that for all $f \in L^1(H)$ one has

$$\|\iota(f)\|_{F^p(G)} \leq \|f\|_{F^p(H)}.$$

Given a contractive representation $\varphi\colon L^1(G) \to \mathcal{B}(E)$ on an L^p-space E, the composition $\psi = \varphi \circ \iota\colon L^1(H) \to \mathcal{B}(E)$ is also contractive, and one clearly has

$$\|\varphi(\iota(f))\| = \|\psi(f)\|.$$

One readily checks, using the definition of $\|\cdot\|_{F^p(G)}$ as a supremum, that the above implies the desired inequality.

The result for ι^p_λ follows immediately from Remark 4.15, together with the fact that $\|\lambda_p^H(f)\otimes \mathrm{id}_E\| = \|\lambda_p^H(f)\|$ for all $f \in L^1(H)$ and all L^p-spaces E. □

For the following result, we will use, without proof, that if N is a closed normal subgroup in a locally compact group G, then there exists a quotient map $\pi\colon L^1(G) \to L^1(G/N)$.

Proposition 4.18. Let G be a locally compact group, let $N \subseteq G$ be a closed normal subgroup, and let $p \in [1,\infty)$.

(1) There is a canonical contractive homomorphism with dense range

$$\pi^p \colon F^p(G) \to F^p(G/N).$$

(2) When $p > 1$, there is a canonical contractive homomorphism with dense range
$$\pi_\lambda^p\colon F_\lambda^p(G) \to F_\lambda^p(G/N)$$
if and only if N is amenable.

Exercise 4.19. Prove part (1) of the previous proposition.

The proof of part (2) (for $p > 1$) uses the theory of weak containment of representations, and we omit it. The map in (2) exists for $p = 1$ regardless of whether N is amenable or not: indeed, this is just the map $\pi\colon L^1(G) \to L^1(G/N)$ described before the proposition.

4.2. **The effect of changing the exponent p.** In this subsection, we look at the question of whether the algebras $F_\lambda^p(G)$ (or $F^p(G)$), for different values of p, are isometrically isomorphic or anti-isomorphic.

Definition 4.20. Let A be a Banach algebra. We define its *opposite algebra* as the Banach algebra A^{opp} whose underlying Banach space structure agrees with that of A, and where $a \cdot_{\mathrm{opp}} b = ba$ for all $a, b \in A$. A representation of A^{opp} is naturally identified with an *anti-representation* of A (namely one which is multiplicative with respect to the opposite multiplication).

The following exercise follows by using the adjoint of an operator (which is an operator on the dual Banach space).

Exercise 4.21. Given $p \in (1, \infty)$, we denote by $p' \in (1, \infty)$ its conjugate exponent. Given a Banach algebra A, show that A is an L^p-operator algebra if and only if A^{opp} is an $L^{p'}$-operator algebra.

Let G be a locally compact group, and denote by $\Delta\colon G \to \mathbb{R}$ its modular function. For $f \in L^1(G)$, let $f^\sharp\colon G \to \mathbb{C}$ be given by $f^\sharp(s) = \Delta(s^{-1})f(s^{-1})$ for all $s \in G$.

Exercise 4.22. Let G be a locally compact group.

(1) Given $f \in L^1(G)$, show that $f^\sharp$ also belongs to $L^1(G)$. We denote by $\sharp\colon L^1(G) \to L^1(G)$ the induced map.
(2) Prove that $\sharp\colon L^1(G) \to L^1(G)$ is an anti-multiplicative isometric linear map of order two.
(3) Let $p \in (1, \infty)$. Prove that $\lambda_p(f)' = \lambda_{p'}(f^\sharp)$ for all $f \in L^1(G)$.

Proposition 4.23. Let G be a locally compact group, and let $p \in (1, \infty)$. Then $\sharp\colon L^1(G) \to L^1(G)$ extends to isometric anti-isomorphisms
$$F^p(G) \cong F^{p'}(G) \;\text{ and }\; F_\lambda^p(G) \cong F_\lambda^{p'}(G).$$

Proof. We prove it for $F^p(G)$. Let $\pi\colon L^1(G) \to \mathcal{B}(E)$ be a contractive representation on an L^p-space E. Denote by $\pi'\colon L^1(G) \to \mathcal{B}(E')$ the linear map given by $\pi'(f) = \pi(f)'$ for all $f \in L^1(G)$. Then E' is an $L^{p'}$-space, and the map π' is contractive (since an operator and its adjoint have the same norm) and anti-multiplicative. Hence $\widetilde{\pi} = \pi' \circ \sharp\colon L^1(G) \to \mathcal{B}(E')$ is a contractive representation satisfying $\|\widetilde{\pi}(f^\sharp)\| = \|\pi(f)\|$ for all $f \in L^1(G)$. Since the norm on $F^p(G)$ is universal with respect to contractive representations of $L^1(G)$ on L^p-spaces, it follows that $\sharp$ extends to an isometric anti-isomorphism $F^p(G) \cong F^{p'}(G)$.

The claim for $F_\lambda^p(G)$ follows immediately from part (2) of Exercise 4.22. $\square$

Since $L^1(G)$ is self-opposite (via the map $\sharp$), it is tempting to guess that the universal completion $F^p(G)$ is self-opposite. This is however not the case in general, as we explain next. Denote by $\mathcal{L}_p$ the class of all L^p-spaces, and for a Banach algebra A, denote by $\overline{A}^{\mathcal{L}^p}$ the universal completion of A with respect to all contractive representations of A on L^p-spaces. (For example, $F^p(G) = \overline{L^1(G)}^{\mathcal{L}^p}$.) It is tempting to claim that $\overline{A^{\mathrm{opp}}}^{\mathcal{L}^p}$ is canonically the opposite algebra of $\overline{A}^{\mathcal{L}^p}$. (If this were true, what we said before about $F^p(G)$ being self-opposite would follow.) Without further assumptions, this does not seem to be true: the norm on the algebra $\overline{A^{\mathrm{opp}}}^{\mathcal{L}^p}$ is constructed using all *anti*-representations of A on L^p-spaces, while the norm on $\overline{A}^{\mathcal{L}^p}$ is defined using genuine representations. Since there is in general no way to relate these two families of maps (given a representation of A on an L^p-space, it is not clear how to get an anti-representation of A on some potentially different L^p-space), we do not see any relationship between $\overline{A^{\mathrm{opp}}}^{\mathcal{L}^p}$ and $(\overline{A}^{\mathcal{L}^p})^{\mathrm{opp}}$.

Remark 4.24. When G is abelian, the anti-isomorphisms in Proposition 4.23 are trivially isomorphisms. Since abelian groups are unimodular, the map $\sharp$ is just inversion on G, which in the abelian case is an isomorphism. By composing again with the inversion, it follows that the identity on $L^1(G)$ extends to isometric isomorphisms between all the relevant completions for p and p'. Except for $p = 2$, it is unclear whether there are any nonabelian groups for which the identity on $L^1(G)$ extends to an isometric isomorphism $F^p_\lambda(G) \to F^{p'}_\lambda(G)$. In fact, in his PhD thesis, Herz conjectured that this is never the case. While the conjecture remains open in general, it has been confirmed for several classes of groups.

In view of Proposition 4.23, one can restrict the attention to group algebras $F^p(G)$ and $F^p_\lambda(G)$ for Hölder exponents p in $[1, 2]$. The remaining question is whether the algebras one gets for different values in $[1, 2]$ are really different. This is indeed the case:

Theorem 4.25. Let G be a nontrivial locally compact group, and let $p, q \in [1.2]$. Then the following are equivalent:

(1) There is an isometric isomorphism $F^p(G) \cong F^q(G)$;
(2) There is an isometric isomorphism $F^p_\lambda(G) \cong F^q_\lambda(G)$;
(3) $p = q$.

The theorem above is not just saying that the norms $\|\cdot\|_{F^p_\lambda(G)}$ and $\|\cdot\|_{F^q_\lambda(G)}$ (or $\|\cdot\|_{F^p(G)}$ and $\|\cdot\|_{F^q(G)}$) on $L^1(G)$ are different: it states that there are no abstract isometric isomorphisms between their completions.

5. Homomorphisms between convolution algebras

In this last subsection, based on Chapter XVI of [20] and on [18], we aim at describing all contractive, unital homomorphisms between two L^p-operator group algebras. In particular, we want to describe all isomorphisms between them. For $p = 2$, this is very complicated, and we illustrate this through some examples.

Example 5.1. The groups $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ have the same group C^*-algebra, namely $\mathbb{C}^4$.

Example 5.2. The group von Neumann algebra of $\mathbb{Z}^n$, for $n = 1, \ldots, \infty$, is $L^\infty([0, 1])$, independently of n.

The following is one of the most important open problems in operator algebras, and is known as the "free factor problem":

Problem 5.3. Is there an isomorphism $W^*(\mathbb{F}_2) \cong W^*(\mathbb{F}_3)$?

A positive answer to the above problem implies that $W^*(\mathbb{F}_2) \cong W^*(\mathbb{F}_n)$ for all $n \in \mathbb{N}$ with $n \geq 2$.

We will see that, for $p \neq 2$, we can obtain a very satisfactory description of the homomorphisms between group algebras, which in particular implies that groups with isomorphic L^p-group algebras must themselves be isomorphic; see Theorem 5.8. In this section, we will work exclusively with F^p_λ, although the results also hold for $PM_p(G)$ and $CV_p(G)$. The situation for $F^p(G)$ is unknown.

We begin with some preparatory results.

Exercise 5.4. Let G be a locally compact group and let $p \in [1, \infty)$. For $s \in G$, let $\mathtt{Rt}_s \in \mathrm{Isom}(L^p(G))$ be the invertible isometry given by

$$\mathtt{Rt}_s(\xi)(t) = \xi(ts)$$

for all $\xi \in L^p(G)$ and all $t \in G$. Show that $\lambda_p(f) \circ \mathtt{Rt}_s = \mathtt{Rt}_s \circ \lambda_p(f)$ for all $f \in L^1(G)$ and all $s \in G$. Deduce that every element of $F^p_\lambda(G)$, $PM_p(G)$ or $CV_p(G)$ commutes with $\mathtt{Rt}_s$.

We will work with discrete groups in the sequel; the results are also valid for general locally compact groups, but the arguments become more complicated and the discrete case is interesting enough.

Lemma 5.5. Let G be a discrete group and let $p \in [1, \infty)$. Define $\mathtt{Lt}\colon G \to \mathrm{Isom}(\ell^p(G))$ by $\mathtt{Lt}_s(\xi)(t) = \xi(s^{-1}t)$ for all $s, t \in G$ and all $\xi \in \ell^p(G)$. Then $F^p_\lambda(G)$ is the subalgebra of $\mathcal{B}(\ell^p(G))$ generated by $\{\mathtt{Lt}_s \colon s \in G\}$.

Proof. We have seen in Exercise 4.5 that the integrated form of $\mathtt{Lt}$ is λ_p. By the first parts of said exercise, the image of λ_p is generated, as a Banach algebra, by the image of $\mathtt{Lt}$, which is what we wanted to show. □

For a unital Banach algebra A, we write

$$\mathrm{Isom}(A) = \{v \in A \colon v \text{ invertible and } \|v\| = \|v^{-1}\| = 1\}.$$

Note that if A is a unital subalgebra of another Banach algebra B, then $\mathrm{Isom}(A)$ is a subgroup of $\mathrm{Isom}(B)$.

Theorem 5.6. Let G be a discrete group and let $p \in [1, \infty) \setminus \{2\}$. Then there is a natural identification of topological groups

$$\mathrm{Isom}(F^p_\lambda(G)) \cong G \times \mathbb{T},$$

where $\mathrm{Isom}(F^p_\lambda(G))$ is endowed with the norm topology, and $G \times \mathbb{T}$ is endowed with the product topology.

Proof. Let $v \in \mathrm{Isom}(F^p_\lambda(G))$. Since $F^p_\lambda(G)$ is a unital subalgebra of $\mathcal{B}(\ell^p(G))$, and since $p \neq 2$, by Lamperti's theorem Theorem 2.13 (and Remark 2.14) there exist a bijection $\varphi\colon G \to G$ and a measurable function $h\colon G \to S^1$ such that

$$v(\xi)(s) = h(s)\xi(\varphi(s))$$

for all $\xi \in \ell^p(G)$ and all $s \in G$. By Exercise 5.4, we have $v \circ \rho_t = \rho_t \circ v$ for all $t \in G$. We evaluate on both sides of this identity:

$$v(\rho_t(\xi))(s) = h(s)(\rho_t(\xi)(\varphi(s))) = h(s)\xi(\varphi(s)t)$$

and

$$(\rho_t \circ v)(\xi)(s) = v(\xi)(st) = h(st)\xi(\varphi(st)).$$

It follows that $h(s)\xi(\varphi(s)t) = h(st)\xi(\varphi(st))$ for all $s, t \in G$. When $s = 1$, we get $h(1)\xi(\varphi(1)t) = h(t)\xi(\varphi(t))$ for all $t \in G$ and all $\xi \in \ell^p(G)$. Setting $\xi = \delta_r$ for some $r \in G$, we deduce that $\varphi(t) = \varphi(1)t$ and $h(1) = h(t)$ for all $t \in G$. With $g_v = \varphi(t)$ and $\alpha_v = h(1)$, this shows that φ is left multiplication by g_v, and h is the constant function α_v. In other words,

$$v(\xi)(s) = \alpha_v \xi(g_v s)$$

for all $\xi \in \ell^p(G)$ and all $s \in G$.

Define $\theta\colon \mathrm{Isom}(F^p_\lambda(G)) \to G \times \mathbb{T}$ by $\theta(v) = (g_v, \alpha_v)$ for all $v \in \mathrm{Isom}(F^p_\lambda(G))$. It is easy to check that θ is a group homomorphism, and it is clearly injective. Moreover, it is clearly surjective, by Lemma 5.5.

The claim about the norm follows from the norm computation in part (4) of Proposition 2.10. □

Corollary 5.7. Let G be a discrete group and let $p \in [1, \infty) \setminus \{2\}$. Then there exists a natural identification $G \cong \mathrm{Isom}(F^p_\lambda(G))/\sim_h$.

The following is the structure theorem for maps between group algebras that we were aiming at:

Theorem 5.8. Let G and H be discrete groups, let $p \in [1, \infty) \setminus \{2\}$, and let $\varphi\colon F^p_\lambda(G) \to F^p_\lambda(H)$ be a unital, contractive homomorphism. Then:

(1) There exist group homomorphisms $\theta\colon G \to H$ and $\gamma\colon G \to S^1$ such that

$$\varphi(\mathtt{Lt}^G_g) = \gamma(g)\mathtt{Lt}^H_{\theta(g)}$$

for all $g \in G$.

(2) The kernel of θ is amenable.

(3) φ is injective if and only if θ is injective if and only if θ is isometric.

In particular, there is an isometric isomorphism $F^p_\lambda(G) \to F^p_\lambda(H)$ if and only if $G \cong H$.

Exercise 5.9. Prove Theorem 5.8.

Exercise 5.10. In the context of Theorem 5.8, can it happen that $F^p_\lambda(G)$ is isomorphic, but not isometrically, to $F^p_\lambda(H)$, even though G and H are not isomorphic?

6. Spatial partial isometries and graph algebras

We begin with a general definition.

Definition 6.1. Let A be an algebra. We say that an element $s \in A$ is a *partial isometry* if there exists $t \in A$ such that st and ts are idempotents.

The prototypical example of a partial isometry in $\mathcal{B}(\mathcal{H})$ is given by a surjective isometry between subspaces; in fact, these are precisely the partial isometries of norm one.

For some purposes in L^p-operator algebras, one needs to work with partial isometries that are in some sense “spatially implemented”, similarly to how invertible isometries are spatially implemented by Lamperti’s theorem. This motivates the following definition.

Definition 6.2. Let $(\mathcal{A}, \mu)$ be a localizable measured algebra. Given $E \in \mathcal{A}$, we set $\mathcal{A}_E = \{E \cap F : F \in \mathcal{A}\}$ and let μ_E denote the restriction of μ to $\mathcal{A}_E$. Then $(\mathcal{A}_E, \mu_E)$ is also localizable. Observe that $L^p(\mu) \cong L^p(\mu_E) \oplus L^p(\mu_{E^c})$.

Given $E, F \in \mathcal{A}$, given an isomorphism $\varphi\colon \mathcal{A}_E \to \mathcal{A}_F$ of Boolean algebras, and given $f \in \mathcal{U}(L^\infty(\mu_F))$, the formula

$$s(\xi) = f(\xi \circ \varphi) \left(\frac{d(\mu_E \circ \varphi^{-1})}{d\mu_F} \right)^{1/p}$$

for all $\xi \in L^p(\mu_E)$, defines an isometric isomorphism $L^p(\mu_E) \to L^p(\mu_F)$, which can be regarded as a contractive map $s\colon L^p(\mu) \to L^p(\mu)$ (vanishing on $L^p(\mu_{E^c})$). We call this map the *spatial partial isometry* associated to (E, F, φ, f).

Spatial partial isometries are partial isometries in the sense of Definition 6.1. In fact, the element t is uniquely determined and is the spatial partial isometry associated to $(F, E, \varphi^{-1}, \overline{f} \circ \varphi^{-1})$. (For $p = 2$, this is just the adjoint of s.)

Exercise 6.3. Verify the claim above, computing st and ts.

Spatiality for partial isometries is defined in terms of the underlying measured algebra. However, for $p \neq 2$ and as long as the measured algebra is localizable, the notion is independent of the underlying algebra.

Exercise 6.4. Let $(\mathcal{A}, \mu)$ and $(\mathcal{B}, \nu)$ be localizable measure algebras, let $p \in [1, \infty) \setminus \{2\}$. Suppose that there exists an isometric isomorphism $u\colon L^p(\mu) \to L^p(\nu)$, define an isometric isomorphism $\varphi\colon \mathcal{B}(L^p(\mu)) \to \mathcal{B}(L^p(\nu))$ by $\varphi(a) = u \circ a \circ u^{-1}$ for all $a \in \mathcal{B}(L^p(\mu))$. Show that an operator $s \in \mathcal{B}(L^p(\mu))$ is a spatial partial isometry if and only if $\varphi(s)$ is a spatial partial isometry. Moreover, show that $t \in \mathcal{B}(L^p(\mu))$ is the reverse of s if and only if $\varphi(t)$ is the reverse of $\varphi(s)$. *(Hint: Lamperti's theorem is valid for isometric isomorphisms between* different *L^p-spaces.)*

In view of the previous exercise, for $p \neq 2$, it makes sense to say that an operator s on an L^p-space E is spatial without fixing a presentation of E as $L^p(\mu)$ for some localizable measure μ.

An idempotent is always a partial isometry (take $s = t$). An idempotent which is additionally a spatial partial isometry is called a *spatial idempotent*.

Exercise 6.5. Let $e \in \mathcal{B}(L^p(\mu))$ be a spatial idempotent. Prove that there exists $E \in \mathcal{A}$ such that e is the multiplication operator by the characteristic function of E.

Exercise 6.6. Let $p \in [1, \infty) \setminus \{2\}$, let E be an L^p-space, and let $s \in B(L^p(\mu))$ be a partial isometry. Prove that s is a spatial partial isometry if and only if it is contractive and there exists $t \in \mathcal{B}(E)$ such that ts and st are spatial idempotents.

Not all contractive partial isometries on an L^p-space, for $p \neq 2$, are spatial (unlike the case of invertible isometries, by Lamperti's theorem). For example, the contractive idempotent $\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \in M_2^p$ is not spatial.

6.1. **Spatial representations of matrix algebras.** For $n \in \mathbb{N}$, we denote by c_n the counting measure on $\{1, \ldots, n\}$. Note that $L^p(c_n) = \ell_n^p$ for all $p \in [1, \infty)$. Spatial partial isometries can be used to characterize the canonical matrix norms:

Proposition 6.7. Let $n \in \mathbb{N}$, let $p \in [1, \infty) \setminus \{2\}$, let E be an L^p-space, and let $\varphi\colon M_n^p \to \mathcal{B}(E)$ be a (not necessarily contractive) unital representation. Then the following are equivalent:

(1) φ is isometric;
(2) $\|\varphi(e_{j,k})\| = 1$ and $\varphi(e_{j,j})$ is a spatial idempotent;
(3) $\varphi(e_{j,k})$ is a spatial partial isometry, for all $j, k = 1, \ldots, n$;
(4) $\varphi(e_{j,k})$ is a spatial partial isometry with reverse $\varphi(e_{k,j})$, for all $j, k = 1, \ldots, n$;
(5) There exist another L^p-space F and an isometric isomorphism

$$u\colon F \otimes \ell_n^p \to E$$

such that $\varphi(a)(u(\xi \otimes \eta)) = u(\xi \otimes a\eta)$ for all $\xi \in F$ and all $\eta \in \ell_n^p$. In other words, φ is (conjugate to) a dilation of the canonical representation of M_n^p.

Proof. It is clear that (5) $\Leftrightarrow$ (4) $\Leftrightarrow$ (3) $\Leftrightarrow$ (2) and that (5) $\Leftrightarrow$ (1). We show that (1) implies (2) and that (2) implies (5).

Assume (1). Then $\|\varphi(e_{j,k})\| = 1$ for all $j, k = 1, \ldots, n$ because $\|e_{j,k}\| = 1$ in M_n^p. Given $z \in S^1$ and $j = 1, \ldots, n$, set $v_{j,z} = 1 - e_{j,j} + ze_{j,j}$. One easily checks that $\|v_{j,z}\| \leq 1$, that $v_{j,z}$ is invertible and that its inverse is $v_{j,\overline{z}}$. It follows that $v_{j,z}$ belongs to $\mathrm{Isom}(M_n^p)$. Since φ is unital, $\varphi(v_{j,z})$ is an invertible isometry on E. Fix a localizable measure algebra $(\mathcal{A}, \mu)$ such that $E \cong L^p(\mu)$. By Lamperti's theorem, there exist $h_{j,z} \in \mathcal{U}(L^\infty(\mu))$ and $\psi_{t,j} \in \mathrm{Aut}(\mathcal{A})$ such that

$$\varphi(v_{j,z}) = m_{h_{j,z}} u_{\psi_{j,z}}.$$

On the other hand, $v_{j,z}$ is homotopic to $v_{j,1} = 1$, so all the automorphisms $\psi_{j,z}$ must be the identity. Set $f_j = h_{j,-1}$ for $j = 1, \ldots, n$. Since $v_{j,-1}$ has order two, we deduce that the range of f_j is contained in $\{1, -1\} \subseteq S^1$. It follows that

$$1 - 2\varphi(e_{j,j}) = \varphi(v_{j,-1}) = m_{f_j}$$

and hence $\varphi(e_{j,j}) = \frac{1 - m_{f_j}}{2}$ is the multiplication operator by the characteristic function of the set where f_j equals -1. This, by definition, is a spatial idempotent.

Assume (2). For $j = 1, \ldots, n$, choose $E_j \in \mathcal{A}$ such that $\varphi(e_{j,j})$ is the multiplication operator by the characteristic function of E_j. Since φ is unital, $\bigsqcup\limits_{j=1}^{n} E_j$ must be the total space in $\mathcal{A}$. It is easy to see that $\varphi(e_{j,k})$ restricts to an isometric isomorphism from $L^p(\mu_{E_k})$ to $L^p(\mu_{E_j})$. Set $F = L^p(\mu_{E_1})$. Identify $F \otimes \ell_n^p$ with the space of n-tuples $(\xi_1, \ldots, \xi_n)$ with $\xi_1, \ldots, \xi_n \in F$ with the norm given by

$$\|(\xi_1, \ldots, \xi_n)\|_p^p = \|\xi_1\|_p^p + \ldots + \|\xi_n\|_p^p.$$

Define $u\colon F \otimes \ell_n^p \to E$ by setting

$$u(\xi_1, \ldots, \xi_n) = \xi_1 + \varphi(e_{2,1})(\xi_2) + \cdots + \varphi(e_{n,1})(\xi_n)$$

for all $(\xi_1, \ldots, \xi_n) \in F \otimes \ell_n^p$. (Observe that $\xi_1 = \varphi(e_{1,1})(\xi_1)$.) Then u is isometric because the summands $\varphi(e_{j,1})\xi_j$ are supported on disjoint subsets. One easily checks that u is bijective and that $\varphi(e_{j,k}) = u(e_{j,k} \otimes 1)u^{-1}$ for $j, k = 1, \ldots, n$. We omit the details. □

A representation φ satisfying the equivalent conditions above is called *spatial*. In particular, we obtain another way of defining the spatial norm on M_n: one defines an (algebraic, unital) representation ρ of M_n on an L^p-space to be *spatial* if

$\rho(e_{j,k})$ is a spatial partial isometry for all $j, k = 1, \ldots, n$. The previous proposition shows that the L^p-operator norm $\|\cdot\|$ on M_n defined by $\|x\| = \|\rho(x)\|$, for a spatial representation ρ, is independent of ρ, and is in fact the same norm from Example 3.3. We will see that this idea can be used in other contexts too, specifically to define L^p-analogs of the Cuntz algebras and, more generally, of graph algebras.

6.2. **Analogs of Cuntz algebras on L^p-spaces.** In this subsection, which is based on [25] (generalizing the case $p = 2$ from [6]), we discuss the L^p-Cuntz algebras.

We begin by defining an algebraic object: the Leavitt algebras.

Definition 6.8. Let $n \in \mathbb{N}$ with $n \geq 2$. We define the *Leavitt algebra* L_n to be the universal complex algebra generated by elements $s_1, \ldots, s_n, t_1, \ldots, t_n$, satisfying

$$t_j s_k = \delta_{j,k} \quad \text{and} \quad \sum_{j=1}^{n} s_j t_j = 1$$

for all $j, k = 1, \ldots, n$.

Leavitt algebras are simple, and they are interesting for many reasons. They were discovered by Leavitt in his attempts to show that there is no way of defining a notion of dimension for free modules over general rings (they case of $\mathbb{Z}$ being well-known to work). Indeed, he showed that L_n, as a free L_n-module, satisfies

$$L_n \not\cong \oplus_{j=1}^{m} L_n \text{ for } 1 < m < n \quad \text{and} \quad L_n \cong \oplus_{j=1}^{n} L_n.$$

In particular, L_n does not have the so-called "Invariant Basis Number Property".

Next, we define a distinguished class of representations of Leavitt algebras on L^p-spaces.

Definition 6.9. Let $n \in \mathbb{N}$ with $n \geq 2$ and let $p \in [1, \infty)$. An algebraic unital representation $\rho\colon L_n \to \mathcal{B}(E)$ on an L^p-space E is said to be *spatial* if $\rho(s_j)$ is a spatial partial isometry with reverse $\rho(t_j)$ for all $j = 1, \ldots, n$.

It is easy to see that spatial representations exist. The following is probably the easiest example. For notational convenience, we describe the representation for $n = 2$.

Exercise 6.10. Let $p \in [1, \infty)$. Define operators s_1, s_2, t_1, t_2 on $\ell^p(\mathbb{Z})$ by setting

$$s_j(e_n) = e_{2n+j} \quad \text{and} \quad t_j(e_n) = \begin{cases} e_{\frac{n-j}{2}} & \text{if } n - j \text{ is even} \\ 0 & \text{if } n - j \text{ is odd} \end{cases}$$

for $j = 1, 2$ and $n \in \mathbb{N}$.

(1) Show that s_1 and s_2 are spatial isometries with reverses t_1 and t_2.
(2) Show that there is a well-defined algebra homomorphism $\rho\colon L_2 \to \mathcal{B}(\ell^p(\mathbb{Z}))$ satisfying $\rho(s_j) = s_j$ and $\rho(t_j) = t_j$ for $j = 1, 2$.
(3) For $p = 2$, show that $s_j^* = t_j$ for $j = 1, 2$.

We now define the L^p-Cuntz algebras $\mathcal{O}_n^p$. For $p = 2$, these are C^*-algebras which are called simply Cuntz algebras and denoted $\mathcal{O}_n$. They were introduced by Cuntz in the late 70's [6] and play a fundamental role in the theory of simple C^*-algebras. Their L^p-analogs are much more recent, and were introduced by Phillips in 2012 [25].

Definition 6.11. Let $n \in \mathbb{N}$ with $n \geq 2$ and let $p \in [1, \infty)$. We define a spatial L^p-operator norm on L_n by setting

$$\|x\| = \sup\{\|\rho(x)\| : \rho\colon L_n \to \mathcal{B}(L^p(\mu)) \text{ spatial representation}\}.$$

We define the *L^p-operator Cuntz algebra* $\mathcal{O}_n^p$ to be the completion of L_n in the above norm.

Besides the group algebras discussed in Definition 4.2, which were introduced in the 70's, the L^p-operator Cuntz algebras were the first class of examples of L^p-operator algebras that was considered. The motivation was the following: Cortiñas and Phillips showed that for a class of C^*-algebras that contains $\mathcal{O}_n$, topological K-theory $K_* = K_*^{\mathrm{top}}$ and algebraic K-theory K_*^{alg} agree naturally. They suspected that their methods were applicable to other Banach algebras, and the search for such examples led Phillips to define the algebras $\mathcal{O}_n^p$. It seems to be still unknown whether the algebraic and topological K-theories of $\mathcal{O}_n^p$ agree:

Question 6.12. Let $n \in \mathbb{N}$ with $n \geq 2$ and let $p \in [1, \infty)$. Is there a natural isomorphism $K_*(\mathcal{O}_n^p) \cong K_*^{\mathrm{alg}}(\mathcal{O}_n^p)$?

As is the case for matrix algebras (Proposition 6.7), any two spatial representations induce the same norm:

Theorem 6.13. Let $n \in \mathbb{N}$ with $n \geq 2$ and let $p \in [1, \infty)$. Then any two spatial L^p-representations of L_n induce the same norm.

In particular, it follows from the previous theorem and Exercise 6.10 that $\mathcal{O}_2^p$ (and in fact any $\mathcal{O}_n^p$) can be isometrically represented on $\ell^p(\mathbb{Z})$.

Exercise 6.14. Find an isometric representation of $\mathcal{O}_2^p$ on $L^p([0, 1])$.

The L^p-Cuntz algebras are remarkable algebras that satisfy a number of very relevant properties:

Theorem 6.15. Let $n \in \mathbb{N}$ with $n \geq 2$ and let $p \in [1, \infty)$. Then

(1) $\mathcal{O}_n^p$ is simple.
(2) $\mathcal{O}_n^p$ is purely infinite: for all $x \in \mathcal{O}_n^p$ with $x \neq 0$, there exist $a, b \in \mathcal{O}_n^p$ with $axb = 1$.
(3) $K_0(\mathcal{O}_n^p) \cong \mathbb{Z}_{n-1}$ and $K_1(\mathcal{O}_n^p) \cong \{0\}$.

None of these results are particularly easy to prove, and the first proofs for $p = 2$ (which appeared much earlier) are very different from the case $p \neq 2$. Indeed, the method originally used by Cuntz to compute $K_*(\mathcal{O}_n)$ breaks down for $p \neq 2$, since it used the fact that the group of unitary matrices in M_n is connected (while the group of invertible isometries in M_n^p is not, by Lamperti's theorem).

The argument which does carry over to the case $p \in [1, \infty)$ consists in expressing $\mathcal{O}_n^p$ as the crossed product of the spatial L^p-operator UHF-algebra of type n^∞ (this is essentially the infinite tensor product of copies of M_n^p) by the shift automorphism. Once this is accomplished, there exists a 6-term exact sequence (the "Pimsner-Voiculescu exact sequence") in K-theory relating the K-groups of an L^p-operator algebra A and the K-groups of its crossed product by $\mathbb{Z}$. We omit the details.

Another result about Cuntz C^*-algebras which proved to be absolutely fundamental in the theory of simple C^*-algebras is the following theorem of Elliott; see [28] for a published proof.

Theorem 6.16. There is an (isometric) isomorphism $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$.

This result motivated the search for analogs of this theorem in other contexts. For Leavitt algebras, Ara and Cortiñas showed that there is no such isomorphism; see [1]:

Theorem 6.17. There is no isomorphism $L_2 \otimes L_2 \cong L_2$.

It remained open whether there is an isomorphism for the L^p-versions of $\mathcal{O}_2$. In joint work with Choi and Thiel (see [3]), we have shown that this is also not the case; a proof will be outlined in the last section. This shows that the C^*-case is really quite special and that there are many more isomorphisms between C^*-algebras than between L^p-operator algebras.

6.3. **L^p-operator algebras of finite directed graphs.** This subsection is based on [4]. Let Q be a finite directed graph, which we write as $Q = Q^{(0)} \cup Q^{(1)}$, where the elements of $Q^{(0)}$ are the *vertices* and the elements of $Q^{(1)}$ are the *edges*. We denote by $d, r\colon Q^{(1)} \to Q^{(0)}$ the *domain* and *range* maps.

We begin by defining the Leavitt path algebra associated to a graph. To avoid technicalities, we assume that for every $v \in Q^{(0)}$ there exists an edge $a \in Q^{(1)}$ such that $r(a) = v$. (In the standard terminology, this means that every vertex in Q is regular.)

Definition 6.18. Let Q be a finite oriented graph. We define its associated *Leavitt path algebra* L_Q to be the universal complex algebra (with a unit) generated by elements e_v, s_a, t_b, for $v \in Q^{(0)}$ and $a, b \in Q^{(1)}$, subject to the following relations:

(1) $e_v e_w = \delta_{v,w} e_v$ for $v, w \in Q^{(0)}$;
(2) $e_{r(a)} s_a = s_a e_{d(a)} = s_a$ for all $a \in Q^{(1)}$;
(3) $t_a e_{r(a)} = e_{d(a)} t_a = t_a$ for all $a \in Q^{(1)}$;
(4) $t_a s_b = e_{d(b)} \delta_{s,b}$ for all $a, b \in Q^{(1)}$;
(5) $e_v = \sum\limits_{\{a \in Q^{(1)}\colon r(a) = v\}} s_a t_a$ for all $v \in Q^{(0)}$.

L^p-operator graph algebras are defined similarly to how L^p-operator Cuntz algebras were defined in Definition 6.11: one considers the completion of the Leavitt path algebra with respect to spatial representations.

Definition 6.19. Let Q be a finite oriented graph and let $p \in [1, \infty)$. Given an L^p-space E and a representation $\varphi\colon L_Q \to \mathcal{B}(E)$, we say that φ is *spatial* if

- $\varphi(e_v)$ is a spatial idempotent for all $v \in Q^{(1)}$;
- $\varphi(s_a)$ and $\varphi(t_a)$ are spatial partial isometries for all $a \in Q^{(1)}$.

We define the associated *L^p-operator graph algebra* $\mathcal{O}^p(Q)$ to be the completion of L_Q in the norm

$$\|x\| = \sup\{\|\varphi(x)\|\colon \varphi \text{ spatial representation on an } L^p\text{-space}\}.$$

Unlike in the case for Cuntz algebras, it is not in general true that *any* two spatial representations of L_Q induce the same norm. In other words, the supremum in the previous definition is actually necessary.

Example 6.20. Let C denote the graph with one vertex and one loop around it. Then $\mathcal{O}^p(C) \cong F^p(\mathbb{Z})$ for all $p \in [1, \infty)$.

Example 6.21. Let $n \in \mathbb{N}$ with $n \geq 2$. Denote by C_n the graph with one vertex and n loops around it. Then $\mathcal{O}^p(C_n) \cong \mathcal{O}^p_n$ for all $p \in [1, \infty)$.

Exercise 6.22. For $n \in \mathbb{N}$, let Q_n be the following graph

$$\bullet_1 \longrightarrow \bullet_2 \longrightarrow \cdots \longrightarrow \bullet_n$$

For $p \in [1, \infty)$, show that there is an isometric isomorphism $\mathcal{O}^p(Q_n) \cong M^p_n$.

There is so far not so much known about L^p-operator graph algebras, although graph C^*-algebras are a very well-studied class with a number of very nice properties. A thorough and systematic study of L^p-operator graph algebras is at this point within reach and certainly very interesting.

7. Crossed products and their isomorphisms

Crossed products are a generalization of group algebras, where one considers an action of a group on a topological space or, more generally, an L^p-operator algebra, and constructs an "enveloping algebra" that encodes dynamical information of the original action.

7.1. **Construction of crossed products.** Let G be a locally compact group, endowed with its Haar measure μ, let A be an L^p-operator algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action (by isometric isomorphisms). For example, one could take $A = C_0(X)$ for X locally compact and Hausdorff, take an action $G \curvearrowright X$ by homeomorphisms, and for $s \in G$ let $\alpha_s\colon C_0(X) \to C_0(X)$ be given by $\alpha_s(f)(x) = f(s^{-1} \cdot x)$ for $f \in C_0(X)$ and $x \in X$.

Our next goal is to define the (reduced) crossed product $F^p_\lambda(G, A, \alpha)$, also denoted $A \rtimes_{\alpha,\lambda} G$ whenever p is clear from the context. Intuitively speaking, and by analogy with the semidirect product of groups, the crossed product $F^p_\lambda(G, A, \alpha)$ is the "smallest" algebra that contains A and G, and where the action of G on A is implemented by conjugation by invertible isometries.

The reduced crossed product will be constructed as a certain completion of the Banach algebra $L^1(G, A, \alpha)$, which as a Banach space agrees with $L^1(G, A)$ and whose product is given by *twisted convolution*:

$$(f * g)(s) = \int_G f(t)\alpha_t(g(s^{-1}t))\, d\mu(t)$$

for all $f, g \in L^1(G, A, \alpha)$.

Definition 7.1. A *covariant representation* of (G, A, α) on an L^p-space E is a pair (φ, u) where $\varphi\colon A \to \mathcal{B}(E)$ is a representation and $u\colon G \to \mathrm{Isom}(E)$ is a group homomorphism, which satisfy

$$u_s\varphi(a)u_s^{-1} = \varphi(\alpha_s(a))$$

for all $s \in G$ and all $a \in A$.

Given a covariant pair (φ, u), we define the associated *integrated representation* $\varphi \rtimes u\colon L^1(G, A, \alpha) \to \mathcal{B}(E)$ by

$$(\varphi \rtimes u)(f)(\xi) = \int_G \varphi(f(s))(u_s(\xi))\, d\mu(s)$$

for all $f \in L^1(G, A, \alpha)$ and all $\xi \in E$.

We will consider a distinguished class of covariant pairs, called the *regular* covariant pairs.

Definition 7.2. Let $\varphi_0\colon A \to \mathcal{B}(E_0)$ be any representation. Set $E = L^p(G, E_0)$, and define the associated *regular covariant pair* (φ, u) by

$$\varphi(a)(\xi)(s) = \varphi_0(\alpha_{s^{-1}}(a))(\xi(s)) \quad \text{and} \quad u_s(\xi)(t) = \xi(s^{-1}t)$$

for all $a \in A$, all $s, t \in G$ and all $\xi \in E$.

Definition 7.3. The *reduced crossed product* $F^p_\lambda(G, A, \alpha)$ is the completion of $L^1(G, A, \alpha)$ in the norm

$$\|f\| = \sup\{\|(\varphi \rtimes u)(f)\|\colon (\varphi, u) \text{ is a regular covariant pair}\}.$$

Exercise 7.4. Let G be a locally compact group and let $p \in [1, \infty)$. Prove that there is a canonical identification $F^p_\lambda(G, \mathbb{C}) \cong F^p_\lambda(G)$.

When A has the form $C(X)$, so that the action α comes from an action of G on X via homeomorphisms, we usually write $F^p_\lambda(G, X)$ instead of $F^p_\lambda(G, C(X))$.

There is also a *full crossed product* $F^p(G, A, \alpha)$ which is defined using all covariant pairs and not just those that are regular. The full crossed product, being universal for all covariant pairs, admits a very nice description in terms of generator and relations. Moreover, when G is amenable, then $F^p(G, A, \alpha)$ and $F^p_\lambda(G, A, \alpha)$ agree canonically; the case $A = \mathbb{C}$ is Theorem 4.10. As a consequence, whenever G is amenable, the reduced crossed product $F^p_\lambda(G, A, \alpha)$ can be described in a very concrete way. A particularly nice case if that of integer actions:

Theorem 7.5. Take $G = \mathbb{Z}$ and X compact and Hausdorff. Then an action of $\mathbb{Z}$ on $C(X)$ is generated by one homeomorphism $h\colon X \to X$. Then $F^p_\lambda(\mathbb{Z}, X)$ is the universal L^p-operator algebra generated by a copy of $C(X)$, an invertible isometry u and its inverse, subject to the relation

$$ufu^{-1} = f \circ h^{-1}.$$

7.2. **Isomorphisms of crossed products.** The study of crossed products, particularly of those of the form $F^p_\lambda(G, X)$, is a very active area of research within C^*-algebras. In this setting, one tries to understand what properties of the dynamics $G \curvearrowright X$ are reflected in the algebraic structure of the crossed product. In this section, which is based on [3], try to answer this question.

Example 7.6. Let G be a finite group, acting on the compact Hausdorff space $X = G$ via left translation. Then $F^p_\lambda(G, G) \cong \mathcal{B}(\ell^p(G))$. In particular, the crossed product of $G \curvearrowright G$ *only* remembers the cardinality of G.

Although it does not remember the group G, we will see that $F^p_\lambda(G, X)$ remembers $C(X)$ (and hence X), and more generally that it remembers the "continuous orbit equivalence" of the action, at least when the action is essentially free[1].

Definition 7.7. Let G and H be countable discrete groups, let X and Y be compact Hausdorff spaces, and let $G \curvearrowright^\sigma X$ and $H \curvearrowright^\rho Y$ be actions. We say that σ and ρ are *continuously orbit equivalent*, written $G \curvearrowright^\sigma X \sim_{\mathrm{coe}} H \curvearrowright^\rho Y$, if there

[1]Recall that an action $G \curvearrowright X$ is said to be *essentially free* if for all $g \in G \setminus \{1\}$, the set $\{x \in X\colon g \cdot x = x\}$ has empty interior.

exist a homeomorphism $\theta\colon X \to Y$ and continuous maps $c_H\colon G \times X \to H$ and $c_G\colon H \times Y \to G$ satisfying

$$\theta(\sigma_g(x)) = \rho_{c_H(g,x)}(\theta(x)) \quad \text{and} \quad \theta^{-1}(\rho_h(y)) = \sigma_{c_G(h,y)}(\theta^{-1}(y))$$

for all $x \in X$, $y \in Y$, $g \in G$ and $h \in H$.

When two essentially free actions as above are continuously orbit equivalent, the maps c_G and c_H from the definition are uniquely determined and satisfy certain cocycle conditions. These cocycle conditions allow one to show that if two essentially free actions are continuously orbit equivalent, then their reduced crossed products are naturally isometrically isomorphic, for all $p \in [1, \infty)$. The following theorem asserts that the converse is true for $p \neq 2$.

Theorem 7.8. Let $p \in [1, \infty) \setminus \{2\}$, let G and H be countable discrete groups, let X and Y be compact Hausdorff spaces, and let $G \curvearrowright X$ and $H \curvearrowright Y$ be topologically free actions. Then the following are equivalent:

(1) There is an isometric isomorphism $F^p_\lambda(G, X) \cong F^p_\lambda(H, Y)$;
(2) $G \curvearrowright X$ and $H \curvearrowright Y$ are continuously orbit equivalent.

We will not prove Theorem 7.8, and we will only explain how to prove that an isometric isomorphism $F^p_\lambda(G, X) \cong F^p_\lambda(H, Y)$ must map the canonical copy of $C(X)$ inside $F^p_\lambda(G, X)$ to the canonical copy of $C(Y)$ inside $F^p_\lambda(H, Y)$. This is attained using the notion of the *C^*-core* of an L^p-operator algebra.

Theorem 7.9. Let $p \in [1, \infty)$, and let A be a unital L^p-operator algebra. Then there is a unique maximal unital C^*-subalgebra $\mathrm{core}(A)$ of A, called the *C^*-core* of A. If $p \neq 2$, then $\mathrm{core}(A)$ is abelian, hence of the form $C(X_A)$ for a (uniquely determined) compact Hausdorff space X_A.

The proof that such an algebra exists is a bit technical for $p \neq 2$, but we give three other ways of identifying it:

- Given any unital isometric representation $\varphi\colon A \to \mathcal{B}(L^p(\mu))$ of A on an L^p-space $L^p(\mu)$, the set
$$A_{\mathrm{h}} = \{a \in A \colon \varphi(a) \in L^\infty(\mu)_{\mathbb{R}}\} \subseteq A$$
can be shown to be independent of φ, and is closed under multiplication and pointwise complex conjugation (as a subset of $L^\infty(\mu)$). In particular, $A_{\mathrm{h}} + iA_{\mathrm{h}}$ is a norm-closed self-adjoint subalgebra of $L^\infty(\mu)$, and is therefore a commutative C^*-algebra. This is the C^*-core of A.
- Given any unital isometric representation $\varphi\colon A \to \mathcal{B}(L^p(\mu))$ of A on an L^p-space $L^p(\mu)$, the subgroup
$$\mathcal{V}(A) = \{u \in \mathrm{Isom}(A) \colon \varphi(u) \in L^\infty(\mu)\} \subseteq \mathrm{Isom}(A)$$
can be shown to be independent of φ. It is clearly commutative because $L^\infty(\mu)$ is commutative. Then $C(X_A)$ is the closed linear span of $\mathcal{V}(A)$.
- Set
$$\mathrm{Herm}(A) = \{a \in A \colon \|e^{ita}\| = 1 \text{ for all } t \in \mathbb{R}\}.$$
Then $\mathrm{Herm}(A) + i\mathrm{Herm}(A) = C(X_A)$. (In fact, $\mathrm{Herm}(A)$ agrees with the set A_{h} from the first bullet point above.)

The algebra $C(X_A)$ plays the role that *maximal abelian subalgebras* play in the context of C^*-algebras, with two differences: it is unique (an advantage), and it may be very small (a disadvantage). For example:

Exercise 7.10. Let G be a discrete group and let $p \neq 2$. Then $X_{F^p_\lambda(G)} = \{*\}$. *(Hint: use the second description of $C(X_A)$ given above together with Theorem 5.6.)*

Exercise 7.11. Let μ be a localizable measure and let $p \neq 2$. Show that the core of $\mathcal{B}(L^p(\mu))$ is $L^\infty(\mu)$. Deduce that $X_{M_n^p} = \{1, \ldots, n\}$.

The following result clarifies how $C(X)$ is abstractly identified inside $F^p_\lambda(G, X)$.

Theorem 7.12. Let G be a discrete group, let X be a compact Hausdorff space, let $G \curvearrowright X$ be an action, and let $p \in [1, \infty) \setminus \{2\}$. Then the C^*-core of $F^p_\lambda(G, X)$ is $C(X)$.

We close this section with some comments on what information about G and H one can deduce from knowing that they admit two continuously orbit equivalent action. One of course does not expect to get an isomorphism of the groups, and in fact the type of equivalence one gets is really very weak (although strong enough to give some interesting applications; see the following section).

Definition 7.13. Let G and H be finitely generated groups, endowed with their word metrics d_G and d_H. We say that G and H are *quasi-isometric*, written $G \sim_{\text{q.i.}} H$, if there exist a function $\varphi\colon G \to H$ and a constant $K > 0$ such that

$$K^{-1} d_G(g, g') - K \leq d_H(\varphi(g), \varphi(g')) \leq K d_G(g, g') + K$$

for all $g, g' \in G$.

Remark 7.14. By Theorem 3.2 in [23], if $G \curvearrowright X$ and $H \curvearrowright Y$ are continuously orbit equivalent, then G is quasi-isometric to H.

8. Tensor products of Cuntz algebras

In this final section, also based on [3], we answer a question of Phillips regarding the existence of an isometric isomorphism $\mathcal{O}_2^p \cong \mathcal{O}_2^p \otimes \mathcal{O}_2^p$ for $p \in [1, \infty) \setminus \{2\}$.

Theorem 8.1. Let $p \in [1, \infty) \setminus \{2\}$, let $n, m \in \mathbb{N}$. Then there is an isometric isomorphism

$$\underbrace{\mathcal{O}_2^p \otimes_p \cdots \otimes_p \mathcal{O}_2^p}_{n} \cong \underbrace{\mathcal{O}_2^p \otimes_p \cdots \otimes_p \mathcal{O}_2^p}_{m}$$

if and only if $n = m$.

The first step in proving the previous theorem is identifying $\mathcal{O}_2^p$ as a crossed product by an essentially free topological action. This is done in the following proposition.

Proposition 8.2. Let $p \in [1, \infty)$. Then there exist an essentially free action of $\mathbb{Z}_2 * \mathbb{Z}_3$ on the Cantor set X an isometric isomorphism

$$F^p_\lambda(\mathbb{Z}_2 * \mathbb{Z}_3, X) \cong \mathcal{O}_2^p.$$

Proof. We only describe the action, and omit the construction of the isomorphism. We identify the Cantor set X as

$$X = \left\{ x\colon \mathbb{N} \to \mathbb{Z}_2 * \mathbb{Z}_3 \;\middle|\; \begin{array}{l} \text{for } k \in \mathbb{N} \text{ there is } j_k \in \{2, 3\} \\ \text{such that } x(k) \in \mathbb{Z}_{j_k} \subseteq \mathbb{Z}_2 * \mathbb{Z}_3 \\ \text{and } j_k \neq j_{k+1} \text{ for all } k \in \mathbb{N} \end{array} \right\}.$$

We denote by $a \in \mathbb{Z}_2$ the nontrivial element, and by $b \in \mathbb{Z}_3$ the canonical generator of order 3. Define an action $\mathbb{Z}_2 * \mathbb{Z}_3 \curvearrowright X$ by

$$(ax)(k) = \begin{cases} x(k+1) & \text{if } x(0) = a; \\ x(k) & \text{if } j_0 = 2, x(0) \neq a, \text{ and } k > 0; \\ ax(0) & \text{if } j_0 = 2, x(0) \neq a, \text{ and } k = 0; \\ x(k-1) & \text{if } j_0 \neq 2, \text{ and } k > 0; \\ a & \text{if } j_0 \neq 2, \text{ and } k = 0, \end{cases}$$

and

$$(bx)(k) = \begin{cases} x(k+1) & \text{if } x(0) = b; \\ x(k) & \text{if } j_0 = 2, x(0) \neq b^n, \text{ and } k > 0; \\ bx(0) & \text{if } j_0 = 2, x(0) \neq b^n, \text{ and } k = 0; \\ x(k-1) & \text{if } j_0 \neq 2, \text{ and } k > 0; \\ b & \text{if } j_0 \neq 2, \text{ and } k = 0. \end{cases}$$

One checks that a acts via a homeomorphism of order 2, and that b acts via a homeomorphism of order 3, so that the previous equations really do define an action of $\mathbb{Z}_2 * \mathbb{Z}_3$ on X. We omit the details. □

It is not difficult to deduce from the previous proposition that $\underbrace{\mathcal{O}_2^p \otimes_p \cdots \otimes_p \mathcal{O}_2^p}_{n}$ is isometrically isomorphic to the crossed product of an essentially free action of $(\mathbb{Z}_2 * \mathbb{Z}_3)^n$ on the Cantor space. We are now ready to finish the proof of Theorem 8.1.

Proof of Theorem 8.1. Suppose that there exists an isometric isomorphism

$$\underbrace{\mathcal{O}_2^p \otimes_p \cdots \otimes_p \mathcal{O}_2^p}_{n} \cong \underbrace{\mathcal{O}_2^p \otimes_p \cdots \otimes_p \mathcal{O}_2^p}_{m}.$$

By the comments above, there are actions of $(\mathbb{Z}_2 * \mathbb{Z}_3)^n$ and $(\mathbb{Z}_2 * \mathbb{Z}_3)^m$ on the Cantor space X such that $F^p_\lambda((\mathbb{Z}_2 * \mathbb{Z}_3)^n, X) \cong F^p_\lambda((\mathbb{Z}_2 * \mathbb{Z}_3)^m, X)$. By Theorem 7.8, this implies that the underlying dynamical systems are continuously orbit equivalent. By Remark 7.14, this implies that $(\mathbb{Z}_2 * \mathbb{Z}_3)^n$ is quasi-isometric to $(\mathbb{Z}_2 * \mathbb{Z}_3)^m$. Finally, it is known that such a quasi-isometry exists if and only if $n = m$. This finishes the proof.

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