PARTIAL GROUP ACTIONS

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Warning: little proofreading has been done.

Contents

Introduction

The concept of partial group actions is relevant in numerous categories, but we shall mostly be concerned with four of these, namely sets, topological spaces, algebras and C^*-algebras. We shall also be interested in the interplay between them. The notion of partial actions is closely tied with that of partial representations of groups, as well as partial group algebras, and we shall try to describe how these concepts relate to each other with the ultimate goal of showing that the structure of certain very important C^*-algebras, including but not limited to the Cuntz-Krieger algebras, may be entirely described with the help of these concepts.

1. Partial actions

We will focus on partial actions of groups on the following categories: sets, topological spaces (usually assumed to be locally compact and Hausdorff), algebras over rings (which includes the categories of algebras over fields and also the category of rings), $*$-algebras, and C^*-algebras.

If G is a discrete group, then an action of G on an object C in a category $\mathcal{C}$ is a group homomorphism $G \to \mathrm{Aut}_{\mathcal{C}}(C)$. These actions are referred to as *global* actions. We turn to *partial* actions.

Definition 1.1. Given a discrete group G and a set X, a *partial action* of G on X consists of a pair of families $(\{D_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ where for all $g \in G$, we have that D_g is a subset of X and $\theta_g \colon D_{g^{-1}} \to D_g$ is a bijection, satisfying

(1) $D_1 = X$ and $\theta_1 = \mathrm{id}_X$.
(2) For all g and h in G, one has $\theta_g \circ \theta_h \subseteq \theta_{gh}$. This inclusion means that the right-hand side is an extension of the left-hand side. Also, the composition is taken whenever possible.

Condition (2) is equivalent to

- $\theta_g(D_{g^{-1}} \cap D_h) = D_g \cap D_{gh}$ for all g and h in G,
- For $a \in D_{h^{-1}} \cap D_{(gh)^{-1}}$, we have $\theta_h(a) \in \theta_h(D_{h^{-1}} \cap D_{(gh)^{-1}}) = D_h \cap D_{g^{-1}}$.

Examples 1.2. Some natural examples of partial actions on sets.

(1) **Flows on manifolds.** Let M be a manifold and let V be a vector field over M. It is a standard result that for all $x_0 \in M$, there exists a path $x(t)$ defined on some interval $(-\beta, \beta)$ around 0 such that $x'(t) = V(x(t))$ for all $t \in (-\beta, \beta)$. This gives rise to a partial map defined the *flow* associated to V, given by

$$\phi_t(x_0) = x(t).$$

This is a partial action of $\mathbb{R}$ on M. It is well-known that if M is compact, then the maximal domain of $\phi(x_0)$ is $\mathbb{R}$. This was proved independently by Fernando Abadie using techniques of partial actions.

(2) **Odometer.** Consider $X = \{0, \dots, 9\}^{\mathbb{N}}$. We exclude the constant sequence 9, since it has no successor. Define a map $\theta_1 \colon X \setminus \{(9, 9, \dots)\} \to X \setminus \{(0, 0, \dots)\}$ by increasing by 1 the first coordinate modulo 10, and increasing the following ones if necessary. Then θ_1 determines a partial $\mathbb{Z}$-action via $\theta_n = (\theta_1)^n$, with $D_{-n} = X \setminus \{(9, 9, \dots), (8, 9, \dots), \dots\}$ and $D_n = X \setminus \{(0, 0, \dots), (1, 0, \dots), \dots\}$ for all $n \in \mathbb{N}$.

(3) **Restriction of a global action.** Let G be a group, let H be a subgroup and let $\beta \colon G \to \mathrm{Aut}(X)$ be a global action. Let

$$D_g = \begin{cases} X, & \text{if } g \in H; \\ \emptyset, & \text{if } g \notin H. \end{cases} \quad \text{and} \quad \theta_g = \begin{cases} \beta_g, & \text{if } g \in H; \\ \emptyset, & \text{if } g \notin H. \end{cases}$$

Then $\theta = (\{D_g\}_{g \in G}, \{\theta_g\}_{g \in G})$ is a partial action on X.

(4) **Partial actions of free groups.** Let X be a set, let $n \in \mathbb{N}$, and for $j = 1, \dots, n$ let A_j and B_j be subsets of X and let $h_j \colon A_j \to B_j$ be a bijection. Let $G = \mathbb{F}_n = \langle g_1, \dots, g_n \rangle$. Given $g \in G$, define

$$\theta_g = \begin{cases} h_j, & \text{if } g = g_j; \\ h_j^{-1}, & \text{if } g = g_j^{-1}; \\ \theta_{g_{k_1}}^{\pm 1} \circ \cdots \circ \theta_{g_{k_m}}^{\pm 1}, & \text{if } g = g_{k_1}^{\pm 1} \cdots g_{k_m}^{\pm 1} \text{ is in reduced form.} \end{cases}$$

(5) **Cuntz-Krieger matrices.** Given $A \in M_n(\{0, 1\})$, assign to it a graph E_A with n vertices and an edge from the vertex i to the vertex j whenever $A_{i,j} = 1$. Consider the Markov space M_A of infinite paths associated to A, this is,

$$M_A = \left\{ x \in \prod_{k \in \mathbb{N}} \{1, \dots, n\} \colon A_{x_j, x_{j+1}} = 1 \text{ for all } j \in \mathbb{N} \right\}.$$

We define a partial action of $\mathbb{F}_n$ on M_A by

$$\theta_j(x_1, x_2, \dots) = (j, x_1, x_2, \dots),$$

with $D_{j^{-1}} = \{x \in M_A \colon A_{j, x_1} = 1\}$.

(6) **Bernoulli partial action.** Let $X = \prod_G \{0, 1\}$ and consider the usual Bernoulli shift of G on X. Note that $X \cong \mathbb{P}(G)$, the power set of G. The Bernoulli action is then

$$\beta_g(E) = gE$$

for all $g \in G$ and all $E \in X$. Put $\Omega = \{E \in \mathbb{P}(G) \colon 1 \in E\}$, and let $D_g = \{E \in \mathbb{P}(G) \colon 1, g \in E\}$. Then $\beta_g(D_{g^{-1}} = D_g$ for all $g \in G$. Hence β_g restricts to a partial action $\theta = \Big(\{D_g\}_{g \in G}, \{\beta_g|_{D_{g^{-1}}}\}_{g \in G}\Big)$.

We now turn to the definition of a partial action of a group on the other categories.

Definition 1.3. Let G be a discrete group and let C be an object in one of the following categories:

(1) Topological spaces;
(2) Algebras over rings;
(3) $*$-algebras;
(4) C^*-algebras.

A *partial action* of G on C is a partial action $\theta = (\{D_g\}_{g \in G}, \{\theta_g\}_{g \in G})$ on the underlying set such that

(1) If C is a topological space, then D_g is open in C;
(2) If C is an algebra over a ring, then D_g is a two-sided ideal;
(3) If C is a $*$-algebra, then D_g is a self-adjoint two-sided ideal;
(4) If C is a C^*-algebra, then D_g is a closed (self-adjoint) two-sided ideal,

and θ_g is an isomorphism in $\mathrm{Hom}_{\mathcal{C}}\left(D_{g^{-1}}, D_g\right)$ for all $g \in G$.

Remark 1.4. A partial action on a topological space X induces a partial action on $C_0(X)$. In fact, given a discrete group G, there is a functorial equivalence between partial actions of G on locally compact Hausdorff spaces and partial actions of G on commutative C^*-algebras.

2. Restriction and globalization

Let X be an object in one of the categories we are looking at, and let $\beta \colon G \to \mathrm{Aut}(X)$ be a global action. Given a subobject $Y \subseteq X$ which in algebraic contexts is assumed to be an ideal, and in topological contexts it is assumed to be an open set, put $D_g = Y \cap \beta_g(Y) \subseteq Y$. Then $\beta_g(D_{g^{-1}}) = D_g$ for all $g \in G$, so we may set $\theta_g \colon D_{g^{-1}} \to D_g$ to be the restriction of β_g to $D_{g^{-1}}$. In this way, we get a partial action from a global action by *restriction.*

We are interested in the inverse problem:

Problem 2.1. Given a partial action θ on an object Y, is it the restriction of a global action β on another space X?

If such a global action exists, we say that β is a *globalization* of θ if moreover $X = \bigsqcup_{g \in G} \beta_g(Y)$. This problem has been thoroughly studied by Fernando Abadie.

Theorem 2.2. (F. Abadie)

(1) Partial actions on sets always admit a unique globalization.

(2) Partial actions on topological spaces always admit a unique globalization, although the enveloping space X may fail to be Hausdorff, even if Y is.
(3) Globalizations may fail to exist for C^*-algebras.

Proof. (1). In $G\times Y$, introduce the equivalence relation $(g,y)\sim(h,z)$ if $y\in D_{g^{-1}h}$ and $\theta_{h^{-1}g}(y)=z$. Set $X=G\times Y/\sim$ and define $\beta_g([h,y])=[gh,y]$ for all $g\in G$ and all $[h,y]\in X$.

(2). This is similar, but one has to be more careful with the topology.

(3). It is enough to find an example of a partial action on a locally compact Hausdorff space Y such that the enveloping space X is not Hausdorff, and then use the fact that a globalization of a partial action on an abelian C^*-algebra must also be abelian. □

Abadie also showed that given a partial action of G on a C^*-algebra A, there always is a Morita equivalent partial action on another C^*-algebra A' which admits a globalization $A'\lhd B$. Moreover, if $A=C_0(Y)$, then $\mathrm{Prim}(B)=X$, where X is the possibly non-Hausdorff space obtained by globalizing the partial action on Y.

3. Crossed products of partial actions

Recall that if α is an action of G on A, then the skew group ring $A[G]$, also denoted $A\rtimes_\alpha G$, is

$$A[G]=\left\{\sum_{g\in G}a_gu_g\colon a_g\in A, a_g=0 \text{ for all but finitely many group elements } g\in G\right\},$$

with the product given by

$$(a_gu_g)(b_hu_h)=a_g\alpha_g(b_h)u_{gh}.$$

Given a partial action α of G on A, we may consider the *skew partial group ring*, also denoted $A\rtimes_\alpha G$, given by

$$A\rtimes_\alpha G=\left\{\sum_{g\in G}a_gu_g\colon a_g\in D_g, a_g=0 \text{ for all but finitely many group elements } g\in G\right\},$$

with the product given by

$$(a_gu_g)(b_hu_h)=\alpha_g(\alpha_{g^{-1}}(a_g)b_h)u_{gh}.$$

We must check that $\alpha_g(\alpha_{g^{-1}}(a_g)b_h)$ belongs to D_{gh}. This follows from the fact that $\alpha_{g^{-1}}(a_g)b_h\in D_{g^{-1}}\cap D_h$ and $\alpha_g(D_{g^{-1}}\cap D_h)=D_g\cap D_{gh}\subseteq D_{gh}$.

The following is a major issue in this theory.

Problem 3.1. When is this product associative?

We have the following partial answer, which includes all cases of interest.

Theorem 3.2. (Dokushaev-Exel) The product on $A\rtimes_\alpha G$ defined above is associative if either
(1) every D_g is idempotent; or
(2) every D_g is non-degenerate, meaning that if $a\in D_g$ and $x\in A\setminus\{0\}$ are such that $ax=0$, then $a=0$.

If A is a $*$-algebra, one can make $A\rtimes_\alpha G$ into a $*$-algebra (assuming associativity holds) by setting $(a_gu_g^*)=\alpha_{g^{-1}}(\alpha_g(a_g^*))u_{g^{-1}}$.

Definition 3.3. If A is a C^*-algebra, then we define the *full crossed product*, denoted $A\rtimes_\alpha G$, by

$$A\rtimes_\alpha G=C^*(A\rtimes_\alpha^{alg}G),$$

this is, the enveloping C^*-algebra with respect to the norm on $A\rtimes_\alpha^{alg}G$ defined using the supremum over all representations.

In order to define reduced crossed products, we will need to introduce Fell bundles first.

Definition 3.4. A *Fell bundle* over a discrete group G is a collection $\mathcal{B}=\{B_g\}_{g\in G}$ of Banach spaces equipped with operations

$$\cdot\colon B_g\times B_h\to B_{gh} \quad \text{and} \quad *\colon B_g\to B_{g^{-1}}$$

satisfying certain axioms. (The idea is to think of the Fell bundle as a graded C^*-algebra $B=\oplus_{g\in G}B_g$ such that B_g is closed in B and $B_gB_h\subseteq B_{gh}$ and $B_g^*=B_{g^{-1}}$ for all g and h in G.)

If $\mathcal{B}=\{B_g\}_{g\in G}$ is a Fell bundle, then it follows from the axioms that B_1 is a C^*-algebra.

Remark 3.5. Given a partial action of G on a C^*-algebra A, one can build a Fell bundle by setting $B_g=D_gu_g$. Using the operations previously defined, $\mathcal{B}=\{B_g\}_{g\in G}$ becomes a Fell bundle.

To any Fell bundle $\mathcal{B} = \{B_g\}_{g\in G}$ we can naturally associate two C^*-algebras: the full and reduced Fell bundle algebras. The full algebra is $C^*(\mathcal{B}) = C^*(\oplus_{g\in G} B_g)$.

To define the reduced algebra, regard each B_g as a right B_1-Hilbert module with $\langle b_g, c_g\rangle = b_g^* c_g \in B_1$ and set $M = \oplus_{g\in G} B_g$ (where the sum is taken as Hilber modules). Define a representation $\lambda\colon B \to \mathcal{L}(M)$ simply my multiplication coordinate-wise. Define the reduced algebra associated to $\mathcal{B} = \{B_g\}_{g\in G}$ by

$$C_r^*(\mathcal{B}) = C^*\left(\bigsqcup_{g\in G} \lambda(B_g)\right).$$

Definition 3.6. Let α be a partial action of a discrete group G on a C^*-algebra A. The *reduced crossed product*, denoted $A \rtimes_{\alpha,r} G$, is the reduced C^*-algebra associated to the Fell bundle $\{D_g u_g\}_{g\in G}$.

Example 3.7. Consider the action $\mathbb{Z} \to \mathrm{Aut}(C_0(\mathbb{Z}))$ by left translation, and restrict it to $\{1,\ldots,n\}$. One has

$$C(\{1,\ldots,n\}) \rtimes \mathbb{Z} \cong M_n.$$

In particular, M_n has a grading over $\mathbb{Z}$, which is given by

$$M_n^{(k)} = \{a \in M_n : a_{i,j} = 0 \text{ whenever } i - j \neq k\}.$$

(These are the diagonal matrices.) In particular, $M_n^{(k)} = \{0\}$ whenever $|k| \geq n$.

One can improve this result greatly.

Theorem 3.8. Every AF-algebra arises as the completely positive of a MASA by a partial action of $\mathbb{Z}$.

F. Abadie gave an explicit description of what the partial homeomorphism looks like in terms of the Bratelli diagram.

Example 3.9. The crossed product of the partial action of $\mathbb{F}_n$ on the Markov space of $A \in M_n(\{0,1\})$ is isomorphic to $\mathcal{O}_A$.

4. Covariant representations

We start by defining the appropriate (partial) representations of the group.

Definition 4.1. Let G be a discrete group and let A be a unital C^*-algebra, often times $\mathcal{B}(\mathcal{H})$. A *partial representation* of G in A is a map $\pi\colon G \to A$ such that

(1) $\pi(e) = 1_A$.
(2) $\pi(g)\pi(h)\pi(h^{-1}) = \pi(gh)\pi(h^{-1})$ for all g and h in G.
(3) $\pi(g^{-1})\pi(g)\pi(h) = \pi(g^{-1})\pi(gh)$ for all g and h in G.
(4) $\pi(g^{-1}) = \pi(g)^*$ for all g in G.

Remark 4.2. Let π be a partial representation of G and let $g \in G$. Then

$$\pi(g)\pi(g)^*\pi(g) = \pi(g) \quad \text{and} \quad \pi(g)^*\pi(g)\pi(g)^* = \pi(g)^*.$$

In particular, $\pi(g)$ is a partial isometry.

Definition 4.3. A *covariant representation* of a partial dynamical system (A, α) in a C^*-algebra C is a pair (ρ, π), where

(1) $\rho\colon A \to C$ is a homomorphisms,
(2) $\rho\colon G \to C$ is a partial representation,

such that

$$\pi(g)\rho(a)\pi(g^{-1}) = \rho(\alpha_g(a))$$

for all $a \in D_{g^{-1}}$ and all $g \in G$.

The following result should be expected.

Proposition 4.4. Given a covariant representation (ρ, π) of a partial dynamical system (A, α) in a C^*-algebra C, the formula

$$(\rho \times \pi)(a_g u_g) = \rho(a_g)\pi(g)$$

induces a homomorphism $A \rtimes_\alpha G \to C$.

In order to prove it, we need a simple lemma first.

Lemma 4.5. In the context of the above proposition, we have

$$\rho(a)\pi(g)\pi(g^{-1}) = \rho(a) = \pi(g)\pi(g^{-1})\rho(a)$$

for all $a \in A$ and all $g \in G$.

Proof. We simply work with the right-hand side. Write $a = \alpha_g(b)$ for some $b \in D_{g^{-1}}$. Then

$$\rho(a)\pi(g)\pi(g^{-1}) = \rho(\alpha_g(b))\pi(g)\pi(g^{-1}) = \pi(g)\rho(b)\pi(g^{-1})\pi(g)\pi(g^{-1}) = \pi(g)\rho(b)\pi(g^{-1}) = \rho(\alpha_g(b)) = \rho(a).$$

The other equality is similar. ∎

In particular, $e_g = \pi(g)\pi(g^{-1})$ is an idempotent for all $g \in G$.

Proof. (of the Proposition) We just show multiplicativity of $\rho \times \pi$, using the Lemma in the second step:

$$\begin{aligned}((\rho \times \pi)(au_g)) \cdot ((\rho \times \pi)(bu_h)) &= \rho(a)\pi(g)\rho(b)\pi(h)\\ &= \pi(g)\pi(g^{-1})\rho(a)\pi(g)\rho(b)\pi(h)\\ &= \pi(g)\rho(\alpha_{g^{-1}}(a)b)\pi(h)\\ &= \pi(g)\rho(\alpha_{g^{-1}}(a)b)\pi(g)\pi(g^{-1})\pi(h)\\ &= \rho(\alpha_g(\alpha_{g^{-1}}(a)b))\pi(gh)\\ &= (\rho \times \pi)\left((au_g) \cdot (bu_h)\right).\end{aligned}$$

□

In the classical setting, there is a one-to-one correspondence between covariant representations of the dynamical system and representations of the completely positive. If we assume a certain non-degeneracy condition for covariant representations of partial dynamical systems, then the same result is true in the context of partial crossed products. One possible non-degeneracy condition for a covariant representation (ρ, π) is that $\pi(g)\pi(g^{-1})$ be the smallest idempotent satisfying the conclusion of the Lemma above.

We now study the case where there is no C^*-algebra.

Definition 4.6. Let G be a discrete group. Define $C^*_{\text{par}}(G)$ to be the universal C^*-algebra generated by symbols u_g for $g \in G$ with relations given by the definition of partial representation.

The C^*-algebra $C^*_{\text{par}}(G)$ is the universal C^*-algebra with respect to partial representations. Indeed, if $\pi\colon G \to C$ is a partial representation, then there exists a unique C^*-algebra homomorphism $\phi\colon C^*_{\text{par}}(G) \to C$ making the following diagram commute:

$$\begin{tikzcd} G \arrow[r] \arrow[dr, "\pi"'] & C^*_{\text{par}}(G) \arrow[d, "\phi"] \\ & C. \end{tikzcd}$$

Remark 4.7. It follows that $C^*(G)$ is a factor of $C^*_{\text{par}}(G)$.

It turns out that $C^*_{\text{par}}(G)$ contains a lot of information about the group G. In fact, we have

Theorem 4.8. If G_1 and G_2 are finite abelian groups and $C^*_{\text{par}}(G_1) \cong C^*_{\text{par}}(G)$, then $G_1 \cong G_2$.

Recall that, in contrast, $C^*(G_1) \cong C^*(G_2)$ in the above Theorem is equivalent to just $|G_1| = |G_2|$.

Given a partial representation $\pi\colon G \to C$, let $e(g) = \pi(g)\pi(g^{-1})$ for all $g \in G$.

Proposition 4.9. For g and h in G, we have $\pi(g)e(h) = e(gh)\pi(g)$.

Proof. We simply compute:

$$\pi(g)e(h) = \pi(g)\pi(h)\pi(h^{-1}) = \pi(gh)\pi(h^{-1}) = \pi(gh)\pi((gh)^{-1})\pi(gh)\pi(h^{-1}) = e(gh)\pi(g).$$

□

Corollary 4.10. For all g and h in G, the idempotents $e(g)$ and $e(h)$ commute.

Proof. Again we compute:

$$e(g)e(h) = \pi(g)\pi(g^{-1})e(h) = \pi(g)e(g^{-1}h)\pi(g^{-1}) = e(h)\pi(g)\pi(g^{-1}) = e(h)e(g).$$

□

In particular, $A = C^*(\{e(g)\colon g \in G\}) \subseteq C$ is a commutative subalgebra. For $g \in G$, denote by D_g the ideal in A generated by $e(g)$, this is, $D_g = e(g)A$. We define a partial action α of G on A as follows. For $g \in G$ and $a \in D_{g^{-1}|}$, set $\alpha_g(a) = \pi(g)a\pi(g^{-1}) \in D_g$. We check that α_g and $\alpha_{g^{-1}}$ are inverses in the appropriate domains, for all $g \in G$:

$$\alpha_g \circ \ alpha_{g^{-1}}(a) = \pi(g^{-1})\pi(g)a\pi(g^{-1})\pi(g) = e(g^{-1})ae(g^{-1}) = a$$

since $e(g^{-1})D_{g^{-1}}e(g^{-1}) = D_{g^{-1}}$.

5. Description of some C^*-algebras as partial crossed products

If one takes π to be the universal partial representation $G \to C^*_{\mathrm{par}}(G)$, we will try to compute $A \subseteq C^*_{\mathrm{par}}(G)$. For this, we compute its spectrum.

A character $\gamma\colon A \to \mathbb{C}$ corresponds to a sequence $(\gamma(e(g)))_{g\in G}$ taking values in $\{0,1\}$, because $\gamma(e(g))$ is an idempotent in $\mathbb{C}$. Note that $e(1)=1$, and hence A is unital. We hance get an injection

$$\widehat{A} \hookrightarrow \{0,1\}^G = \mathbb{P}(G).$$

(Note that $\mathbb{P}(G)$ is homeomorphic to the Cantor set.) This map is *never* surjective, since $\gamma(e_1)=1$ for all characters $\gamma\in\widehat{A}$. When regarding $\widehat{A}$ as a subset of $\mathbb{P}(G)$, this is equivalent to $1\in E$ whenever $E\in\widehat{A}$. It turns out that the converse is also true, and hence we can describe the spectrum Ω of A by

$$\Omega = \{E\subseteq G\colon 1\in E\}.$$

The corresponding partial action on Ω is the Bernoulli shift, this is, $D_g=\{E\subseteq G\colon 1,g\in E\}$ and $\theta_g(E)=gE$ for $g\in G$ and $E\in\Omega$.

Theorem 5.1. There is an isomorphism $C(\Omega)\rtimes_\alpha G \cong C^*_{\mathrm{par}}(G)$.

Example 5.2. Denote by s_1 and s_2 the canonical generators of $\mathcal{O}_2$. Define a map $\pi\colon \mathbb{F}_2\to\mathcal{O}_2$ by

$$g_j\mapsto s_j \quad g_j^{-1}\mapsto s_j^* \quad 1\mapsto 1$$

for $j=1,2$, and extend to reduced words. One then checks that π is a partial representation, and hence there is a homomorphism $\varphi\colon C^*_{\mathrm{par}}(\mathbb{F}_2)\to\mathcal{O}_2$. Using the expression $C(\Omega)\rtimes_\alpha\mathbb{F}_2\cong C^*_{\mathrm{par}}(\mathbb{F}_2)$, we may look at the commutative subalgebra $\varphi(C(\Omega))\subseteq\mathcal{O}_2$. It can be shown that $\varphi(C(\Omega))$ is the MASA in $\mathcal{O}_2$. We must therefore have $\varphi(C(\Omega))=C(X)$ for some $X\subset\Omega\subseteq\mathbb{P}(\mathbb{F}_2)$.

We can generalize this example as follows.

Let $n\in\mathbb{N}\cup\{\infty\}$ and let B be the universal C^*-algebragenerated by partial isometries $\{s_1,\ldots,s_n\}$ subject to some relations. (Notice that there are many such examples in the literature.) Assume that the multiplicative semigroup generated by $\{s_j\}\cup\{s_j^*\}$ consists of partial isometries. Define $\pi\colon\mathbb{F}_n\to B$ by

$$g_j\mapsto s_j \quad g_j^{-1}\mapsto s_j^* \quad 1\mapsto 1$$

for $j=1,2$, and extend to reduced words.

Theorem 5.3. The map π defined above is a partial representation, and hence there is a homomorphism $C^*_{\mathrm{par}}(\mathbb{F}_n)\to B$.

This leads to the very natural question of whether the completely positive expression for $C^*_{\mathrm{par}}(\mathbb{F}_n)$ is inherited by B, this is, whether B is a completely positive by a partial action on a commutative C^*-algebra. The following is a partial answer to this question.

Theorem 5.4. (Exel-Laca-Quigg) Let G be a group and let B be the universal unital C^*-algebra generated by a set $\{\pi(g)\colon g\in G\}$ subject to the relations

(1) $g\mapsto\pi(g)$ is a $*$-partial representation of G,
(2) Relations only involving the projections $e(g)$ for $g\in G$, this is, relations of the form

$$\sum_j\prod_k\lambda_{j,k}e(g_{j,k})=0.$$

Then B is naturally isomorphic to $C(\Omega_R)\rtimes G$, where $\Omega_R\subseteq\Omega$ is a closed invariant subspace for the Bernoulli partial action. Moreover,

$$\Omega_R=\{E\in\Omega\colon f(g^{-1}E)=0 \text{ for all } g\in E \text{ and all functions derived from the relations}\},$$

where we identify relations with continuous functions in such a way that every $e(g)$ corresponds to the Boolean-valued function $[g\in E]$. Hence the relation $\sum_j\prod_k\lambda_{j,k}e(g_{j,k})=0$ turns into $\sum_j\prod_k\lambda_{j,k}[g_{j,k}\in E]=0$.

Example 5.5. One has $\mathcal{O}_2\cong C^*(\pi(g_1),\pi(g_2))$, where $\langle g_1,g_2\rangle=\mathbb{F}_2$ and moreover the relation $s_j^*s_j=1$ becomes $e(g_j^{-1})=1$ for $j=1,2$, and $s_1s_1^*+s_2s_2^*=1$ becomes $e(g_1)+e(g_2)=1$. Thus $\mathcal{O}_2$ satisfies the hypotheses of the Theorem above (and so do all Cuntz-Krieger algebras). Since the relations are $e(g_1^{-1})=1$, $e(g_2^{-1})=1$ and $e(g_1)+e(g_2)=1$, the functions are

$$[g_1^{-1}\in E]=1 \qquad [g_2^{-1}\in E]=1 \qquad \text{and} \qquad [g_1\in E]+[g_2\in E]=1.$$

We proceed to describe all such sets $E\subseteq G$. If an element $g\in G$ belongs to E, then gg_1^{-1} and gg_2^{-1} also belong to E, and exactly one of either gg_1 or gg_2 belongs to E. In particular, since $1\in E$, the set E contains exactly one of either g_1 or g_2. Inductively, one constructs words that will form the set E. The other condition says that whenever $g\in E$, all of the "previous" words also belong to E.

Unfortunately, the resulting space

$$\Omega_R=\{E\subseteq G\colon 1\in E \text{ and for all } g\in E, f(g^{-1}E)=0 \text{ for all functions derived from the relations}\}$$

is not invariant for the global Bernoulli action since the condition $1 \in E$ will not in general be preserved (the other one clearly is). Nevertheless, the space

$$\widetilde{\Omega}_R = \{E \subseteq G\colon \text{ for all } g \in E, f(g^{-1}E) = 0 \text{ for all functions derived from the relations}\}$$

is globally invariant, and the Bernoulli action on it is the globalization of the partial action on Ω_R.

6. Applications to Fell bundles

Recall that given a partial action θ, we can construct a Fell bundle $\mathcal{B} = \{D_g u_g\}_{g\in G}$. This Fell bundle is called the *semidirect product bundle* for θ. One may ask for the following.

Question 6.1. Is every bundle a semidirect product for some partial action?

Fix a Fell bundle $\mathcal{B} = \{B_g\}_{g\in G}$ over a discrete group G. Then B_1 is the candidate for algebra where G will act partially. If we had $B_g = D_g u_g$, then

$$B_g B_{g^{-1}} = B_g (B_g)^* = D_g u_g D_{g^{-1}} u_{g^{-1}}$$

and if $a_g u_g \in B_g$ and $b_{g^{-1}} u_{g^{-1}} \in B_{g^{-1}}$, then

$$(a_g u_g)(b_{g^{-1}} u_{g^{-1}}) = \alpha_g(\alpha_{g^{-1}}(a_g) b_{g^{-1}}) u_1 \in D_g u_1.$$

One easily sees that this is an arbitrary element in $D_g u_1$. Hence $B_g B_{g^{-1}} = D_g u_1$ for all $g \in G$. In particular, one may recover the ideals D_g by setting

$$D_g = \{b \in B_1\colon b u_1 \in B_g B_{g^{-1}}\} \lhd B_1.$$

The obstruction to the Fell bundle coming from a partial action is the fact that D_g may not be isomorphic to $D_{g^{-1}}$. Nevertheless, they always are Morita equivalent via the imprimitivity Hilbert B_1-module B_g.

Example 6.2. The ideals D_g and $D_{g^{-1}}$ may not be isomorphic in general. Indeed, consider $A = M_3(\mathbb{C})$ with the grading

$$B_1 = \begin{pmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ * & 0 & 0 \end{pmatrix} \quad , \quad B_{-1} = \begin{pmatrix} 0 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B_0 = \begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}.$$

Then

$$B_1 B_1^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \cong M_2(\mathbb{C}) \quad \text{and} \quad B_1^* B_1 = \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cong \mathbb{C}$$

are not isomorphic.

If B_1 is separable and stable, then every ideal in B_1 is also separable and stable, and hence $D_g \cong D_{g^{-1}}$ for all $g \in G$ by a result of Brown-Douglas-Rieffel, which moreover provides a specific isomorphism. One must do some work to arrange these isomorphisms to obtain a partial action and hence obtain the following.

Theorem 6.3. (Exel-Senhen 2013) If $\mathcal{B} = \{B_g\}_{g\in G}$ is a Fell bundle over a countable discrete group G such that B_1 is stable and separable, then there exists a partial action of G on B_1 such that $\mathcal{B}$ is its semidirect product bundle.

7. Computation of the K-theory

Let α be a partial action of $\mathbb{Z}$ on a C^*-algebra A. Assume that α is semisaturated in the sense that $\alpha_n = (\alpha_1)^n$ for all $n \in \mathbb{Z}$. Denote by I the ideal D_{-1}.

Theorem 7.1. There is an exact sequence

$$\begin{tikzcd} K_0(I) \arrow[r, "1-(\alpha_1)_*"] & K_0(A) \arrow[r] & K_0(A \rtimes_\alpha Z) \arrow[d] \\ K_1(A \rtimes_\alpha \mathbb{Z}) \arrow[u] & K_1(A) \arrow[l] & K_1(I). \arrow[l, "1-(\alpha_1)_*"] \end{tikzcd}$$

The original proof uses partial actions, but it can also be obtained using F. Abadie's work on Morita equivalent globalizable actions.

8. Ideal structure

Theorem 8.1. (Exel-Laca-Quigg) Let α be a partial action of a discrete countable group G on a compact metric space X. Assume that α is topologically free, this is, that for all $g \in G$ with $g \neq 1$, the interior of the set

$$\{x \in D_{g^{-1}} : \alpha_g(x) = x\}$$

is empty. Then any non-zero ideal $J \lhd C(X) \rtimes_{\alpha,r} G$ satisfies $J \cap C(X) \neq \{0\}$.

It follows that $J \cap C(X) = C_0(U)$ for some open set U in X. Moreover, U is G-invariant in the sense that $\alpha_g(D_{g^{-1}} \cap U) \subseteq U$ for all $g \in G$.

Corollary 8.2. Let α be a partial action of a discrete countable group G on a compact metric space X. Assume that α is topologically free and minimal, this is, that every G-invariant open must be trivial. Then $C(X) \rtimes_{\alpha,r} G$ is simple.

In general, if $U \subseteq X$ is a G-invariant open set, then $C_0(U)$ generates an ideal in $C(X) \rtimes_{\alpha,r} G$ which is isomorphic to $C_0(U) \rtimes_{\alpha,r} G$. Nevertheless, there may be other ideals J in $C(X) \rtimes_{\alpha,r} G$ whose intersection with $C(X)$ is also $C_0(U)$. To rule this out, we must require the partial action to be topologically free when restricted to every invariant closed subset of X. We then get:

Theorem 8.3. Let α be a partial action of a discrete countable group G on a compact metric space X. Assume that α is topologically free when restricted to every G-invariant closed subset. Then the ideals in $C(X) \rtimes_{\alpha,r} G$ are in one-to-one correspondence with the G-invariant subsets U of X.

9. KMS states

Let A be a C^*-algebra and let σ be an action of $\mathbb{R}$ on A, also called a *flow*.

Definition 9.1. An element $a \in A$ is said to be *analytic* if the map $\sigma^a \colon \mathbb{R} \to A$ given by $\sigma^a(t) = \sigma_t(a)$ for all $t \in \mathbb{R}$ extends to an entire function on $\mathbb{C}$.

Proposition 9.2. The analytic elements form a dense subset of A.

Definition 9.3. Let $\beta \in \mathbb{R}$. A state $\varphi \colon A \to \mathbb{C}$ is said to be a *KMS state at inverse temperature* β if

$$\varphi(a\sigma_{i\beta}(b)) = \varphi(ba)$$

for all $a \in A$ and every analytic element b in A.

Theorem 9.4. (Exel-Laca) Let (A, α, G) be a partial dynamical system and let $N \colon G \to (\mathbb{R}_+, \cdot)$ be a group homomorphism. Then there is a flow σ on $A \rtimes_\alpha G$ such that

$$\sigma_t(au_g) = N(g)^{it} au_g$$

for all $t \in \mathbb{R}$, all $a \in A$ and all $g \in G$.

Moreover, one gets β-KMS states of the form

$$\sum_{g \in G} a_g u_g \mapsto \tau(a_e)$$

for a trace $\tau \colon A \to \mathbb{C}$ satisfying $\tau(\alpha_g(a)) = N(g)^{-\beta}\varphi(a)$ for all $g \in G$ and all $a \in D_{g^{-1}}$.

Remark 9.5. In some cases, it can be shown that all β-KMS states have the form described in the Theorem.

Remark 9.6. If α is a global action and A is unital, then the conditions force the flow σ to be trivial, since we must have $N(g) = 1$ for all $g \in G$.