SOFIC GROUPS, ENTROPY AND OPERATOR ALGEBRAS.

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Warning: little proofreading has been done.

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Introduction.

The study of amenability, both for both groups and for operator algebras, revolves around the idea of approximation by internal finite or finite-dimensional structures. The classical theory of Kolmogorov-Sinai entropy is built on such internal finite approximation, which provides a mechanism for averaging in an asymptotic way. By externalizing the approximation, one arrives at the notion of tracial microstates in the context of operator algebras and at the notion of soficity in the context of groups. Soficity was formalized by Gromov and Weiss and is the basis of a far-reaching extension of classical entropy theory that was pioneered a few years ago by Bowen.

1. Motivation: Gottschalk's surjunctivity problem

The notion of a sofic group goes back to the definition of a finite set. (*Sofic* means *finite* in Hebrew.) We should therefore begin by recalling the following.

Definition 1.1. Let X be a set.
(1) We say that X is *finite* if there exist $n \in \mathbb{N}$ and a bijection $X \to \{1, \ldots, n\}$.
(2) We say that X is *Dedekind finite* if every injective map $X \to X$ is automatically surjective.

Remark 1.2. For a set X, finiteness implies Dedekind finiteness, but without the Axiom of Choice, Dedekind finiteness in general does not imply finiteness.

Both notions generalize to different notions in operator algebras and in other areas such as group theory. We turn to one of the main motivations of soficity.

Definition 1.3. Let G be a discrete group and give $\{1, \ldots, k\}^G \cong \prod_{g \in G} \{1, \ldots, k\}$ the product topology and the G-action by shifts. The group G is said to be *surjunctive* if for every $k \in \mathbb{N}$, every G-equivariant injective continuous map $\{1, \ldots, k\}^G \to \{1, \ldots, k\}^G$ is automatically surjective.

The following is known as Gottschalk's surjunctivity problem. It is an equivariant version of Dedeking finiteness.

Question 1.4. Which countable groups are surjunctive?

Entropy is a very useful tool to show surjunctivity for some groups. We point out that there are no known examples of non-surjunctive groups. We will illustrate this fact by showing surjunctivity for $\mathbb{Z}$ using entropy. We need some definitions first.

Definition 1.5. Let $T \colon X \to X$ be a homeomorphism of a compact metric space (X, d). Given $n \in \mathbb{N}$ and $\varepsilon > 0$, we say that a subset $E \subseteq X$ is (n, ε)-*separated* if for every x and y in E with $x \neq y$, there is $k \in \{0, \ldots, n-1\}$ such that

$$d(T^k(x), T^k(y)) \geq \varepsilon.$$

Define the *topological entropy* of T by

$$\mathrm{h_{top}}(T) = \sup_{\varepsilon>0} \limsup_{n\to\infty} \left[\frac{1}{n} \log \left(\max_{E \text{ is } (n,\varepsilon)\text{-separated}} |E| \right) \right].$$

Remark 1.6. The expression $\limsup_{n\to\infty}$ is necessary to get a meaningful invariant. Indeed,

$$\sup_{\varepsilon>0} \left[\log \left(\max_{E \text{ is } (n,\varepsilon)\text{-separated}} |E| \right) \right] = \log |X|,$$

if X is finite, and infinite otherwise.

This is an invariant under conjugation of dynamical systems, but it is rather difficult to compute.

Proposition 1.7. The entropy does not depend on the choice of the metric. In fact, one gets the same value for any continuous pseudometric ρ which is dynamically generating in the sense that for every x, y in X with $x \neq y$, there exists $n \in \mathbb{N}$ such that $\rho(T^n(x).T^n(y)) > 0$.

Proof. Let ρ and ρ' be continuous dynamically generating pseudometrics on X. Let $\varepsilon > 0$. Using compactness of X, choose $n \in \mathbb{N}$ and $\varepsilon' > 0$ such that whenever x and y in X are such that

$$\max_{k=-N+1,\ldots,N} \rho'(T^k(x), T^k(y)) < \varepsilon',$$

then $\rho(x, y) < \varepsilon$. Let $E \subseteq X$ be (ρ, n, ε)-separated. Then E is $(\rho', 2N + n, \varepsilon')$-separated and so

$$\frac{1}{n} \log \left(\max_{E \text{ is } (\rho,n,\varepsilon)\text{-separated}} |E| \right) \leq \frac{2N+n}{n} \cdot \frac{1}{2N+n} \log \left(\max_{E \text{ is } (\rho',2N+n,\varepsilon')\text{-separated}} |E| \right).$$

Taking $\limsup_{n\to\infty}$ and $\sup_{\varepsilon>0}$, one concludes that $\mathrm{h}^{\rho}_{\mathrm{top}}(T) \leq \mathrm{h}^{\rho'}_{\mathrm{top}}(T)$, and by symmetry they are equal. □

Example 1.8. Define a pseudometric ρ on $\{1, \ldots, k\}^G$ by

$$\rho\left((a_g)_{g\in G}, (b_g)_{g\in G}\right) = \begin{cases} 0, & \text{if } a_e = b_e; \\ 1, & \text{if } a_e \neq b_e. \end{cases}$$

One checks that ρ is a dynamically generating pseudometric. One then reduces the computation of the entropy on $\{1, \ldots, k\}^{\mathbb{Z}}$ to a combinatorial problem. The entropy measures the exponential growth of the number of strings over the window $\{0, \ldots, n-1\}$, which is just k^n. Hence $\mathrm{h_{top}}(T) = \log(k)$.

What makes the computation possible is the fact that $\frac{2N+n}{n}$ converges to 1 as n goes to ∞, which is essentially due to the existence of Følner sets in $\mathbb{Z}$. This suggests that the definition can be generalized to all amenable groups.

2. Measurable dynamics

Throughout this section we assume that G is a discrete group acting on a probability measure space (X, μ) preserving the measure μ. The basic examples are:

(1) The group G acting by shifts on $X = \{1, \ldots, k\}^G$ with $\mu = \prod_{g\in G} \nu$, where ν is some probability measure on $\{1, \ldots, k\}$.
(2) Given $\theta \in \mathbb{R}$, let $T_\theta \colon \mathbb{T} \to \mathbb{T}$ denote the rotation by angle θ.

Then there is a dichotomy

multiplicative structure	and	additive structure
(weak mixing)		(compactness)

Compactness can be relaxed to get a certain Rokhlin-type property, and this weakening can in fact coexist with weak mixing. Whenever this is true for the Bernoulli action of G on $\prod_{g\in G} G$, the group is said to be *amenable.*

Definition 2.1. The *Koogman representation* of the measurable system $G \curvearrowright (X, \mu)$ is the unitary representation $\pi\colon G \to \mathcal{U}(L^2(X, \mu))$ given by

$$(\pi_g(f))\,(x) = f(g^{-1} \cdot x)$$

for all $g \in G$, all $f \in L^2(X, \mu)$ and all $x \in X$.

Examples 2.2. Some examples of Koogman representations.

(1) For G acting on $\{1, \ldots, k\}^G$, the Koogman representation is the left regular representation λ with multiplicity equal to $|G|$ (which may be infinity), direct sum with one copy of the trivial representation. Notice that one cannot recover k from the Koogman representation.
(2) For $T_\theta\colon \mathbb{T} \to \mathbb{T}$, the Hilbert space $L^2(\mathbb{T})$ has a complete set of eigenvectors $\zeta \mapsto \zeta^n$, with eigenvalues $e^{2\pi i n\theta}$, for $n \in \mathbb{Z}$.

Definition 2.3. We say that the action $G \curvearrowright (X, \mu)$ is *ergodic* if $\mu(A) = 0$ or $\mu(A) = 1$ for any G-invariant measurable subset of X.

Proposition 2.4. Let G act on (X, μ). Then the following are equivalent:

(1) The action is ergodic.
(2) For every pair of subsets A and B of X with positive measure, there exists $g \in G$ such that $\mu(g \cdot A \cap B) > 0$.
(3) There are no non-trivial G-invariant vectors in $L^2(X, \mu)$. (The trivial ones are the constant functions.)

Definition 2.5. The action $G \curvearrowright (X, \mu)$ is said to be *essentially free* if there exists a G-invariant measurable set $X_0 \subseteq X$ with $\mu(X_0) = 1$ and such that the restriction of G to X_0 is free in the ordinary sense.

In the following theorem, we do not assume that the action is either ergodic nor essentially free. The result follows follows using infiniteness of G together with $\mu(X) < \infty$.

Theorem 2.6. (Poincaré recurrence) If G is infinite and A is a subset of X with positive measure, then for almost every $x \in A$ the set

$$\{g \in G \colon g \cdot x \in A\}$$

is infinite.

Question 2.7. The above result raises some natural questions.

(1) How frequently and with what degree of overlap does recurrence occur asymptotically across orbits of sets? The answer to this question leads to *asymptotic properties*: weak mixing and entropy (involving the multiplicative structure).
(2) Is recurrence part of a more complete picture of the dynamics at a certain scale? The answer to this question leads to the notions of amenability and socificty.

Given a unitary representation $\pi \colon G \to \mathcal{U}(\mathcal{H})$, we denote by $\overline{\pi} \colon G \to \mathcal{U}(\overline{\mathcal{H}})$ the conjugate representation of π on the Hilbert space $\overline{\mathcal{H}}$ conjugate of $\mathcal{H}$.

Weak mixing, defined below, is arguably the most important notion in measurable dynamics.

Definition 2.8. Let $\pi \colon G \to \mathcal{U}(\mathcal{H})$ be a representation.

(1) We say that π is *ergodic* if there are no non-zero G-invariant vectors in $\mathcal{H}$.
(2) We say that π is *weak mixing* if $\pi \otimes \overline{\pi}$ is ergodic.
(3) We say that π is *mixing* if $\langle \pi(g)\xi, \zeta \rangle \to 0$ as $g \to \infty$ for all ξ and ζ in $\mathcal{H}$.

Denote the set of all Hilbert-Schmidt operators on $\mathcal{H}$ by

$$HS(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) \colon Tr(T^*T) < \infty\}.$$

Remark 2.9. The representation $\pi \otimes \overline{\pi}$ is unitarily equivalent to conjugation $g \mapsto \pi(g)T\pi(g)^*$ by π on $HS(\mathcal{H})$.

Remark 2.10. Mixing implies weak mixing, which in turn implies ergodicity.

Denote the C^*-subalgebra of $\ell^\infty(G)$ consisting of all *weakly almost periodic* functions on G by

$$\mathrm{WAP}(G) = \{f \in \ell^\infty(G) \colon \text{ the } G-\text{orbit of } f \text{ has compact closure in the weak operator topology in } \ell^\infty(G)\} \subseteq \ell^\infty(G).$$

Definition 2.11. Let S be a subset of G.

(1) We say that S is *syndetic* if there exist $n \in \mathbb{N}$ and group elements $g_1, \ldots, g_n$ in G such that $\bigcup_{j=1}^n g_j S = G$.
(2) We say that S is *thickly syndetic* if for all $n \in \mathbb{N}$ and all group elements $g_1, \ldots, g_n$ in G, the set $\bigcap_{j=1}^n g_j S$ is syndetic.

For example, $2\mathbb{Z}$ is syndetic in $\mathbb{Z}$ since $2\mathbb{Z} \cup (2\mathbb{Z} + 1) = \mathbb{Z}$. (The same is true for $n\mathbb{Z}$ for any $n \in \mathbb{Z}$.) On the other hand, it is not thickly syndetic since $2\mathbb{Z} \cap (2\mathbb{Z} + 1) = \emptyset$.

Definition 2.12. A vector $\xi \in \mathcal{H}$ is said to be *compact* if the norm closure of its G-orbit in $\mathcal{H}$ is compact.

Theorem 2.13. Let $\pi \colon G \to \mathcal{U}(\mathcal{H})$ be a representation. The following are equivalent

(1) The representation π is weak mixing.
(2) If m denotes the unique G-invariant state (mean) on $\mathrm{WAP}(G)$, then $m\left(|f_{\xi,\zeta}|\right) = 0$ for all ξ and ζ in $\mathcal{H}$. (When G is amenable, the von Neumann algebra $\ell^\infty(G)$ has many G-invariant states, and they all restrict to the unique G-invariant state on $\mathrm{WAP}(G)$.)
(3) For every finite subset Ω of $\mathcal{H}$ and for every $\varepsilon > 0$, the set
$$\{g \in G \colon \, |\langle \pi(g)\xi, \zeta \rangle| < \varepsilon \text{ for all } \xi, \zeta \in \Omega\}$$
is thickly syndetic.
(4) There exists a sequence $(s_n)_{n\in\mathbb{N}}$ in G such that
$$\lim_{n\to\infty} \langle \pi(s_n)\xi, \zeta \rangle = 0$$
for all ξ and ζ in $\mathcal{H}$.
(5) There does not exist a non-zero compact vector ξ in $\mathcal{H}$.

(6) The representation π has no finite dimensional subrepresentations.

These statements translate to statements about measurable dynamics via the Koogman representation. For example, (4) becomes asymptotic independece: the sequence $(s_n)_{n\in\mathbb{N}}$ satisfies

$$\lim_{n\to\infty} \mu(S_n \cdot A \cap B) = \mu(A)\mu(B)$$

for all subsets A and B of X.

Definition 2.14. A representation $\pi\colon G \to \mathcal{U}(\mathcal{H})$ is said to be *compact* if every vector in $\mathcal{H}$ is compact.

Proposition 2.15. Every representation $\pi\colon G \to \mathcal{U}(\mathcal{H})$ can be decomposed as a direct sum $\pi \cong \pi_{wm} \oplus \pi_{cpt}$ of a weak mixing representation π_{wm} and a compact representation π_{cpt}.

Corollary 2.16. A probability measure preserving action $G \curvearrowright (X, \mu)$ is either weak mixing or has a nontrivial compact factor.

This yields a structure theorem for ergodic actions.

Theorem 2.17. (Furstenberg-Zimmer) If $G \curvearrowright (X, \mu)$ is ergodic, then there is a countable ordinal α and extensions

$$X \longrightarrow Y_\alpha \longrightarrow \cdots \longrightarrow Y_n \longrightarrow Y_{n-1} \longrightarrow \cdot \longrightarrow Y_0 = *$$

such that $X \to Y_\alpha$ has a relative version of weak mixing, and such that the extensions $Y_n \to Y_{n-1}$ have a relative version of compactness.

3. Amenability

Having no non-trivial finite dimensional subrepresentations is rather weak: there are groups whose only finite dimensional representations are sums of the trivial one. We are lead to the weaker notion of having almost invariant finite dimensional subspaces.

Definition 3.1. Let $F \subseteq G$ be a finite subset and let $\delta > 0$. We say that a finite dimensional subspace $V \subseteq \mathcal{H}$ is *almost invariant with respect to* (F, δ) if the orthogonal projection of $\mathcal{H}$ onto V satisfies

$$\|\pi(g)P\pi(g)^* - P\|_2 \leq \delta\|P\|_2,$$

where $\|\cdot\|_2$ denotes the Hilbert-Schmidt norm on $\mathcal{B}(\mathcal{H})$. (Note that the right-hand sice is ∞ whenever V is infinite dimensional.)

The representation π is said to have *almost invariant finite dimensional subspaces* if for every pair (F, δ) as above, there exists a finite dimensional subspace V of $\mathcal{H}$ which is almost invariant with respect to (F, δ).

As we will se below, this notion is very closely related to amenability of G, which we proceed to define now.

Definition 3.2. A discrete group G is said to be amenable is there exists a G-invariant positive lineal functional σ on $\ell^I(G)$, this is, such that $\sigma(g \cdot f) = \sigma(f)$ for all $g \in G$ and all $f \in \ell^I(G)$, where $g \cdot f(h) = f(g^{-1}h)$ for $h \in G$.

Examples 3.3. Finite groups, the integers $\mathbb{Z}$, abelian groups and compact groups are examples of amenable groups.

Theorem 3.4. Let G be a discrete group. The following are equivalent:

(1) The group G is amenable.
(2) There exist a G-invariant additive measure μ on G with $\mu(G) = 1$.
(3) The group G has a Følner sequence $(F_n)_{n\in\mathbb{N}}$, this is, each F_n is finite and

$$\lim_{n\to\infty} \frac{|gF_n \triangle F_n|}{|F_n|} = 0$$

for all $g \in G$.

The following result relates the notion of having almost invariant finite dimensional subspaces to amenability and property (T) of the group.

Theorem 3.5. Let G be a discrete group.

(1) Every weak mixing representation of G has almost invariant finite dimensional subspaces if and only if G is amenable.
(2) No weak mixing representation of G has almost invariant finite dimensional subspaces if and only if G is property (T).

Remark 3.6. Finite groups are both amenable and property (T), since they have no weak mixing representations.

4. Entropy for amenable group actions and the surjunctivity conjecture

Definition 4.1. Let G be an amenable discrete group acting on a space X. Choose a Følner sequence $(F_n)_{n\in\mathbb{N}}$ for G. Let ρ be a continuous dynamically generating pseudometric on X. Define the *topological entropy* of $G \curvearrowright X$ by

$$\mathrm{h_{top}}(G,X) = \sup_{\varepsilon>0} \limsup_{n\to\infty} \left[\frac{1}{|F_n|} \log \left(\max_{E \text{ is } (n,\varepsilon)\text{-separated}} |E| \right) \right].$$

As in the case of integer actions, the definition of entropy does not depend on the pseudometric ρ. It also does not depend on the choice of the Følner sequence $(F_n)_{n\in\mathbb{N}}$ for G.

A combinatorial argument similar to the one used to compute the entropy in the case of integer actions shows the following.

Example 4.2. Let G act on $X = \{1,\ldots,k\}^G$. Then $\mathrm{h_{top}}(G,X) = \log(k)$.

We return to the surjunctivity conjecture.

Theorem 4.3. Let G be a discrete amenable group. Then G is surjunctive.

Proof. Let $k \in \mathbb{N}$. Consider first the case $G = \mathbb{Z}$.

Let $f\colon \{1,\ldots,k\}^{\mathbb{Z}} \to \{1,\ldots,k\}^{\mathbb{Z}}$ be an equivariant embedding. It follows that the image system as the same entropy, since this quantity is invariant under conjugation. We show that g is onto by showing that any proper subsystem of $\{1,\ldots,k\}^G$ must have strictly smaller entropy.

Let $X \subseteq \{1,\ldots,k\}^{\mathbb{Z}}$ be a proper closed $\mathbb{Z}$-invariant set. Let $T\colon X \to X$ be the restricted system. Since X is proper, there exist a positive integer $N \in \mathbb{N}$ and a string of length N that is excluded from among the elements of X. Given $n \in \mathbb{N}$, that we think of as being much bigger than N, we may write the interval $[0, n-1]$ as the disjoint union

$$[0, N-1) \cup [N-1, 2N-1) \cup \cdots \cup [\left\lfloor \frac{n}{N} \right\rfloor (N-1) - 1, \left\lfloor \frac{n}{N} \right\rfloor N - 1) \cup [\left\lfloor \frac{n}{N} \right\rfloor N - 1), n-1].$$

For $\varepsilon \in (0,1)$, a combinatorial argument then yields the estimate

$$\max_{E \text{ is } (n,\varepsilon)\text{-separated}} |E| \leq (k^N - 1)^{\left\lfloor \frac{n}{N} \right\rfloor} k^{n - N \left\lfloor \frac{n}{N} \right\rfloor}.$$

Hence

$$\frac{1}{n} \log \left(\max_{E \text{ is } (n,\varepsilon)\text{-separated}} |E| \right) < \log(k)$$

and so is its $\limsup$ as $n \to \infty$. This implies that $\mathrm{h_{top}}(T) < \log(k)$, and thus surjunctivity of $\mathbb{Z}$ follows by the argument above.

Now suppose that G is amenable and let $G \curvearrowright X$ be a proper subshift. It follows that there is an excluded string over a finite subset K of G. Using amenability, for all $\delta > 0$ one can find a finite subset F of G such that

$$\frac{|gF \triangle F|}{|F|} \leq \delta$$

for all g in K. Equivalently: there exists a non-empty finite subset F of G such that

$$\left| \bigcap_{g\in K} gF \right| \geq (1-\delta)|F|.$$

If $s \in \bigcap_{g\in K} gF$, then $g^{-1}s \in F$ for all $g \in K$, and hence (assuming that $K = K^{-1}$, which we may) it follows that $Ks \subseteq F$. Set $F_0 = \bigcap g \in KgF$ and note that $tK \subseteq F$ for all $t \in F_0$. Take a maximal invariant subset $E \subseteq F_0$ such that $sK \cap tK = \emptyset$ for all s and t in E distinct. (This can fail for at most $|K \cdot K^{-1}|$ pairs (s,t).) Hence

$$|E| \geq \frac{|F_0|}{|K \cdot K^{-1}|} \geq \frac{|F_0|}{|K|^2} \geq \frac{(1-\delta)|F|}{|K|^2}.$$

This is a lower bound for the number of tiles and it is proportionally independent of $|F|$. With an argument similar to the one used in the case of $G = \mathbb{Z}$, one shows that the entropy of the system $G \curvearrowright X$ is strictly smallar than $\log(k)$. This shows that G is surjunctive. $\square$

5. Sofic groups

The proof of Theorem 4.3 does not use the full strength of amenability, and it can therefore be pushed beyond the amenable case. This is the motivation for the definition of sofic groups, in which Følner sets are replaced by approximately equivariant maps from abstract finite G-sets into X.

Given $d \in \mathbb{N}$, denote by S_d the group of symmetries of the set $\{1,\ldots,d\}$.

Definition 5.1. A discrete group G is said to be *sofic* if there are a sequence of positive integers $(d_n)_{n\in\mathbb{N}}$ and maps $\sigma_n\colon G \to S_{d_n}$ such that

(1) the sequence $(\sigma_n)_{n\in\mathbb{N}}$ is an *asymptotic group homomorphism*, this is,

$$\lim_{n\to\infty}\frac{1}{d_n}\left|\{v\in\{1,\dots,d_n\}\colon \sigma_n(s)\sigma_n(t)v=\sigma_n(st)v \text{ for all } s,t\in G\}\right|=1.$$

(2) the sequence $(\sigma_n)_{n\in\mathbb{N}}$ is *asymptotically free*, this is,

$$\lim_{n\to\infty}\frac{1}{d_n}\left|\{v\in\{1,\dots,d_n\}\colon \sigma_n(s)v\neq\sigma_n(t)v \text{ whenever } s,t\in G, s\neq t\}\right|=1.$$

Philosophically, when working with sofic groups, one replaces averaging over a Følner sets F_n with averaging over the sets $\{1,\dots,d_n\}$. In some sense, soficity is a local property: it is determined by the behavior of the product in G on finite subsets.

The following question remains open.

Question 5.2. Is there a non-sofic group?

Examples 5.3. The following are examples of sofic groups.

(1) Amenable groups; use Følner sets as $\{1,\dots,d_n\}$.
(2) Residually finite groups (and in particular free groups, which are not amenable except for $\mathbb{Z}$); use finite quotients as $\{1,\dots,d_n\}$. The first condition in the definition of sofic group will hold exactly at every finite stage.
(3) Gromov's example of a non-exact group (if correct!), is likely to be sofic. The relationship between soficity and exactness is unclear.

We turn to the definition of entropy for sofic group actions. As in the case of amenable groups, entropy is crucial in showing that sofic groups are surjunctive.

Definition 5.4. Let G be a sofic group acting on a space X. Let ρ be a continuous dynamically generating pseudometric on X. For $d\in\mathbb{N}$, define on the set of all maps $\{1,\dots,d\}\to X$ the 2-pseudometric

$$\rho_2(\varphi,\psi)=\left[\frac{1}{d}\sum_{v=1}^{d}\rho(\varphi(d),\psi(d))^2\right]^{1/2}.$$

Let F be a finite subset of G and let $\delta>0$. Given a map $\sigma\colon G\to S_d$, denote by $Map(\rho,F,\delta,\sigma)$ the set of all maps $\varphi\colon\{1,\dots,d\}\to X$ such that $\rho_2(\varphi\circ\sigma_g,g\cdot\varphi)<\delta$ for all $g\in F$. Choose a sofic approximation $(\sigma_n\colon G\to S_{d_n})_{n\in\mathbb{N}}$ of G. Define the *topological entropy* of the system (G,X) by

$$\mathrm{h_{top}}(G,X)=\sup_{\varepsilon>0}\inf_{F,\delta>0}\limsup_{n\to\infty}\left[\frac{1}{|d_n|}\log\left(\max_{E \text{ is } (n,\varepsilon)\text{-separated in } Map(\rho,F,\delta,\sigma_n)}|E|\right)\right].$$

The definition does not depend on the choice of ρ or on the choice of the sofic approximation $(\sigma_n\colon G\to S_{d_n})_{n\in\mathbb{N}}$ of G.

Remark 5.5. The term $\sup_{\varepsilon>0}$ tells us how well we can distinguish maps $\{1,\dots,d\}\to X$, and the term $\inf_{F,\delta>0}$ tells us how well these maps model the dynamics.

We now turn to the computation of the entropy of the shift $G\curvearrowright\{1,\dots,k\}^G$. Again, consider the continuous dynamically generating pseudometric ρ on $\{1,\dots,k\}^G$ given by

$$\rho\left((a_g)_{g\in G},(b_g)_{g\in G}\right)=\begin{cases}0, & \text{if } a_e=b_e;\\ 1, & \text{if } a_e\neq b_e.\end{cases}$$

Choose a sofic approximation $(\sigma_n\colon G\to S_{d_n})_{n\in\mathbb{N}}$ of G. Let F be a finite subset of G and let $\delta>0$. For each $w\in\{0,1\}^{d_n}$, choose a map $\varphi_w\colon\{1,\dots,d_n\}\to\{1,\dots,k\}^G$ such that for all $g\in F$ and all $v\in\{1,\dots,d_n\}$, we have

$$\varphi_w(v)_{g^{-1}}=w(\sigma(g)v).$$

The map φ_w will automatically be equivariant on the set F. The choice of φ_w is possible because we are looking at the full subshift and hence there is complete freedom in the choice of the strings. The outcome of the calculation is again $\log(2)$ (and $\log(k)$ in general).

For a proper subsystem, one uses an argument similar to the one used in the amenable case, replacing the Følner sets F_n by the abstract finite set $\{1,\dots,d_n\}$, to conclude that the entropy is strictly smaller than $\log(k)$. This shows that

Theorem 5.6. Let G be a discrete sofic group. Then G is surjunctive.

Beyond the surjunctivity problem, the main examples of applications of entropy for sofic groups are to algebraic actions $G\curvearrowright\widehat{\frac{\mathbb{Z}G}{\mathbb{Z}Gf}}$.