C^*-ALGEBRAS (MATH 684) COURSE NOTES.

Course given in the Winter term of 2013 by Prof. Chris Phillips.
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Abstract. These are lecture notes from a course given by Chris Phillips at the University of Oregon. The main purpose of writing these notes is to have some kind of study guide for my Oral Examination, so in particular:

- Little proof-reading was done.
- Some proofs will be roughly outlined, and some will be just skipped.

There's some material in these notes that wasn't covered in class, and there's some material that was covered in class but is not here (mainly proofs of various results). Some sections are presented differently here than they were in class.

Contents

1. Basics of C^*-algebras.

Definition 1.1. A *Banach $*$-algebra* A, is a Banach algebra with a conjugate linear (continuous) involution $*\colon A \to A$. If $a \in A$, then a^* is usually called the *adjoint* of a. If the norm on A also satisfies $\|a^*a\| = \|a\|^2$, then A is called a *C^*-algebra*.

Notice that since $\|x\|^2 = \|x^*x\| \leq \|x\|\|x^*\|$, we get $\|x\| \leq \|x^*\|$ whenever $x \neq 0$. Since $x^{**} = x$, we get $\|x^*\| = \|x\|$, for all $x \in A$.

Examples 1.2. Some C^*-algebras.

1. If $\mathcal{H}$ is a Hilbert space, then $\mathcal{B}(\mathcal{H})$ is a C^*-algebra, with the adjoint of T being characterized by $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x, y \in \mathcal{H}$.
2. More generally, any closed $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ is naturally a C^*-algebra.
3. If $\mathcal{H}$ has finite dimension n, then $\mathcal{B}(\mathcal{H}) \cong M_n(\mathbb{C})$. Hence $M_n(\mathbb{C})$ is a *finite dimensional simple* $C*$-algebra.
4. More generally, $\oplus_{j=1}^m M_{n_j}(\mathbb{C})$ is a *finite dimensional* C^*-algebra.
5. If X is a locally compact and Hausdorff space, then $C_0(X)$ is a *commutative* C^*-algebra. Notice that $C_0(X)$ is unital if and only if X is compact, in which case $C_0(X) = C(X)$.

In the course of the following sections, we will prove that:

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- Every C^*-algebra is (isomorphic to) a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Hence, every C^*-algebra is as in (2).
- Every simple, finite-dimensional C^*-algebra is (isomorphic to) $M_n(\mathbb{C})$ for $n = \sqrt{\dim_{\mathbb{C}} A}$. Hence, every simple, finite-dimensional C^*-algebra is as in (3).
- Every finite-dimensional C^*-algebra is (isomorphic to) a sum of matrices with coefficients in $\mathbb{C}$. Hence, every finite-dimensional C^*-algebra is as in (4).
- Every commutative C^*-algebra is isomorphic to $C_0(X)$ for some locally compact Hausdorff space X. Hence, every commutative C^*-algebra is as in (5).

Examples 1.3. The following Banach $*$-algebras are *not* C^*-algebras.

(1) $A(\mathbb{D})$ with $f^*(z) = \overline{f(\overline{z})}$. In this case, $f(z) = z$ is self-adjoint, but $\mathrm{sp}(f) = \mathbb{D}$ (the spectrum of a self-adjoint element in a C^*-algebra is contained in $\mathbb{R}$: see Proposition 3.6).

(2) $L^1(G)$ for a locally compact group G, with convolution and $f^*(g) = \Delta(g)^{-1}\overline{f(g^{-1})}$.

Definition 1.4. Let A be a C^*-algebra. An element $a \in A$ is called *normal* if $a^*a = aa^*$, and it is self-adjoint if $a = a^*$. We denote the set of all self-adjoint elements of A by A_{sa}.

For each $a \in A$, the elements $\frac{1}{2}(a + a^*)$ and $-\frac{i}{2}(a - a^*)$ are self-adjoint, and are called the *real and imaginary parts* of a. It follows that A_{sa} is a closed real subspace of A and each element $a \in A$ has a unique decomposition as $a = b + ic$, with $b, c \in A_{sa}$. In particular, the self-adjoint elements of A linearly span A.

Definition 1.5. An element $p \in A$ is called a *projection* if $p^*p = p$. We denote the set of all projections on A by $\mathcal{P}(A)$. If A is unital, an element $u \in A$ is called *unitary* if $u^*u = uu^* = 1$. We denote the set of all unitaries in A by $\mathcal{U}(A)$.

2. Unitizations of C^*-algebras.

There are many ways to construct a unital C^*-algebra that contains a given C^*-algebra as an essential ideal. We present first the "maximal" construction.

Definition 2.1. Let A be a Banach algebra. A *double centralizer* on A is a pair $(L, R) \in \mathcal{B}(A) \oplus \mathcal{B}(A)$ such that for all a, b in A, it is true that

(1) $L(ab) = L(a)b$
(2) $R(ab) = aR(b)$
(3) $R(a)b = aL(b)$.

Examples 2.2. Some centralizers.

(1) Given $x \in A$, consider $L_x(a) = xa$ and $R_x(a) = ax$ for all $a \in A$.
(2) Consider $(L, R) = (\mathrm{id}_A, \mathrm{id}_A)$. If A has a unit, $\mathrm{id}_A = L_1 = R_1$.

We denote by $\mathcal{M}(A)$ the set of all double centralizers on A.

Lemma 2.3. With the product $(L_1, R_1) \cdot (L_2, R_2) = (L_1L_2, R_2R_1)$, it follows that $\mathcal{M}(A)$ is a closed subalgebra of $\mathcal{B}(A) \oplus \mathcal{B}(A)$, with the maximum norm on the direct sum.

Lemma 2.4. Let A be a Banach $*$-algebra. Define an involution on $\mathcal{B}(A)$ by $T^*(a) = T(a^*)^*$ for $a \in A$. Then the map $(L, R) \mapsto (R^*, L^*)$ is an involution on $\mathcal{M}(A)$.

Notice that for $x \in A$, we have $L_x^* = R_{x^*}$ and $R_x^* = L_{x^*}$.

Remark 2.5. If A is a C^*-algebra, then for all $a \in A$ we have

$$\|a\| = \sup_{\|x\| \leq 1} \|ax\| = \sup_{\|x\| \leq 1} \|xa\|.$$

Proof. In any Banach algebra it is true that $\|a\| \geq \sup_{\|x\| \leq 1} \|ax\|$ and $\|a\| \geq \sup_{\|x\| \leq 1} \|xa\|$. If $a = 0$, this is immediate, and if $a \neq 0$ take $x = \frac{1}{\|a\|}a^*$. □

Something stronger than a Banach algebra is needed: if X is any Banach space, define $ab = 0$ for all $a, b \in X$. Then X is a Banach algebra and the above remark is false in this case.

Lemma 2.6. Let A be a C^*-algebra and let $(L, R) \in \mathcal{M}(A)$. Then $\|L\| = \|R\|$. Also, $\|L_x\| = \|R_x\| = \|x\|$ for all $x \in A$.

Proof. Let $a \in A$. Then

$$\|L(a)\| = \sup_{\|x\| \leq 1} \|xL(a)\| = \sup_{\|x\| \leq 1} \|R(x)a\| \leq \sup_{\|x\| \leq 1} \|R\|\|x\|\|a\| = \|R\|\|a\|,$$

so $\|L\| \leq \|R\|$, and likewise $\|R\| \leq \|L\|$. The second statement follows from Remark 2.5 above. □

Definition 2.7. If $z = (L, R) \in \mathcal{M}(A)$, we define its norm by $\|z\| = \|L\| (= \|R\|)$.

Lemma 2.8. Let A be a Banach algebra with involution such that $\|x^2\| \geq \|x\|^2$ for all $x \in A$. Then A is a C^*-algebra.

Proof. We only need to show that $\|x\| = \|x^*\|$, and it is enough to show that $\|x\| \geq \|x^*\|$. Assume $x \neq 0$, then $\|x\|^2 \leq \|x^*x\| \leq \|x^*\|\|x\|$, and the claim follows. $\square$

Theorem 2.9. Let A be a C^*-algebra. Then $\mathcal{M}(A)$ is a unital C^*-algebra.

Proof. That $\mathcal{M}(A)$ is unital is clear: $(\mathrm{id}_A, \mathrm{id}_A) \in \mathcal{M}(A)$ is its unit. For the first claim, we need to verify the C^*-condition, so let $z \in \mathcal{M}(A)$. Write $z = (L, R)$, so that $z^*z = (R^*L, RL^*)$. Let $a \in A$ with $\|a\| \leq 1$. Then

$$\|L(a)\|^2 = \|L(a)^*L(a)\| = L^*(a^*)L(a)\| = \|R(L^*(a^*))a\| \leq \|RL^*\|\|a^*\|\|a\| \leq \|RL^*\| = \|z^*z\|.$$

Since $\|z\| = \|L\|$, it follows that $\|z\|^2 \leq \|z^*z\|$ and by Lemma 2.8 above, the result follows. $\square$

Remark 2.10. The map $A \to \mathcal{M}(A)$ given by $a \mapsto (L_a, R_a)$ is an isometric $*$-homomorphism, and its range is a closed two-sided ideal.

Unitization of a C^*-algebra. Let A be a C^*-algebra. Set $A^+ = A \oplus \mathbb{C} = \{(a, \lambda) \colon a \in A, \lambda \in \mathbb{C}\}$. Define addition and scalar multiplication coordinate-wise on A^+, and define multiplication and involution by

$$(a, \lambda) \cdot (b, \mu) = (ab + \mu \cdot a + \lambda \cdot b, \lambda\mu) \qquad (a, \lambda)^* = (a^*, \overline{\lambda}).$$

Then A^+ is a unital $*$-algebra, with unit $(0, 1)$. Via the injective homomorphism $\iota \colon A \to A^+$ given by $\iota(a) = (a, 0)$, we regard A as sitting inside of A^+. We need to define a norm on A^+ making it a C^*-algebra.

Corollary 2.11. Let A be a C^*-algebra. Then there is a unique norm on A^+ which makes it a C^*-algebra.

Proof. Uniqueness was done before (on any $*$-algebra, there is at most one norm making it a C^*-algebra). For existence, we divide the discussion into two cases. If A is unital, then $A^+ \cong A \oplus \mathbb{C}$ as $*$-algebras, the unit of $\mathbb{C}$ being $1_{A^+} - 1_A$. In this case, $A \oplus \mathbb{C}$ has a C^*-norm. If A is non-unital, then in $\mathcal{M}(A)$ we have $\mathbb{C} \cdot 1 \cap A = \{0\}$, so $A + \mathbb{C} \cdot 1 \subseteq \mathcal{M}(A)$ is a subalgebra isomorphic to A^+. Since it is closed (because A is closed and $\mathbb{C}$ is finite dimensional), it inherits a norm from $\mathcal{M}(A)$. $\square$

We can also unitize maps between C^*-algebras. The proof of the following lemma is straightforward.

Lemma 2.12. Let A, B be C^*-algebras, let $\varphi \colon A \to B$ be a homomorphism. Then there is a unique unital homomorphism $\varphi^+ \colon A^+ \to B^+$ extending φ. Moreover, if B is unital, then there is a unique extension of φ to a homomorphism $A^+ \to B$.

3. Some spectral theory for C^*-algebras.

Recall that given a unital Banach algebra A and $a \in A$, the spectrum of a in A is defined to be $\mathrm{sp}(a) = \{\lambda \in \mathbb{C} \colon \lambda - a \text{ is not invertible}\}$.

Definition 3.1. For a non-unital C^*-algebra A, the spectrum of $a \in A$, denoted by $\mathrm{sp}(a)$, is the spectrum of a as an element of A^+.

Lemma 3.2. If A is a C^*-algebra and $a \in A$ is a normal element, then $\|a\| = r(a)$.

Proof. Notice that if $a \in A_{sa}$, then $\|a^2\| = \|a\|^2$, so $r(a) = \lim \left\|a^{2^n}\right\|^{\frac{1}{2^n}} = \|a\|$. For a general normal element a, we get

$$\left\|a^{2^n}\right\|^2 = \left\|(a^*)^{2^n} a^{2^n}\right\| = \left\|(a^*a)^{2^n}\right\| = \|a\|^{2^{n+1}}.$$

Notice that we used that a is normal in the second equality. Finally, $\|a\| = \left\|a^{2^n}\right\|^{\frac{1}{2^n}} \to r(a)$, so $\|a\| = r(a)$. $\square$

Corollary 3.3. Let $\varphi \colon A \to B$ be a unital homomorphism of unital C^*-algebras. Then $\|\varphi\| \leq 1$ and $\|\varphi\| = 1$ if φ is bijective.

Proof. Since $\mathrm{sp}(\varphi(a)) \subseteq \mathrm{sp}(a)$, we get that

$$\|\varphi(a)\|^2 = \|\varphi(a^*a)\| = r(a^*a) \leq r(\varphi(a^*a)) = \|\varphi(a^*a)\| = \|\varphi(a)\|^2.$$

Finally, if φ is bijective, then $\mathrm{sp}(b) = \mathrm{sp}(\varphi(b))$ and we get equality. $\square$

Later in this chapter we will prove a stronger result: any homomorphism between C^*-algebras is norm-decreasing, and it is an isometry if and only if it is injective.

Corollary 3.4. There exists at most one norm on a $*$-algebra making it a C^*-algebra.

Proof. In a C^*-algebra A, for every $a \in A$ we have $\|a\|^2 = \|a^*a\| = r(a^*a)$, and the spectral radius is independent of the norm. So there is at most one option, given by $\|a\| = \sqrt{r(a^*a)}$. $\square$

The proof of the following proposition also follows from the Gelfand Theorem (and more easily). To prove it now we will need a technical lemma.

Lemma 3.5. If A is a unital Banach algebra, $a \in \mathrm{Inv}(A)$ and $\lambda \neq 0$, then $\lambda \in \mathrm{sp}(a)$ if and only if $\lambda^{-1} \in \mathrm{sp}(a^{-1})$.

Proof. It is enough to prove one direction, so assume that $\lambda \in \mathrm{sp}(a)$. Then $\lambda a(\lambda^{-1} - a^{-1}) = a - \lambda$ and hence

$$\lambda^{-1} - a^{-1} = \lambda^{-1} a^{-1} (a - \lambda)^{-1},$$

so $\lambda^{-1} - a^{-1}$ is a product of invertible elements, so it is invertible itself. Therefore, $\lambda^{-1} \in \mathrm{sp}(a^{-1})$. □

Proposition 3.6. Let A be a C^*-algebra and $a \in A_{sa}$. Then $\mathrm{sp}(a) \subseteq \mathbb{R}$. If moreover A is unital and $u \in \mathcal{U}(A)$, then $\mathrm{sp}(u) \subseteq S^1$.

Proof. We shall assume that A is unital (or take its unitization). If $u \in \mathcal{U}(A)$, then $\|u\|^2 = \|u^*u\| = 1$, so $\|u\| \leq 1$. Let $\lambda \in \mathrm{sp}(u)$. Then $\lambda^{-1} \in \mathrm{sp}(u^{-1}) = \mathrm{sp}(u^*)$, and since u^* is also unitary, we get that $|\lambda|, |\lambda^{-1}| \leq 1$. Hence $|\lambda| = 1$ and $\mathrm{sp}(u) \subseteq S^1$.

If $a \in A_{sa}$, the function $e^{iz} = \sum_{n=0}^{\infty} \frac{(iz)^n}{n!}$ is entire. By applying holomorphic functional calculus we see that $u = e^{ia}$ is unitary, with $u^* = e^{-ia}$. Now, if $\lambda \in \mathrm{sp}(a)$ and $b = \sum_{n=1}^{\infty} \frac{i^n (a-\lambda)^{n-1}}{n!}$, then

$$e^{ia} - e^{i\lambda} = e^{i(a-\lambda)-1} e^{i\lambda} = (a - \lambda) b e^{i\lambda}.$$

Since b commutes with a and $a - \lambda$ is not invertible, hence $e^{ia} - e^{i\lambda}$ is not invertible either. Thus $e^{i\lambda} \in \mathrm{sp}(e^{ia}) \subseteq S^1$, and $\lambda \in \mathbb{R}$. □

Example 3.7. The converse of the above proposition is not true. Indeed, $a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in M_2(\mathbb{C})$ is:

- not unitary, although $\mathrm{sp}(a) = \{1\} \subseteq S^1$.
- not self-adjoint, although $\mathrm{sp}(a) = \{1\} \subseteq \mathbb{R}$.

However, if we assume that a is normal, then a is self-adjoint if and only if $\mathrm{sp}(a) \subseteq \mathbb{R}$ and a is unitary if and only if $\mathrm{sp}(a) \subseteq S^1$. This follows from the Gelfand theorem.

4. Commutative C^*-algebras and the Gelfand transform.

The goal of this section is to prove that if A is a commutative C^*-algebra, then there exists a locally compact Hausdorff space X such that $A \cong C_0(X)$. If we start with an isomorphism $A \cong C_0(X)$, we can recover X from A by considering the set of maximal ideals of A, because each maximal ideal M has the form $M = \{f \in C_0(X) \colon f(x_0) = 0\}$ for some $x_0 \in X$. This is the motivating idea for this section.

Definition 4.1. Let A be a Banach algebra. A *character* on A is a nonzero homomorphism from A into $\mathbb{C}$. If A is unital, we require this homomorphism to be unital.

The set of all characters on A is called the *maximal ideal space* of A, and it is denoted by $\mathrm{Max}(A)$.

Recall the following standard result for Banach algebras.

Theorem 4.2. Let A be a unital Banach algebra.

(1) If $\omega \in \mathrm{Max}(A)$, then $\|\omega\| = 1$.
(2) The map $\omega \mapsto \ker \omega$ defines a bijection from $\mathrm{Max}(A)$ onto the set of all maximal ideals of A.

Remark 4.3. If A is a unital Banach algebra, then $\mathrm{Max}(A)$ is nonempty. However, $\mathrm{Max}(A)$ may be empty if A is nonunital, although this is still true for commutative C^*-algebras.

Endow $\mathrm{Max}(A)$ with the weak* topology (pointwise convergence). Since $\mathrm{Max}(A)$ is a closed subset (that is, the pointwise limit of a net of characters is again a character – only algebra is involved here) of the unit ball of A^*, the following theorem is a consequence of Alaoglu's theorem.

Theorem 4.4. If A is a unital and commutative Banach algebra, then $\mathrm{Max}(A)$ is a nonempty compact Hausdorff space.

Definition 4.5. Let A be unital and commutative Banach algebra. If $a \in A$, define $\widehat{a} \colon \mathrm{Max}(A) \to \mathbb{C}$ by

$$\varepsilon_a(\omega) = \omega(a). \tag{1}$$

Lemma 4.6. If A is a unital, commutative Banach $*$-algebra and $\omega \colon A \to \mathbb{C}$ is a character, then $\omega(a^*) = \overline{\omega(a)}$.

Proof. Write $a = x + iy$ for $x, y \in A_{sa}$. Then $\omega(a^*) = \omega(x - iy) = \omega(x) - i\omega(y)$ so it is enough to show that $\omega(a) \in \mathbb{R}$ if $a \in A_{sa}$. Indeed, if this is the case, we get that

$$\omega(a^*) = \omega((x + iy)^*) = \overline{\omega(x) + i\omega(y)} = \overline{\omega(x) + i\omega(y)} = \overline{\omega(a)}.$$

To show this, notice that $a - \omega(a) \in \ker(\omega)$, which is a maximal ideal, so $a - \omega(a)$ is not invertible. We conclude that $\omega(a) \in \mathrm{sp}(a) \subseteq \mathbb{R}$. □

Proposition 4.7. Let A be a commutative Banach $*$-algebra. Then the map defined by (1) is a $*$-homomorphism $\mathcal{G}_A\colon A \to C_0(\mathrm{Max}(A))$, given by $a \mapsto \widehat{a}$.

Proof. Let's check that $\widehat{a}$ is continuous. Notice that if $F \subseteq \mathbb{C}$ is closed, then $\widehat{a}^{-1}(F) = \{\omega \in \mathrm{Max}(A)\colon \omega(a) \in F\}$ is weak* closed (recall that $\omega_i \to \omega$ weak* if $\omega_i(b) \to (\omega(b))$). Hence $\widehat{a} \in C(\mathrm{Max}(A))$.

$\mathcal{G}_A$ is a unital homomorphism because

$$\widehat{ab}(\omega) = \omega(ab) = \omega(a)\omega(b) = \widehat{a}(\omega)\widehat{b}(\omega) \qquad \widehat{1}(\omega) = \omega(1) = 1.$$

Finally, $\mathcal{G}_A$ preserves the involution: $\widehat{a^*}(\omega) = \omega(a^*) = \overline{\omega(a)} = \overline{\widehat{a}(\omega)}$. ☐

The unital $*$-homomorphism $\varepsilon\colon A \to C_0(\mathrm{Max}(A))$ is called the *Gelfand transform*.

Theorem 4.8. (Gelfand Transform) Let A be commutative C^*-algebra. Then $\mathcal{G}_A\colon A \to C_0(\mathrm{Max}(A))$ is an isometric $*$-isomorphism. Hence

$$A \cong C_0(\mathrm{Max}(A)).$$

Proof. Assume first that A is unital. Thanks to the Open Mapping theorem, we only need to show that $\mathcal{G}_A$ is isometric and surjective. Notice that an isometry is automatically injective.

Isometry: If a is invertible, then $\mathcal{G}_A(a)$ is invertible, and its inverse is $\mathcal{G}_A(a^{-1})$. Conversely, if a is not invertible, then the ideal $I = aA$ (recall that A is commutative) is proper and thus is contained in some maximal ideal M (because A is assumed to be unital). Since $M = \ker \omega$ for some character ω, we get that $\widehat{a}(\omega) = \omega(a) = 0$ and hence $\widehat{a}$ is not invertible in $C(\mathrm{Max}(A))$, because it vanishes at ω.

From the above paragraph we conclude that a and $\widehat{a}$ have the same spectrum, which must agree with the range of $\widehat{a} \in C(\mathrm{Max}(A))$. Since the norm on $C(\mathrm{Max}(A))$ is the supremum norm, we conclude that $\|a\|_A = r(a) = r(\widehat{a}) = \|\widehat{a}\|_\infty$ (since a is normal, as A is commutative). Hence $\mathcal{G}_A$ is an isometry.

Surjectivity: Notice that $\mathcal{G}(A)$ is a closed subalgebra of $C(\mathrm{Max}(A))$ (because it is isometrically isomorphic to A, which is complete), that separates points of $\mathrm{Max}(A)$ (because two different characters on A must differ at some point of A), and contains the constants. Hence the Stone-Weierstrass theorem implies that $\varepsilon(A) = C(\mathrm{Max}(A))$.

For the nonunital case, we get that $\mathcal{G}_{A^+}\colon A^+ \to C(\mathrm{Max}(A^+))$ is an isometric isomorphism. Since $A^+/A \cong \mathbb{C}$, $\mathcal{G}_{A^+}(A)$ is a maximal ideal of $C(\mathrm{Max}(A^+))$, and hence it has the form

$$\mathcal{G}_{A^+}(A) = C_0(\mathrm{Max}(A^+) \setminus \{\pi\})$$

for some $\rho \in \mathrm{Max}(A^+)$. It is clear that $\mathrm{Max}(A^+) = \mathrm{Max}(A) \cup \{\pi\}$ where $\pi\colon A^+ \to A^+/A \cong \mathbb{C}$ is the quotient map. Hence $\rho = \pi$ and the restriction of $\mathcal{G}_{A^+}$ to A, which agrees with $\mathcal{G}_A$, is an isometric $*$-isomorphism between A and $C_0(\mathrm{Max}(A)) = C_0(\mathrm{Max}(A^+) \setminus \{\pi\}) = \mathcal{G}_{A^+}(A)$. ☐

The following theorem reduces the study of *commutative* C^*-algebras to the study of topological spaces. Therefore the study of C^*-algebras is usually thought of as the study of *noncommutative topology*.

Theorem 4.9. Define a contravariant functor C from the category of compact Hausdorff spaces and continuous maps to the category of commutative unital C^*-algebras and unital homomorphisms, by $X \mapsto C(X)$. Then C is an equivalence of categories.

Proof. Need a quasi-inverse functor. Consider the functor Max from the category of commutiative, unital C^*-algebras to the category of compact Hausdorff spaces. For a homomorphism $\varphi\colon A \to B$ and $\omega \in \mathrm{Max}(B)$ (this is, a character $\omega\colon B \to \mathbb{C}$), define $\mathrm{Max}(\varphi)(\omega) = \omega \circ \varphi$. One checks that $\mathrm{Max}(\varphi)\colon \mathrm{Max}(B) \to \mathrm{Max}(A)$ is weak*-continuous.

For every commutative unital C^*-algebra A and every compact Hausdorff space X, one needs isomorphisms $\mathcal{G}_A\colon A \to C(\mathrm{Max}(A))$ and $\varepsilon_X\colon X \to \mathrm{Max}(C(X))$. That $\mathcal{G}_A$ is an isomorphism was proved in the Theorem above. The map $\varepsilon_X\colon X \to \mathrm{Max}(C(X))$ is defined by $\varepsilon_X(x)(\omega) = \omega(x)$ for all $x \in X$ and all $\omega \in \mathrm{Max}(C(X))$. One needs to show that ε_X is continuous, injective and surjective. Since X is compact, this will imply that it is an homeomorphism.

Continuity: suppose that $(x_\lambda)_{\lambda\in\Lambda}$ is a net in X converging to $x \in X$. In order to show that $\varepsilon_X(x_\lambda) \to \varepsilon_X(x)$ in $\mathrm{Max}(C(X))$, weakly-*, one has to show that the convergence is true opon evaluating at $f \in \mathrm{Max}(C(X))$, this is, $f(x_\lambda) \to f(x)$. This follows from continuity of f.

Injectivity: two non-identical functions defined on $\mathrm{Max}(C(X))$ must differ in some $x \in \mathrm{Max}(C(X))$.

Surjectivity: every maximal ideal of $C(Y)$ has the form $\{f \in C(Y)\colon f(y_0) = 0\}$ for some $y_0 \in Y$. ☐

Warning 4.10. Not every homomorphism $C(X) \to C(Y)$ comes from a continuous map $Y \to X$. It needs to be unital.

Let's look at the locally compact case. A homomorphism $C_0(X) \to C_0(Y)$ corresponds to a unital homomorphism $C(X^+) \to C(Y^+)$ that makes the diagram

$$\begin{tikzcd} C(Y^+) \arrow[rr] \arrow[dr, "\mathrm{ev}_\infty"'] & & C(X^+) \arrow[dl, "\mathrm{ev}_\infty"] \\ & \mathbb{C} & \end{tikzcd}$$

commutes. Such maps correspond to continuous maps $X^+ \to Y^+$ sending ∞ to ∞.

Remark 4.11. Note that:

- A continuous map $X \to Y$ need not extend to a map $X^+ \to Y^+$ sending ∞ to ∞, and it does so if and only if it is proper.
- Not all maps $X^+ \to Y^+$ come from maps $X \to Y$. Indeed, some points in X could get mapped to ∞ in Y^+.

Consider the category of locally compact Hausdorff spaces X, with $\mathrm{Mor}(X, Y)$ being all pairs (U, h) with $U \subseteq X$ open and $h\colon U \to Y$ continuous and proper. Equivalently,

$$\mathrm{Mor}(X, Y) = \{f\colon X^+ \to Y^+ \text{ continuous with } f(\infty) = \infty\}.$$

(To obtain the pair (U, h), set $U = f^{-1}(Y)$ and $h = f|_U$.)

Example 4.12. The map $C([0,1]) \hookrightarrow C([0,1]) \oplus C([0,1])$ given by $f \mapsto (f, 0)$ comes from the morphism $(U, h) = ([0,1] \times \{0\}, h(x, 0) = x)$.

5. Continuous functional calculus.

Let A be a unital C^*-algebra and let $a \in A$ be a normal element. Then $B = C^*(1, a)$ is a commutative unital C^*-algebra, and hence there is a compact space X such that $B \cong C(X)$.

Lemma 5.1. In the situation described above, we have $X \cong \mathrm{sp}_B(a)$.

Proof. The map $\mathrm{ev}_a\colon \mathrm{Max}(B) \to X$ is continuous and surjective. Let $\omega_1, \omega_2\colon B \to \mathbb{C}$ be characters that agree on a. Since they are $*$-homomorphisms, they agree on all of B, and hence $\omega_1 = \omega_2$. This shows that $\mathcal{G}_B(a)$ is injective, and hence it is a homeomorphism. □

Theorem 5.2. Let A be a unital C^*-algebra and let B be a subalgebra of A. For $a \in B$, we have:

(1) If A and B are unital with the same unit, then $\mathrm{sp}_A(a) = \mathrm{sp}_B(a)$.
(2) If A and B are non-unital, then $\mathrm{sp}_A(a) = \mathrm{sp}_B(a)$ and both with contain $\{0\}$.
(3) If A is non-unital and B is unital, then
   - If $a \in \mathrm{Inv}(B)$, then $\mathrm{sp}_A(a) = \mathrm{sp}_B(a) \cup \{0\}$ and $\{0\}$ is not contained in $\mathrm{sp}_B(a)$.
   - If $a \notin \mathrm{Inv}(B)$, then $\mathrm{sp}_A(a) = \mathrm{sp}_B(a)$ and both with contain $\{0\}$.
(4) If A and B are unital with $1_A \neq 1_B$, then
   - If $a \in \mathrm{Inv}(B)$, then $\mathrm{sp}_A(a) = \mathrm{sp}_B(a) \cup \{0\}$ and $\{0\}$ is not contained in $\mathrm{sp}_B(a)$.
   - If $a \notin \mathrm{Inv}(B)$, then $\mathrm{sp}_A(a) = \mathrm{sp}_B(a)$ and both with contain $\{0\}$.

Example 5.3. The result is not true for Banach $*$-algebras. Let $A = C(S^1)$ and let $B = A(\mathbb{D})$ be the disc algebra. Regard $B \hookrightarrow A$ via $f \mapsto f|_{S^1}$. Let $z \in A(\mathbb{D})$ be given by $z(\zeta) = \zeta$ for all $\zeta \in \mathbb{D}$. Then $\mathrm{sp}_B(z) = \mathbb{D}$ and $\mathrm{sp}_A(z) = S^1$.

We have proved the following.

Theorem 5.4. (Functional Calculus) Let A be a unital C^*-algebra and let $a \in A$ be a normal element. then there is an isometric isomorphism $C^*(a) \cong C_0(\mathrm{sp}(a) \setminus \{0\})$ that sends a to the identity function on $\mathrm{sp}(a)$.

Corollary 5.5. If a is normal and nilpotent or quasinilpotent ($\|a^n\|^{\frac{1}{n}} \to 0$, which implies $r(a) = 0$ and $\mathrm{sp}(a) = \{0\}$), then $a = 0$.

Definition 5.6. If a is a normal element of A and $f \in C_0(\mathrm{sp}(a) \setminus \{0\})$, then we denote by $f(a)$ the element of A corresponding to f via the isomorphism given in the preceding theorem. This is known as the *continuous functional calculus for normal elements.*

Proposition 5.7. (Spectral mapping theorem) If $f \in C_0(\mathrm{sp}(a) \setminus \{0\})$, then $\mathrm{sp}(f(a)) = f(\mathrm{sp}(a))$.

Proof. We have

$$\mathrm{sp}(f(a)) = \mathrm{sp}(\varphi(f)) = \mathrm{sp}(f) = f(\mathrm{sp}(a)),$$

using that φ is an isomorphism and that $\mathrm{sp}(f) = \mathrm{range}(f)$. □

Proposition 5.8. If in addition $g \in C_0(\mathrm{sp}(a) \setminus \{0\})$, then

$$g \circ f(a) = g(f(a)).$$

Proof. Consider the maps $C(\mathrm{sp}(f(a))) \to A$ given by $g \mapsto (g \circ f)(a)$ and $g \mapsto g(f(a))$. Both agree on $z\colon \mathrm{sp}(f(a)) \to \mathbb{C}$ since they both send it to $f(a)$. Since z generates $C(\mathrm{sp}(f(a)))$, they are equal. $\square$

Theorem 5.9. Let $A = C(X), B = C(Y)$ commutative C^*-algebras, where X, Y are compact Hausdorff spaces. Then $A \cong B$ if and only if $X \cong Y$.

The Gelfand representation theorem for commutative algebras is fundamentally important. Even in a noncommutative C^*-algebra A, we often obtain useful information of A via the study of certain commutative subalgebras.

The last theorem shows that the study of commutative C^*-algebras is equivalent to the study of compact Hausdorff spaces, as these are their maximal ideal spaces. Therefore, commutative C^*-algebra theory is the theory of topology. General C^*-algebra theory is usually referred to as *noncommutative topology.*

6. Positive cones.

The ultimate goal of this and the following sections is to prove Gelfand-Naimark representation theorem: every C^*-algebra is a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. For that, we need to study the order structure on C^*-algebras.

Recall that an operator $T \in \mathcal{B}(\mathcal{H})$ is *positive* if $\langle Tx, x\rangle \geq 0$ for all $x \in \mathcal{H}$. In the case that $\mathcal{H}$ is finite-dimensional, T is positive if and only if T is self-adjoint and all its eigenvalues are nonnegative. The following is its straightforward generalization.

Definition 6.1. Given a C^*-algebra A, an element $a \in A$ is said to be *positive* if $a \in A_{sa}$ and $\mathrm{sp}(a) \subseteq \mathbb{R}^+$. The set of all positive elements of A is denoted by A_+.

Theorem 6.2. Let A be a C^*-algebra and $a \in A$. Then the following are equivalent:

(1) $a \geq 0$.
(2) There exists $b \in A$ such that $a = b^*b$.
(3) There exists $b \in A_{sa}$ such that $a = b^2$.
(4) There exists a unique $b \in A_+$ such that $a = b^2$.
(5) $a = a^*$ and $\|t - a\| \leq t$ for any $t \geq \|a\|$.
(6) $a = a^*$ and $\|t - a\| \leq t$ for some $t \geq \|a\|$.

Proof. (a) implies (c). Apply functional calculus in $C^*(1, a)$.
(c) implies (e). Since $t - a$ is normal (it is indeed self-adjoint), we have

$$\|t - a\| = \sup\{|\lambda|\colon \lambda \in \mathrm{sp}(t - a) = t - \mathrm{sp}(a)\} \leq t.$$

(e) implies (f). Clear.
(f) implies (a). Again, $\mathrm{sp}(t - a) = t - \mathrm{sp}(a)$. If $|t - \lambda| = t - \lambda \leq t$ for some $t \geq \|a\|$, then $\lambda \geq 0$.
(a) implies (d). Apply functional calculus in $C^*(1, a)$.
(d) implies (c) implies (b). Clear.
(b) implies (a). One needs to show that elements of the form b^*b are always positive. This takes the most work and its proof is omitted. $\square$

Definition 6.3. Let $a \in A_{sa}$. Then $a^2 \in A_+$. Set $|a| = (a^2)^{\frac{1}{2}}$. Define the *positive and negative parts* of a by

$$a_+ = \frac{|a| + a}{2} \quad \text{and} \quad a_- = \frac{|a| - a}{2}.$$

Then $|a|, a_+$ and a_- are positive and $a = a_+ - a_-$. Moreover, $a_+a_- = 0 = a_-a_+$.

Corollary 6.4. If A is a unital C^*-algebra, then A is the linear span of its unitaries.

Proof. Let $a \in A_{sa}, \|a\| \leq 1$. Then $1 - a^2 \in A_+$. Put

$$u = a + i(1 - a^2)^{\frac{1}{2}} \quad \text{and} \quad v = a - i(1 - a^2)^{\frac{1}{2}}.$$

Then $u^* = v$ and $u^*u = uu^* = 1$. Hence u and v are unitaries and $a = \frac{u+v}{2}$. Finally, any element in A can be decomposed as the sum of its real and imaginary parts, both of which are self-adjoint. $\square$

Proposition 6.5. Let A be a C^*-algebra. Then

(1) The set A_+ is a closed cone, that is:
  - A_+ is a closed subset of A.
  - $A_+ + A_+ \subseteq A_+$

- $A_+ \cap (-A_+) = \{0\}$.

(2) $a \in A_+$ if and only if there exists $x \in A$ such that $a = x^*x$.

Proof. Part (a). That A_+ is closed follows from condition (e) in the above lemma. Now, let $a, b \in A_+$. Set $\alpha = \|a\|$ and $\beta = \|b\|$. Then

$$\|(\alpha+\beta) - (a+b)\| = \|(\alpha - a) + (\beta - b)\| \leq \|\alpha - a\| + \|\beta - b\| \leq \alpha + \beta,$$

so condition (f) of the above lemma holds, and $a + b \in A_+$. For the last item, assume that $a \geq 0$ and $-a \geq 0$. Then we conclude that $a = 0$ by looking at the commutative algebra $C^*(1, a) \cong C(\mathrm{sp}(a))$.

Part (b). It is a weaker form of the existence of (positive) square roots. □

Definition 6.6. In A_{sa}, we write $a \leq b$ if $b - a \in A_+$. Since $A_{sa} = A_+ - A_+$ and $A_+ \cap -A_+ = \{0\}$, A_{sa} becomes a partially ordered real vector space.

Remark 6.7. $a, b \geq 0$ doesn't imply $ab \geq 0$. In fact, unless a, b commute, ab need not be selfadjoint.

Theorem 6.8. Let A be a C^*-algebra. Then

(1) $A_+ = \{a^*a \colon a \in A\}$.
(2) $a, b \in A_{sa}, a \leq b$ and $c \in A$ imply that $c^*ac \leq c^*bc$.
(3) $0 \leq a \leq b$ implies $\|a\| \leq \|b\|$.
(4) If A is unital and $a \in A_+$ is invertible, and $a \leq b$ implies that b is invertible and $0 \leq b^{-1} \leq a^{-1}$.

Proof. (a) Follows from the previous theorem.

(b) Notice that $c^*bc - c^*ac = c^*(b-a)c = c^*x^*xc = (xc)^*(xc)$, where $b - a = x^*x \in A_+$.

(c) We may assume that A is unital. Recall that $a \leq \|a\|$ for every $a \in A_{sa}$. Now, $0 \leq a \leq b$ implies $a \leq \|b\|$ and hence by looking at $C^*(a, 1)$ (where both a and $\|b\|$ belong), we conclude that $\|a\| \leq \|b\|$.

(d) Notice that if $b \geq 0$ is invertible, then $b^{-1} \geq 0$ too (consider $C^*(1, b) \cong C(\mathrm{sp}(b))$). Conjugate both sides of the inequality $a \leq b$ by $a^{-\frac{1}{2}}$ to get $1 \leq a^{-\frac{1}{2}} b a^{-\frac{1}{2}}$. Now, these two elements commute and by Gelfand theorem, $a^{-\frac{1}{2}} b a^{-\frac{1}{2}}$ is invertible, which implies that b is invertible itself. Now, since the result is true in the commutative case, we get that $0 \leq a^{\frac{1}{2}} b^{-1} a^{\frac{1}{2}} \leq 1$, and again conjugating by $a^{-\frac{1}{2}}$ we get $0 \leq b^{-1} \leq a^{-1}$. □

Theorem 6.9. Let A be a C^*-algebra. If $a, b \in A_+$ and $a \leq b$, then $a^\alpha \leq b^\alpha$ for $0 < \alpha \leq 1$.

On the other hand, if $\alpha > 1$ is fixed and A is a C^*-algebra in which $0 \leq a \leq b$ implies $0 \leq a^\alpha \leq b^\alpha$, then A is commutative.

Example 6.10. It is not true that $0 \leq a \leq b$ implies $a^2 \leq b^2$ (in noncommutative C^*-algebras). For example, let $A = M_2(\mathbb{C})$ and take

$$p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad q = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then p and q are projections, $p \leq p + q$ but $p^2 = p \nleq (p+q)^2 = p + q + qp + pq$ since

$$q + pq + qp = \frac{1}{2}\begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$$

is not positive.

Warning 6.11. If a, b, x are elements of a C^*-algebra A such that $a^*a \leq b^*b$ and $x \geq 0$, it does not follow that $a^*xa \leq b^*xb$.

Warning 6.12. Unless A is commutative, A_+ will in general not be a lattice. Indeed, the minimum or maximum of two positive elements need not even exist.

7. Basic structure of C^*-algebras.

7.1. **Approximate identities.**

To deal with nonunital C^*-algebras one can often embed A into A^+. However, more often, one has to work in the original nonunital C^*-algebra. Therefore, the notion of approximate identity is essential.

Definition 7.1. Give a C^*-algebra A, an *approximate identity* for A is an increasing net $\{e_\lambda\}_{\lambda \in \Lambda}$ of positive elements in the closed unit ball of A such that $a = \lim_\lambda a e_\lambda$ for all $a \in A$. Notice that this is equivalent to $a = \lim_\lambda (e_\lambda a)$.

It is not clear that approximate identities always exist in non-unital C^*-algebras. Let's see a couple of examples.

Examples 7.2. Approximate identities in $\mathcal{K}$ and $C_0(X)$.

(1) Let $\mathcal{H}$ be a Hilbert space with orthonormal basis $\{v_n\}_{n\in\mathbb{Z}_{>0}}$. Then $\mathcal{K} \leq \mathcal{B}(\mathcal{H})$ is a nonunital C^*-algebra. Let's find an approximate identity for $\mathcal{K}$. Let $p_n\colon \mathcal{H} \to \operatorname{span}\{v_1, \dots, v_n\}$ be the orthogonal projection. Then $p_n \in \mathcal{K}$ and $(p_n)_{n\in\mathbb{Z}_{>0}}$ is an increasing sequence of projections in $\mathcal{K}$. Moreover, for any $x \in \mathcal{K}$,

$$\|p_n x - x\|, \|x p_n - x\| \to 0 \quad \text{as } n \to \infty.$$

This follows from the fact that every element in $\mathcal{K}$ can be approximated (in the operator norm) by operators of finite rank.

(2) Let X be a $\sigma-$compact, noncompact, locally compact Hausdorff space. Then $C_0(X) = A$ is a nonunital C^*-algebra. Moreover, $X = \cup_{n\in\mathbb{Z}_{>0}} X_n$, where each X_n is compact and X_{n+1} contains a neighborhood of X_n. There exists a sequence of positive functions $f_n \in C_0(X)$ such that $0 \leq f_n \leq 1$, $f_n(t) = 1$ on X_n and $f_n(t) = 0$ is $t \notin X_{n+1}$. One checks that, for every $g \in C_0(X)$,

$$\|g f_n - g\|, \|f_n g - g\| \to 0 \quad \text{as } n \to \infty.$$

Lemma 7.3. Let A be a C^*-algebra and denote by Λ the set of all elements $a \in A_+$ with $\|a\| < 1$. Then Λ is an upwards directed set, that is, if $a, b \in \Lambda$, then there exists $c \in \Lambda$ such that $a, b \leq c$.

Proof. Suppose $a \in A_+$. Then $1 + a \in \operatorname{Inv}(A^+)$ and $a(1+a)^{-1} = 1 - (1+a)^{-1}$.

We claim that $a, b \in A_+$ and $a \leq b$ imply $a(1+a)^{-1} \leq b(1+b)^{-1}$. In fact, if $a \leq b$, then $1 + a \leq 1 + b$, which implies that $(1+b)^{-1} \leq (1+a)^{-1}$. Consequently, $1 - (1+a)^{-1} \leq 1 - (1+b)^{-1}$, which is the same as $a(1+a)^{-1} \leq b(1+b)^{-1}$.

Note that if $a \in A_+$, then $a(1+a)^{-1} \in \Lambda$ (it is positive and its norm is less than 1 using functional calculus and the fact that $\frac{z}{z+1} < 1$ for all $z \in \mathbb{R}$). For $a, b \in \Lambda$, put

$$x = a(1-a)^{-1}, \quad y = b(1-b)^{-1} \quad \text{and} \quad c = (x+y)(1+x+y)^{-1}.$$

Then $x, y \in A_+$, and hence also $x + y \in A_+$. Note that $(1+x)^{-1} = 1 - a$ and $x(1+x)^{-1} = a$. Hence, the above claim implies that $a = x(1+x)^{-1} \leq c$, since $x \leq x + y$. Similarly, $b \leq c$. Finally, notice that $c \in A_+$ and $\|c\| < 1$ and $c \in \Lambda$. ☐

Theorem 7.4. Every C^*-algebra A admits an approximate identity. Indeed, if Λ is the upwards directed set of all positive elements $a \in A_+$ with $\|a\| < 1$, and $e_\lambda = \lambda$ for all $\lambda \in \Lambda$, then $\{e_\lambda\}_{\lambda\in\Lambda}$ forms an approximate identity for A.

Corollary 7.5. If A is separable, we can choose a countable approximate identity; in fact, a sequence.

Proof. Let $\{x_1, x_2, \dots\}$ be a countable dense subset and let $(e_\lambda)_{\lambda\in\Lambda}$ an approximate identity. Choose $\lambda_1 \in \Lambda$ such that $\|e_\lambda x_1 - x_1\| < \frac{1}{2}$ for all $\lambda \geq \lambda_1$. Now choose λ_2 such that $\lambda_2 \geq \lambda_1$ and $\|e_\lambda x_j - x_j\| < \frac{1}{4}$ for $j = 1, 2$. Proceeding inductively, we get a sequence $\{\lambda_n\}_{n\in\mathbb{Z}_{>0}}$ such that

$$\|e_{\lambda_n} x_k - x_k\| \to 0 \quad \text{as } n \to \infty, \quad \text{for all } k \in \mathbb{Z}_{>0}.$$

Since $\|e_{\lambda_n}\| \leq 1$, a routine $\frac{\varepsilon}{3}$-argument shows that $\|e_{\lambda_n} x - x\| \to 0$ as $n \to \infty$ for all $x \in A$. ☐

Definition 7.6. A C^*-algebra is said to be *σ-unital* if it admits a countable approximate identity.

The above corollary shows that every separable C^*-algebra is σ-unital.

7.2. **Hereditary subalgebras.**

Definition 7.7. A C^*-subalgebra B of A is said to be *hereditary* if for any $a \in A_+, b \in A_+$, $a \leq b$ implies that $a \in B$.

The subalgebras A and $\{0\}$ of A are clearly hereditary. Moreover, the intersection of an arbitrary family of hereditary subalgebras is again an hereditary subalgebra. If $S \subseteq A$, then the hereditary subalgebra *generated* by S is the smallest hereditary subalgebra of A containing S. Equivalently, it is the intersection of all hereditary subalgebras of A that contain S. It is denoted by $\operatorname{Her}(S)$.

Example 7.8. Let $p \in A$ be a projection. Then $pAp \leq A$ is an hereditary subalgebra.

Proof. Assume that $0 \leq b \leq pap$ for some $b \in A$. Then $0 \leq (1-p)b(1-p) \leq (1-p)pap(1-p) = 0$, so $(1-p)b(1-p) = 0$. Thus, $\|b^{\frac{1}{2}}(1-p)\|^2 = \|(1-p)b(1-p)\| = 0$, and hence $b(1-p) = 0$. This implies that $b = bp$, and similarly $b = pb$. Hence $b = pbp \in pAp$, so pAp is hereditary. ☐

Theorem 7.9. Let A be a C^*-algebra and let B be a subalgebra. Then the following are equivalent:

(1) B is a hereditary subalgebra.
(2) There exists a closed left ideal L of A such that $L \cap L^* = B$.
(3) There exist closed left and right ideals L and R of A such that $L \cap R = B$.
(4) For all $a \in A$ and for all $b, c \in B$, the element bac belongs to B.

(5) For all $a \in A_+$ and all $b \in B_+$, the element bab belongs to B_+.

Moreover, the correspondence $L \mapsto L \cap L^*$ is an order preserving bijection from closed left ideals to hereditary subalgebras. Given B, we set $L = \{a \in A \colon a^*a \in B\}$.

Corollary 7.10. If I is an ideal in A, then I is an hereditary subalgebra of A.

Proposition 7.11. Let A be a C^*-algebra and $a \in A_+$. Then $\overline{aAa}$ is the hereditary subalgebra generated by a.

Proof. The subset aAa is a $*$-subalgebra and hence its closure is a C^*-algebra. We have $a^3 \in \overline{aAa}$ and since $a \geq 0$, it follows that $a = (a^3)^{1/3} \in \overline{aAa}$ by functional calculus. Now,

$$(\overline{aAa})A(\overline{aAa}) \subseteq \overline{aAaAaAa} \subseteq \overline{aAa},$$

and by condition (d) in the Theorem, $\overline{aAa}$ is hereditary. □

Proposition 7.12. Suppose that $B \leq A$ is hereditary and B is separable. Then there exists $a \in A$ such that $B = \overline{aAa}$.

Proof. Let $(u_n)_{n\in\mathbb{Z}_{>0}}$ be a *countable* approximate identity for B. Then $a = \sum_{n\in\mathbb{Z}_{>0}} \frac{u_n}{2^n} \in B^+$, and hence $\overline{aAa} \subseteq B$. Since $0 \leq u_n \leq a$, we get that $u_n \in \overline{aAa}$. Now, if $b \in B$, then $b = \lim_n u_n b u_n \in \overline{aAa}$, so $B = \overline{aAa}$. □

For the pair $\mathcal{K}(\mathcal{H}) \leq \mathcal{B}(\mathcal{H})$, the converse is also true: there exists $u \in \mathcal{K}^+$ such that $\mathcal{K} = u\mathcal{B}(\mathcal{H})u$ if and only if $\mathcal{H}$ is separable. Notice that $\mathcal{K}(\mathcal{H})$ is a hereditary subalgebra of $\mathcal{B}(\mathcal{H})$ since it is an ideal.

Example 7.13. Let $\mathcal{H}$ be a Hilbert space and assume that there exists $u \in \mathcal{K}(\mathcal{H})^+$ such that $\mathcal{K}(\mathcal{H}) = u\mathcal{B}(\mathcal{H})u$. If $x \in \mathcal{H}$, then $\theta_{x,x} = \lim_n uv_nu$ for some sequence $(v_n)_{n\in\mathbb{Z}_{>0}} \subseteq \mathcal{B}(\mathcal{H})$. Hence $x = \theta_{x,x}\left(\frac{x}{\|x\|^2}\right) \in \overline{u(\mathcal{H})}$. In particular, $\mathcal{H} = \overline{u(\mathcal{H})}$, and hence $\mathcal{H}$ is separable, because so is the range of any compact operator.

In particular, if $\mathcal{H}$ is not separable, the hereditary subalgebra $\mathcal{K}(\mathcal{H})$ is not singly generated as an hereditary subalgebra.

Theorem 7.14. Let A be a C^*-algebra, let B be a hereditary subalgebra and let J be an ideal in B. Then there exists an ideal I of A such that $I \cap B = J$.

Corollary 7.15. If A is simple and $B \leq A$ is hereditary, then B is simple.

Example 7.16. The above result is not true if B is not hereditary. For example, let $A = M_2(\mathbb{C})$ and $B = \mathbb{C} \oplus \mathbb{C}$. Then B is not simple although A is.

Theorem 7.17. A C^*-algebra A is σ-unital if and only if there is $a \in A_+$ such that $\mathrm{Her}(a) = A$.

Proof. The second claim in the above theorem shows that $\mathrm{Her}(a)$ is always σ-unital.

Conversely, assume that A is σ-unital and let $(e_n)_{n\in\mathbb{Z}_{>0}}$ be a countable approximate identity. Take $a = \sum_{n\in\mathbb{Z}_{>0}} \frac{e_n}{2^n}$ and $B = \mathrm{Her}(a)$. Since $0 \leq e_n \leq 2^n a$, it follows that $e_n \in B$, and in particular $(e_n)_{n\in\mathbb{Z}_{>0}}$ is an approximate identity for B. Now, if $b \in A$, we have that $e_n b e_n \to b$ in norm, but since $e_n b e_n \in B$, it follows that $b \in B$. Hence, $A = B = \mathrm{Her}(a)$. □

7.3. **Ideals and quotients.**

The most important consequence of the following theorem is that ideals are hereditary C^*-algebras. In particular, they are self-adjoint.

Comment: this is actually automatic by Theorem 7.9. In a better version of these notes, this should be fixed.

Theorem 7.18. Let I be an ideal of A. Then I is a hereditary subalgebra of A. Moreover, if $(e_\lambda)_{\lambda\in\Lambda}$ is an approximate identity for I, then

$$\|\pi_I(a)\| = \|a + I\| = \inf_{b\in I} \|a + b\| = \lim_\lambda \|a - e_\lambda a\| = \lim_\lambda \|a - ae_\lambda\|.$$

Proof. Let $B = I \cap I^*$. Then B is a C^*-subalgebra of A. Let $(e_\lambda)_{\lambda\in\Lambda}$ be an approximate identity for B. Note that $e_\lambda \in B \subseteq I$. If $a \in I$, then $\lim_\lambda \|a^*a - a^*ae_\lambda\| = 0$. Hence

$$\|a - ae_\lambda\| = \lim_\lambda \|(1 - e_\lambda)a^*a(1 - e_\lambda)\| \leq \lim_\lambda \|a^*a - a^*ae_\lambda\| \to 0 \quad \text{as } \lambda \to \infty.$$

Therefore $a = \lim_\lambda ae_\lambda$ and also $a^* = \lim_\lambda a^*e_\lambda$. Since $e_\lambda \in I$, we conclude that $a^* \in I$. In particular $B = I$ and I is a C^*-subalgebra.

We still need to show that I is hereditary. Let $0 \leq a \leq b$ with $b \in I_+$. We want to show that $a \in I$ as well. By definition, $c \in \mathrm{Her}(a) = \overline{aAa}$. But since $\overline{aAa} \subseteq I$, we conclude that $c \in I$ and I is an hereditary subalgebra.

The claim about the norms is just a clever computation. □

Corollary 7.19. Let A be a C^*-algebra, let I be an ideal of A, and let J be an ideal of I. Then J is an ideal of A.

Proof. Let $(e_\lambda)_{\lambda\in\Lambda}$ be an approximate identity for I. Let $a \in A$ and let $x \in J$. Then $ax \in I$ since I is an ideal. In particular, $ax = \lim_\lambda ae_\lambda x$ belongs to J. By taking adjoints, xa also belongs to J. $\square$

Example 7.20. The above corollary is not true in purely algebraic situations. `What is the example?`

Corollary 7.21. If I is an ideal of A, then A/I equipped with its natural operations is a C^*-algebra.

Proof. Notice that A/I is a Banach $*$-algebra (because $I = I^*$). Hence it remains to show that $\|\overline{a}\|^2 = \|\overline{a}^*\overline{a}\|$:

$$\|\overline{a}^*\overline{a}\| = \lim_\lambda \|a^*a(1-e_\lambda)\| \geq \lim_\lambda \|(1-e_\lambda)a^*a(1-e_\lambda)\| = \lim_\lambda \|(1-e_\lambda)a\|^2 = \|\overline{a}\|^2.$$

For the reverse inequality, notice that $\|\overline{a}^*\overline{a}\| \leq \|\overline{a}^*\|\|\overline{a}\| = \|\overline{a}\|^2$, since the involution in A/I is an isometry (this is clear from the formula for the norm in A/I in the previous theorem, and the fact that the involution in A is an isometry). $\square$

Theorem 7.22. If $\varphi\colon A \to B$ is a homomorphism of C^*-algebras, then φ is norm-decreasing and $\varphi(A)$ is a C^*-subalgebra of B. Moreover, φ is injective if and only if it is isometric.

Proof. If $b \in A_{sa}$, then $\mathrm{sp}(\varphi(b)) \setminus \{0\} \subseteq \mathrm{sp}(b) \setminus \{0\}$. Since $\|b\| = r(b)$, we conclude that for any $a \in A$,

$$\|\varphi(a)\|^2 = \|\varphi(a^*a)\| = r(\varphi(a^*a)) \leq r(a^*a) = \|a^*a\| = \|a\|^2,$$

so φ is norm-decreasing.

Assume that φ is injective. To show that φ is isometric, if suffices to show it for positive elements since $\|\varphi(a)\|^2 = \|\varphi(a^*a)\|$ (if the latter equals $\|a^*a\| = \|a\|^2$, then the result follows). So let $a \in A_+$, and by restricting φ to $C^*(a)$, we may assume that A is commutative. Also, by considering unitizations of A, B and φ, we may assume that A, B are unital. Thus, we may assume that $A = C(X), B = C(Y)$ for some compact Hausdorff spaces X, Y (the latter assumption follows from the fact that $\varphi(A)$ is commutative, and by taking the closure we may assume that $\overline{\varphi(A)} = B$).

Define $\varphi^*\colon Y \to X$ by $\varphi^*(y) = \varepsilon_y \circ \varphi$. Since φ is injective, φ^* is surjective. Hence, for each $g \in C(Y)$,

$$\|g\| = \sup_{y\in Y} |g(y)| = \sup_{x\in X} |g(\varphi^*(x))| = \|\varphi(g)\|,$$

and thus φ is an isometry.

To complete the proof, if φ is any homomorphism, then $I = \ker\varphi$ is a closed ideal of A, and hence φ induces an isometric isomorphism between A/I and $\varphi(A)$, which is therefore a C^*-subalgebra of B. Finally, since $\varphi = \widehat{\varphi}\circ\pi_I$, where π_I is norm-decreasing and $\widehat{\varphi}$ is isometric, it follows that φ is norm-decreasing. $\square$

Corollary 7.23. If $I \lhd A$ and $B \leq A$, then $B + I \leq A$ and

$$\frac{B+I}{I} \cong \frac{B}{B\cap I}.$$

8. Positive linear functionals.

Definition 8.1. A linear map $\phi\colon A \to B$ is said to be *positive* if $\phi(A^+) \subseteq B^+$.

If $B = \mathbb{C}$, a positive map $\phi\colon A \to \mathbb{C}$ is usually called *positive linear functional.* If moreover ϕ is bounded and $\|\phi\| = 1$, then we say that ϕ is a *state* on A.

A positive linear functional $\tau\colon A \to \mathbb{C}$ is called a *trace* if $\tau(ab) = \tau(ba)$ for all $a, b \in A$. A trace which is also a state is called a *tracial state.*

Examples 8.2. Positive linear functionals in some C^*-algebras.

(1) Let $A = C(X)$, where X is a compact Hausdorff space, and let μ be a finite positive Borel measure on X. Then the linear functional $\tau\colon A \to \mathbb{C}$ given by

$$\tau(f) = \int_X f\ d\mu$$

is positive. Since A is commutative, τ is also a trace. Finally, since $\|\tau\| = \mu(X)$, it follows that τ is a tracial state if and only if $\mu(X) = 1$.

(2) Let $A = M_n(\mathbb{C})$. Define a linear functional Tr on A by

$$Tr\left((a_{ij})_{ij}\right) = \sum_{i=1}^n a_{ii}.$$

Then Tr is a trace, and it is usually called the *standard trace* on $M_n(\mathbb{C})$. The *normalized trace* on $M_n(\mathbb{C})$, denoted by tr, is defined by $tr = \frac{1}{n}Tr$.

(3) Let $A \leq \mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space, and $v \in \mathcal{H}, v \neq 0$. Define $f(a) = \langle a(v), v\rangle$ for $a \in A$. Then f is positive. Indeed, if $a = c^*c$, then $f(a) = \langle a^*a(v), v\rangle = \langle a(v), a(v)\rangle = \|a(v)\|^2 \geq 0$. This functional is in general not a trace.

Lemma 8.3. Every positive linear functional on a C^*-algebra A is bounded.

Proof. Let $\phi\colon A \to B$ be a positive linear functional. It is enough to show that there exists $M \geq 0$ such that $\|\phi(a)\| \leq M\|a\|$ whenever $a \in A_+$, since every element in A is a linear combination of four positive elements. Assume by contradiction that no such M exists. Then, for each $n \in \mathbb{Z}_{>0}$, there exists $a_n \in A_+, \|a_n\| = 1$ such that $\|\phi(a_n)\| > 2^{2n}$. Consider $a = \sum_{n\in\mathbb{Z}_{>0}} \frac{a_n}{2^n} \in A_+$. Then $a \geq \frac{a_n}{2^n}$ and hence $\|\phi(a)\| \geq \|\phi(\frac{a_n}{2^n})\| > 2^n$. Since this is true for all n, it follows that $\|\phi(a)\| = \infty$, which is a contradiction. Hence, ϕ is bounded. □

The following is the Cauchy-Schwarz inequality.

Lemma 8.4. If $\phi\colon A \to \mathbb{C}$ is a positive linear functional on a C^*-algebra A, then for all $a, b \in A$

$$|\phi(b^*a)|^2 \leq \phi(b^*b)\phi(a^*a).$$

Theorem 8.5. Let $\phi \in A^*$. Then the following are equivalent:

(1) ϕ is positive.
(2) $\lim_\lambda \phi(e_\lambda) = \|\phi\|$ for any approximate identity.
(3) $\lim_\lambda \phi(e_\lambda) = \|\phi\|$ for some approximate identity.

Proof. (a) implies (b). Let $\phi \geq 0$ and $(e_\lambda)_{\lambda\in\Lambda}$ be any approximate identity. Then $\{\phi(e_\lambda)\}_{\lambda\in\Lambda}$ is increasing, and since it is bounded (by $\|\phi\| < \infty$), it has a limit $L \leq \|\phi\|$. For each $a \in A, \|a\| \leq 1$, Cauchy-Schwarz inequality gives us

$$|\phi(e_\lambda a)|^2 \leq \phi(e_\lambda^2)\phi(a^*a) \leq \phi(e_\lambda)\|\phi\| \leq L\|\phi\|.$$

Since ϕ is continuous (being positive), making $\lambda \to \infty$ we get $|\phi(a)|^2 \leq L\|\phi\|$, so $\|\phi\| \leq L$. Since the other inequality holds by definition of L, we get $\|\phi\| = L$.

(b) implies (c). Immediate.

(c) implies (a). Suppose that $(\phi(e_\lambda))_{\lambda\in\Lambda}$ converges to $\|\phi\|$. We want to show that ϕ is positive, that is, that if $a \geq 0$ then $\phi(a) \geq 0$ as well. Let's start by proving that $\phi(A_{sa}) \subseteq \mathbb{R}$. Let $a \in A_{sa}$ with $\|a\| \leq 1$. Write $\phi(a) = \alpha + i\beta$. By multiplying by -1 we can assume that $b \geq 0$. We want to show that actually $\beta = 0$.

Given $n \in \mathbb{Z}_{>0}$, choose e_λ such that $\|e_\lambda a - ae_\lambda\| < \frac{1}{n}$. Then

$$\|ne_\lambda - ia\|^2 = \|n^2e_\lambda + a^2 - in(ae_\lambda - e_\lambda a)\| \leq n^2 + 2.$$

On the other hand, $\lim_\lambda |\phi(ne_\lambda - ia)|^2 = (n\|\phi\| + \beta)^2 + \alpha^2$. Combining these two inequalities, we get

$$(n\|\phi\| + \beta)^2 + \alpha^2 \leq (n^2 + 2)\|\phi\|^2$$

for all $n \in \mathbb{Z}_{>0}$. Hence $2n\|\phi\|\beta + \beta^2 + \alpha^2 \leq 2\|\phi\|^2$. If $\beta \neq 0$, we would get $\|\phi\| = \infty$, so $\beta = 0$ and hence $\phi(A_{sa}) \subseteq \mathbb{R}$.

Now, if $a \in A_+$ with $\|a\| \leq 1$, then $e_\lambda - a \in A_{sa}$ and $e_\lambda - a \leq e_\lambda \leq 1$. Hence $\phi(e_\lambda - a) \leq \|\phi\|$. Taking limit on λ, we get $\|\phi\| - \phi(a) \leq \|\phi\|$, which implies $\phi(a) \geq 0$. This proves that $\phi \geq 0$. □

Corollary 8.6. If A is unital, then a linear map $\phi : A \to \mathbb{C}$ is positive if and only if $\|\phi\| = \phi(1)$.

Corollary 8.7. Let A be a C^*-algebra and let ϕ_1 and ϕ_2 be two positive functionals on A. Then $\|\phi_1 + \phi_2\| = \|\phi_1\| + \|\phi_2\|$.

Corollary 8.8. If A is unital and $\phi \in A^*$, then any two of the following imply the other:

(1) $\phi(1) = 1$.
(2) $\|\phi\| = 1$.
(3) ϕ is positive.

Remark 8.9. The set of all states on A, denoted $S(A)$, is convex, and if A is unital then it is weak*-closed, and hence weak*-compact.

Example 8.10. If A is not unital, then $S(A)$ may not be closed. For example, let $A = C_0(\mathbb{R})$ and for $x \in \mathbb{R}$, set $\tau_x = \mathrm{ev}_x$. Then τ_x is a state, and the weak limit of τ_x as $x \to \infty$ is 0, which is not a state.

Definition 8.11. A state ϕ on A is said to be *pure* if it is an extreme point in $S(A)$.

We will see that pure states on A correspond to irreducible representations of A.

Proposition 8.12. Let X be a locally compact Hausdorff space. Then the states on $C_0(X)$ are exactly the Borel regular probability measures (via the Riesz Representation Theorem), and the pure states are the point mass measures.

Lemma 8.13. Let A be a C^*-algebra. For each positive linear functional $\phi\colon A \to \mathbb{C}$ define an extension $\phi^+\colon A\colon \to \mathbb{C}$ by setting $\phi^+(1) = \|\phi\|$. Then ϕ^+ is positive on A^+ and $\|\phi^+\| = \|\phi\|$. Moreover, the extension is unique.

Proof. We only need to show that $\|\phi^+\| = \|\phi\|$, because positiveness will follow from the previous theorem. Let $(e_\lambda)_{\lambda\in\Lambda}$ be an approximate unit in A.

Let $a + \alpha \cdot 1 \in A^+$, with $\|a + \alpha \cdot 1\| < 1$. Then

$$|\phi^+(a + \alpha \cdot 1)| = |\phi(a) + \alpha\|\phi\|| = |\phi(a) + \alpha \lim_\lambda \phi(e_\lambda)| = \lim_\lambda |\phi(a + \alpha \cdot e_\lambda)|$$

Since $a + \alpha \cdot e_\lambda \in A$ converges to $a + \alpha \cdot 1$, which has norm strictly less than 1, it follows that there exists $\lambda_0 \in \Lambda$ such that for all $\lambda \geq \lambda_0$, $\|a + \alpha \cdot e_\lambda\| \leq 1$. Hence

$$|\phi^+(a + \alpha \cdot 1)| = \lim_\lambda |\phi(a + \alpha \cdot e_\lambda)| \leq \|\phi\|,$$

and $\|\phi^+\| \leq \|\phi\|$. Since the other inequality holds by definition of ϕ^+ (an extension of ϕ), we get $\|\phi^+\| = \|\phi\|$. $\square$

Lemma 8.14. Let ϕ be a state on A, and denote by ϕ^+ the unique state on A^+ such that $\phi^+|_A = \omega$. Then ϕ^+ is pure if and only if ϕ is pure.

Proposition 8.15. Let B be a C^*-subalgebra of A. For each positive linear functional ϕ on B there exists a positive linear functional $\widetilde{\phi}$ on A extending ϕ and such that $\|\widetilde{\phi}\| = \|\phi\|$. Moreover, if ϕ is pure, then there exists a pure extension. Finally, if furthermore B is hereditary, the extension is unique.

Proof. In view of the preceding lemma, we may assume that A and B are unital, and that $1 \in B$. By the Hahn-Banach theorem, there is a linear functional $\widehat{\phi}$ defined on A extending ϕ such that $\|\widehat{\phi}\| = \|\phi\|$. Since $\widehat{\phi}(1) = \phi(1) = \|\phi\| = \|\widehat{\phi}\|$, it follows that $\widehat{\phi}$ is positive.

To prove the second part, assume that B is hereditary and let $(e_\lambda)_{\lambda\in\Lambda}$ be an approximate identity for B. If ψ is a positive linear functional on A extending ϕ with $\|\phi\| = \|\psi\|$, then $\|\psi\| = \lim_\lambda \phi(e_\lambda)$. Hence $\lim_\lambda \psi(1 - e_\lambda) = 0$. It follows that

$$\psi\left((1_A - e_\lambda)^2\right) \leq \psi(1_A - e_\lambda) \to 0.$$

Therefore, for any $c \in A$ we have $|\psi\left((1 - e_\lambda)ac\right)|^2 \leq \psi\left((1 - e_\lambda)^2\right)\psi(c^*a^*ac) \to 0$. Therefore, by taking $c = e_\lambda$, we get that $\psi(a) = \lim_\lambda \psi(e_\lambda a e_\lambda)$. Since B is hereditary and $e_\lambda \in B$, $e_\lambda A e_\lambda \subseteq B$ for all $\lambda \in \Lambda$. Hence

$$\psi(a) = \lim_\lambda \psi(e_\lambda a e_\lambda) = \lim_\lambda \phi(e_\lambda a e_\lambda)$$

for every $a \in A$. Therefore, $\psi = \widehat{\phi}$. $\square$

In short, the idea of the proof is that whenever (e_λ) is an approximate identity for B, then $\psi(a)$ can be obtained from the values that ψ takes in $\cup_\lambda e_\lambda A e_\lambda$ (this is essentially because $\lim_\lambda \psi(e_\lambda) = \psi(1)$). When B is hereditary, $\cup_\lambda e_\lambda A e_\lambda \subseteq B$, where ψ agrees with ϕ. Hence there is only one choice for $\psi(a)$, which must be $\widetilde{\psi}(a)$.

Remark 8.16. If the extension of a pure state is not unique, then not every extension will be pure. For example, if ϕ is pure and μ, ν are distinct extensions, then $\frac{\mu+\nu}{2}$ is a non-pure extension of ϕ.

Problem 8.17. (Kasidon-Singer, 1950's) Let $A = \mathcal{B}(\ell^2(\mathbb{Z}))$ and $B = \ell^\infty(\mathbb{Z})$, embedded in A as multiplicative operators. Let ϕ be a pure state of B. Then it extends to a pure state μ of A. Is the extension unique?

Corollary 8.18. Let $a \in A$ be a normal element. Then there is a state ϕ on A such that $|\phi(a)| = \|a\|$.

Proof. By the preceding proposition, we can assume that A is commutative, by restricting to $C^*(a) \cong C_0(X)$. Let $x_0 \in X$ such that $\|a\| = |a(x_0)|$ and define a Borel probability measure μ by $\mu = \delta_{x_0}$, and define a state $\tau\colon A \to \mathbb{C}$ via $\tau(f) = \int_X f(x)d\mu = f(x_0)$ for $f \in C_0(X)$. Then $|\tau(a)| = |a(x_0)| = \|a\|$. $\square$

9. Basic representation theory for C^*-algebras and the Gelfand-Naimark Theorem.

Recall the following definition given at the beginning of the section.

Definition 9.1. A *representation* of the C^*-algebra A is a pair $(\pi, \mathcal{H})$, where $\pi\colon A \to \mathcal{B}(\mathcal{H})$ is a homomorphism of C^*-algebras. We usually say that π is a representation of A (by bounded operators) on $\mathcal{H}$.
We say that $(\mathcal{H}, \pi)$ is

- *faithful*, if π is injective.
- *non-degenerate*, if $\{\pi(a)x\colon a \in A, x \in \mathcal{H}\}$ is dense in $\mathcal{H}$.
- *cyclic*, if there exists $v \in \mathcal{H}$ such that $\{\pi(a)v\colon a \in A\}$ is dense in $\mathcal{H}$.

Remark 9.2. If π has a cyclic vector, then π is non-degenerate.

An application of Zorn's lemma shows that every non-degenerate representation can be written as the direct sum of cyclic representations.

Definition 9.3. Let A be a $*$-algebra and let $\pi\colon A \to \mathcal{B}(\mathcal{H})$ be a representation. A closed subspace $E \subseteq \mathcal{H}$ is said to be *invariant* if for all $a \in A$ and all $\xi \in E$, we have that $\pi(a)\xi \in E$.

The following proposition is straightforward.

Proposition 9.4. Let A be a $*$-algebra, let $\pi\colon A \to \mathcal{B}(\mathcal{H})$ be a representation and let $E \subseteq \mathcal{H}$ be an inveriant subspace. Then $E^\perp$ is also invariant.

Definition 9.5. A representation $\pi\colon A \to \mathcal{B}(\mathcal{H})$ is said to be *irreducible* if there are no non-trivial closed invariant subspaces. Equivalently, there is no non-trivial direct sum decomposition $\pi = \pi_1 \oplus \pi_2$.

Fact: If A is a C^*-algebra and π is irreducible, then there are no non-trivial, not necessarily closed, invariant subspaces.

Remark 9.6. π is irreducible if and only if every non-zero vector is cyclic.

Lemma 9.7. A closed subspace E is invariant if and only if the orthogonal projection $p \in \mathcal{B}(\mathcal{H})$ onto E commutes with $\pi(a)$ for all $a \in A$.

Proof. This is also straightforward. $\square$

We present a number of examples.

Example 9.8. Let $A = \mathbb{C} \oplus \mathbb{C}$, let $\mathcal{H} = \mathbb{C}^3$, and consider $\pi\colon A \to M_3$ given by $\pi(\lambda_1, \lambda_2) = \mathrm{diag}(\lambda_1, \lambda_1, \lambda_2)$. We can recover the summands associated with the 1-dimensional irreducible representations $\sigma_1(\lambda_2, \lambda_2) = \lambda_1$ and $\sigma_2(\lambda_1, \lambda_2) = \lambda_2$ by $E_1 = \mathrm{im}(\pi(1,0))$ and $E_2 = \mathrm{im}(\pi(0,1))$. These spaces are uniquely determined by π_1 in the sense that if $\pi = \sigma_1 \oplus \sigma_1 \oplus \sigma_2$, then the range of $\sigma_1 \oplus \sigma_1$ must be E_1 and the range of σ_2 must be E_2. Notice that E_2 is irreducible but E_1 is not.

Example 9.9. Let X be any locally compact Hausdorff space, and set $A = C_0(X)$ and $\mathcal{H}_\mu = L^2(X, \mu)$ for some Borel measure μ. Let $\pi_\mu\colon A \to \mathcal{B}(\mathcal{H}_\mu)$ be the multiplication operator.

If μ is the counting measure on a countable set $S \subseteq X$, then $\pi_\mu \cong \oplus_{s\in S}\mathrm{ev}_s$, with no multiplicities. If S is dense, then π_μ is isometric. It is a fact that irreducible representations of $C_0(X)$ correspond to point evaluations.

Let μ be the Lebesgue measure for $X \subseteq \mathbb{R}^m$. The invariant subspaces of π_μ are exactly the subspaces of the form $L^2(Y, \mu)$ for $Y \subseteq X$ measurable. So π_μ has many invariant subspaces, all of them infinite dimensional, except the zero subspace. None of these are irreducible, except for 0. Indeed, if $\mu(E) > 0$, there exist disjoint subsets $E_1, E_2 \subseteq E$ such that $E_1 \cup E_2 = E$ and $\mu(E_1), \mu(E_2) > 0$.

Theorem 9.10. (Gelfand-Naimark-Segal construction) Let A be a C^*-algebra. Then for every state ω on A there is a triple $(\mathcal{H}_\omega, \pi_\omega, \xi_\omega)$ consisting of a Hilbert space $\mathcal{H}_\omega$, a representation π_ω of A on $\mathcal{H}_\omega$, and a cyclic vector ξ_ω such that $\|\xi_\omega\| = 1$ and

$$\omega(a) = \langle \pi_\omega(a)\xi_\omega, \xi_\omega\rangle$$

for all $a \in A$, and such that ξ is a cyclic vector for π, this is, $\overline{\pi(A)\xi} = \mathcal{H}$.

Moreover,

(1) This triple is unique up to unitary equivalence.
(2) The representation π_ω is irreducible if and only if ω is pure.
(3) Every triple $(\mathcal{H}, \pi, \xi)$ as above comes in this way from a state.

Proof. We only sketch the main steps of the proof.

Existence. Define a sesquilinear form $\langle \cdot, \cdot\rangle_\omega$ on A by $\langle a, b\rangle_\omega = \omega(b^*a)$ for $a, b \in A$. Set

$$N_\omega = \{a \in A\colon \langle a, b\rangle_\omega = 0 \text{ for all } b \in A\} = \{a \in A\colon \langle a, a\rangle_\omega = 0\}.$$

Standard methods give a non-degenerate scalar product on A/N_ω. Complete it to get a Hilbert space which we denote $\mathcal{H}_\omega$. The action of A on A/N_ω by left multiplication induces an action of A on $\mathcal{H}_\omega$ which we denote by π_ω. One shows that it is bounded and preserves the adjoints, so that it is a C^*-algebra representation.

If A is unital, the cyclic vector is $\xi_\omega = 1 + N \in \mathcal{H}_\omega$ (notice that $\pi(a)\xi = a + N$ for all $a \in A$, and hence $\overline{\pi(A)\xi} = \mathcal{H}_\omega$). In the general case, if $(e_\lambda)_{\lambda\in\Lambda}$ is an approximate unit for A, then one shows that $(e_\lambda + N)_{\lambda\in\Lambda}$ is a Cauchy net in $\mathcal{H}_\omega$, and its limit ξ_ω is a cyclic vector since $\pi(a)\xi = \lim_\lambda ae_\lambda + N = a + N$ for all $a \in A$.

(1) Uniqueness. Given $(\mathcal{H}_1, \pi_1, \xi_1)$ and $(\mathcal{H}_2, \pi_2, \xi_2)$, define $u\colon \mathcal{H}_1 \to \mathcal{H}_2$ by $u(\pi_1(a)\xi_1)\pi_2(a)\xi_2$. One checks that u is well-defined and it is unitary, and that $u(\xi_1) = \xi_2$.

(2) Uses a proposition below, and its proof is omitted.

(3) Define $\omega(a) = \langle \pi(a)\xi, \xi\rangle$ for $a \in A$. One checks that ω is a state on A and that $(\mathcal{H}_\omega, \pi_\omega, \xi_\omega) = (\mathcal{H}, \pi, \xi)$. $\square$

Theorem 9.11. (Gelfand-Naimark Representation Theorem) Every C^*-algebra has a faithful representation. In other words, every C^*-algebra is (isomorphic to) a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. If A is separable, then $\mathcal{H}$ can be chosen to be separable.

Proof. Let S be a dense subset of A. For $a \in S$, choose a state ω_a on A such that $|\omega_a(a)| = \|a\|$. Take $\pi = \oplus_{a\in S}\pi_{\omega_a}$, and note that

$$\langle \pi_{\omega_a}\xi_{\omega_a}, \xi_{\omega_a}\rangle = |\omega_a(a)| = \|a\| \leq \|\pi_{\omega_a}(a)\|\|\xi_{\omega_a}\|^2 = \|\pi_{\omega_a}(a)\|,$$

so $\|a\| \leq \|\pi_{\omega_a}(a)\|$. That $\|\pi_{\omega_a}(a)\| \leq \|a\|$ is clear. Hence $\|\pi_{\omega_a}(a)\| = \|a\|$ for all $a \in A$, and hence for all $a \in A$.

If A is separable, choose S to be countable. □

Corollary 9.12. Let A be a C^*-algebra. Make $M_n(A)$ into a $*$-algebra in the obvious way. Then there is a unique norm on $M_n(A)$ making it a C^*-algebra.

Proof. Uniqueness was done before. For existence, if A is represented on $\mathcal{H}$, represent $M_n(A)$ on $\mathcal{H}^n$. □

Suppose X is a subset of $\mathcal{B}(\mathcal{H})$. Set $X' = \{a \in \mathcal{B}(\mathcal{H})\colon ax = xa \text{ for all } x \in X\}$. If X is self-adjoint, then X' is a C^*-algebra. It is in fact a von Neumann algebra.

Proposition 9.13. Let A be a $*$-algebra and let $\pi\colon A \to \mathcal{B}(\mathcal{H})$ be a representation. Then π is irreducible if and only if $\pi(A)' = \mathbb{C}\cdot 1_{\mathcal{H}}$.

10. Examples.

10.1. **Algebra of compact operators and Calkin algebra.**

Throughout this section, $\mathcal{H}$ will denote a Hilbert space.

Recall that $T \in \mathcal{B}(\mathcal{H})$ is said to be *compact* if the closure of the image of the closed unit ball is compact. The C^*-algebra of all compact operators in $\mathcal{H}$ is denoted by $\mathcal{K}(\mathcal{H})$ or simply $\mathcal{K}$ (especially when $\mathcal{H}$ is separable). Clearly every finite-rank operator on $\mathcal{H}$ is compact (in this case the closure is not needed!), and we moreover have the following result.

Lemma 10.1. Let $\mathcal{F}(\mathcal{H})$ denote the set of all finite-rank operators on $\mathcal{H}$. Then $\mathcal{K}(\mathcal{H}) = \overline{\mathcal{F}(\mathcal{H})}$.

Proof. It suffices to show that every positive element $0 \leq x \in \mathcal{K}$ is in the closure of $\mathcal{F}(\mathcal{H})$. We use the fact that 0 is the only possible limit point in $\mathrm{sp}(x)$. Hence, there is a sequence $t_n \in (0,1]$ such that $t_n \searrow 0$ and $p_n = f_{t_n}(x)$ are projections. These projections necessarily belong to $\mathcal{K}$ because so does x. Since any compact projection has finite rank, $p_n \in \mathcal{F}(\mathcal{H})$ and moreover $\|p_n x - x\| = \|f_{t_n} - Id_{\mathrm{sp}(x)}\|_\infty \to 0$. Since $\mathcal{F}(\mathcal{H})$ is an algebraic ideal of $\mathcal{B}(\mathcal{H})$, we conclude that $p_n x \in \mathcal{F}(\mathcal{H})$ and $x \in \overline{\mathcal{F}(\mathcal{H})}$. □

Definition 10.2. The *Calkin algebra* of $\mathcal{H}$ is $\mathcal{Q}(\mathcal{H}) = \mathcal{B}(\mathcal{H})/\mathcal{K}$.

The goal of this subsection is to prove that $\mathcal{K}(\mathcal{H})$ and $\mathcal{Q}$ are simple algebras, that is, that they have no non-trivial closed two-sided ideals.

Theorem 10.3. The algebra of compact operators $\mathcal{K}(\mathcal{H})$ is a simple C^*-algebra.

Proof. It suffices to show that if $I_0 \subseteq \mathcal{K}(\mathcal{H})$ is a non-zero algebraic ideal, then $\mathcal{F}(\mathcal{H}) \subseteq I_0$. Let $0 \neq x \in I_0$. Then there is $v \in \mathcal{H}$ with $\|v\| = 1$ such that $x(v) \neq 0$. If $p \in \mathcal{B}(\mathcal{H})$ is a rank one projection, we may write $p(w) = \langle w, e\rangle e$ for some unit vector $e \in \mathcal{H}$. Define $y(w) = \langle w, x(v)\rangle e$ and $z(w) = \langle w, v\rangle v$. Then $y, z \in \mathcal{F}(\mathcal{H})$ and $p = (yx)z(x^*y^*) \in I_0$. Since p is arbitrary, $\mathcal{F}(\mathcal{H}) \subseteq I_0$. □

Theorem 10.4. Let $\mathcal{H}$ be a separable Hilbert space. Then $\mathcal{Q}$ is simple.

Proof. Let $I \lhd \mathcal{B}(\mathcal{H})$ such that $\mathcal{K} \subsetneq I$. Let $0 \leq x \in I$ be a non-compact operator. For any $\varepsilon > 0$ let p_ε be the projection in $C^*(x) \subseteq I$ associated with $\chi_{[\varepsilon,\|x\|]}$. Then $f_{\frac{\varepsilon}{2}} \geq \chi_{[\varepsilon,\|x\|]}$ (because $f_{\frac{\varepsilon}{2}}$ is nonzero between $\frac{\varepsilon}{2}$ and ε). Hence $0 \leq p_\varepsilon \leq f_{\frac{\varepsilon}{2}}$. Since I is an hereditary subalgebra and $f_{\frac{\varepsilon}{2}} \in I$, it follows that $p_\varepsilon \in I$ as well. Moreover, $p_\varepsilon x - x\| < \varepsilon$, and hence p_ϵ is not compact (because otherwise x could be approximated by elements in $\mathcal{K}$). Hence p_ε is a projection onto an infinite-dimensional subspace of $\mathcal{H}$, which must be isomorphic to $\mathcal{H}$ because it is separable. Therefore, there exists an isometry $v \in \mathcal{B}(\mathcal{H})$ such that $v^*v = I_{\mathcal{H}}$ and $vv^* = p$. In particular $v^*p_\varepsilon v = v^*vv^*v = I_{\mathcal{H}} \in I$, and hence $I = \mathcal{B}(\mathcal{H})$. □

10.2. **Group C^*-algebras.**

Definition 10.5. Let G be a locally compact Hausdorff topological group. A *unitary representation* of G on a Hilbert space $\mathcal{H}$ is a strongly continuous homomorphism $\pi\colon G \to \mathcal{U}(\mathcal{H})$. Continuity means that the map $G \to \mathcal{H}$, given by $g \mapsto \pi(g)\xi$ is continuous (for every fixed $\xi \in \mathcal{H}$).

A unitary representation $\pi\colon G \to \mathcal{B}(\mathcal{H})$ is said to be *irreducible* if $\pi(G)$ doesn't commute with any proper projection in $\mathcal{B}(\mathcal{H})$. This is equivalent to asking that $C^*(\pi(G))$ has no non-trivial invariant subspaces.

Definition 10.6. Let $\prod$ be the collection of unitary representations of G. The *group C^*-algebra of G*, denoted by $C^*(G)$, is the closure of the universal representation $\bigoplus_{\pi\in\prod}\pi$ of G.

For every locally compact group G there exists a regular Borel measure μ_G such that $\mu_G(gE) = \mu_G(E)$ for all Borel subsets $E \subseteq G$ and all $g \in G$. It is unique up to scalar multiples, and it is known as the *left Haar measure*. Moreover, μ_G is finite if and only if G is compact, in which case we normalize it so that $\mu_G(G) = 1$. If G is infinite and discrete, then we normalize it so that $\mu_G(\{e\}) = 1$, were e is the unit of the group G.

Examples 10.7. Examples of group C^*-algebras.

(1) If $G = \mathbb{Z}$, then $C^*(\mathbb{Z}) \cong C(S^1)$.
(2) More generally, if G is abelian, then $C^*(G) \cong C(\widehat{G})$.
(3) If F_n is the free group on n generators, then $C^*(F_n)$ is a simple C^*-algebra with a unique tracial state and no non-trivial projections. It is known that $C^*(F_n) \cong C^*(F_m)$ if and only if $n = m$, since $K_1(C^*(F_n)) = \mathbb{Z}^n$. However, if we take their weak closures, it is not known whether $L(F_n) \cong L(F_m)$ for some $n \neq m$.

Theorem 10.8. Let G be a locally compact group. Then the unitary representations of G are in one-to-one correspondence with the non-degenerate representations of $C^*(G)$.

Remark 10.9. Every locally compact Hausdorff group has a distinguished representation, called the *left regular representation* on $L^2(G, \mu_G)$, which is defined by $\lambda_t(f)(s) = f(t^{-1}s)$, for $g, h \in G$ and $f \in L^2(G, \mu_G)$. Then

$$\langle \lambda_t(f), \lambda_t(g)\rangle_{L^2(G)} = \int_G f(t^{-1}s)\overline{g(t^{-1}s)}d\mu_G(s) = \int_G f(r)\overline{g(r)}d\mu_G(r) = \langle f, g\rangle_{L^2(G)}.$$

Hence λ_t is unitary, and one can check that λ is strongly continuous. Hence λ is a unitary representation.

The *reduced group C^*-algebra* is $C^*(\lambda(G))$, and it is denoted by $C^*_r(G)$. There is a surjective homomorphism $C^*(G) \to C^*_r(G)$, because $C^*_r(G)$ is a factor of $C^*(G)$. This surjection is an isomorphism exactly when the group G is *amenable*. (See Definition 10.10 and Theorem 10.13 below.)

Definition 10.10. Let G be a locally compact group. A *mean* on G is a positive linear functional $\Lambda \in \mathrm{Hom}(L^\infty(G), \mathbb{R})$ of norm 1. A mean Λ is said to be *left invariant* (respectively, *right invariant*), if $\Lambda(g \cdot a) = \Lambda(a)$ (respectively, $\Lambda(a \cdot g) = \Lambda(a)$) for all $g \in G, a \in L^\infty(G)$, where the action of G on $L^\infty(G)$ is given by $g \cdot a(h) = a(g^{-1}h)$ for $h \in G$ (respectively, $a \cdot g(h) = a(hg)$ for $h \in G$).

The group G is said to be *amenable* if it admits a left (or right) invariant mean.

Remark 10.11. Suppose that G is discrete. Then G is amenable if and only if there is a *finitely* additive probability measure μ (also called a mean), such that $\mu(gA) = \mu(A)$ for all $g \in G$ and all $A \subseteq G$.

Having a measure μ on G allows us to define integration of bounded functions on G via $f \mapsto \int_G f d\mu$. This is sometimes called *Lebesgue integration*.

Examples 10.12. Some examples of amenable groups.

(1) Finite groups are amenable: use the (normalized) counting measure.
(2) More generally, compact groups are amenable: use the Haar measure.
(3) Subgroups of amenable groups are amenable.
(4) The direct product of finitely many amenable groups is amenable. The infinite direct product of amenable groups need not be amenable.
(5) The group $\mathbb{Z}$ is amenable.

Proof. We will show the existence of a shift invariant, finitely additive probability measure on $\mathbb{Z}$. Let S be the shift operator on $\ell^\infty(\mathbb{Z})$, given by $Sx_j = x_{j+1}$ for $j \in \mathbb{Z}$. Let $u \in \ell^\infty(\mathbb{Z})$ be the constant sequence $u_j = 1$ for all $j \in \mathbb{Z}$. Any element $y \in Y = \mathrm{range}(S - I)$ has a distance larger or equal to 1 from u, since otherwise $y_j = x_{j+1} - x_j$ would be positive an bounded away from 0, hence x_j could not be bounded. This implies that there is a well defined norm one linear functional on the subspace $\mathbb{R} \cdot u + Y$ given by $tu + y \mapsto t$. By the Hahn-Banach theorem, this functional can be extended to a norm-one linear functional on all of $\ell^\infty(\mathbb{Z})$, which is by construction a shift-invariant finitely additive probability measure on $\mathbb{Z}$. $\square$

(6) A group is amenable if and only if all its finitely generated subgroups are.
(7) In particular, any abelian group is amenable.

Theorem 10.13. A locally compact group G is amenable if and only if the natural map $C^*(G) \to C^*_r(G)$ is an isomorphism.

10.3. **Toeplitz and Cuntz algebras, and their generalizations.**

Example 10.14. Fix $n \in \{2, 3, \dots\}$, and choose isometries $s_1, \dots, s_n \in \mathcal{B}(L^2([0,1]))$ such that the range of s_j is $L^2([\frac{j-1}{n}, \frac{j}{n}])$. Denote by $\mathcal{O}_n$ the C^*-algebra generated by $s_1, \dots, s_n$ inside of $\mathcal{B}(L^2([0,1]))$. Then $\mathcal{O}_n$ is simple, purely infinite, separable and traceless. For different values of n, they are not isomorphic since $K_0(\mathcal{O}_n) = \mathbb{Z}_{n-1}$.

Example 10.15. Let $s \in \mathcal{B}(\ell^2(\mathbb{Z}_{>0}))$ be the unilateral shift. Let $\tau = C^*(s) \leq \mathcal{B}(\ell^2(\mathbb{Z}_{>0}))$. Then $\mathcal{K} \subseteq \tau$ is an ideal, and $\tau/\mathcal{K} \cong C(S^1)$. Fact: there is no unitary $u \in \tau$ such that $\pi(u) = \xi$, where $\pi \colon \tau \to C(S^1)$ is the canonical surjection, and $\xi \in C(S^1)$ is the canonical unitary that generates $C(S^1)$.

Example 10.16. Generalizations of the above algebras include Cuntz-Krieger algebras and graph algebras.

11. Inductive limits of C^*-algebras.

Definition 11.1. Let $\mathcal{C}$ be a category and let D be a directed set (this is, for all $\mu, \lambda \in D$ there exists $\nu \in D$ such that $\nu \geq \mu, \lambda$). A *directed system* in $\mathcal{C}$ is a pair of families $((A_\lambda)_{\lambda \in D}, (\varphi_{\lambda,\mu})_{\mu,\lambda \in D, \lambda \geq \mu})$ where $(A_\lambda)_\lambda$ is a family of objects of $\mathcal{C}$ indexed by D and $(\varphi_{\lambda,\mu})_{\mu,\lambda \in D, \lambda \geq \mu}$ is a family of morphisms $\varphi_{\lambda,\mu} \colon A_\mu \to A_\lambda$ such that $\varphi_{\lambda,\mu} \circ \varphi_{\mu,\nu} = \varphi_{\lambda,\nu}$ for all $\lambda \geq \mu \geq \nu$.

The *direct limit* of the directed system $((A_\lambda)_{\lambda \in D}, (\varphi_{\lambda,\mu})_{\mu,\lambda \in D, \lambda \geq \mu})$ is an object $A = \varinjlim A_\lambda$ in $\mathcal{C}$ together with maps $\varphi_{\infty,\lambda} \colon A_\lambda \to A$ for $\lambda \in D$ such that $\varphi_{\infty,\lambda} \circ \varphi_{\lambda,\mu} = \varphi_{\infty,\mu}$ for all $\lambda \geq \mu$, that satisfy the following universal property. Given an object B of $\mathcal{C}$ and morphisms $\psi_\lambda \colon A_\lambda \to B$ for all λ such that $\psi_\lambda \circ \varphi_{\lambda,\mu} = \psi_\mu$ for all $\lambda \geq \mu$, then there exists a unique morphism $\psi \colon A \to B$ such that $\psi \circ \varphi_\lambda = \psi_\lambda$ for all $\lambda \in D$.

$$
\begin{tikzcd}
A_\mu \arrow[r, "\varphi_{\lambda,\mu}"] \arrow[rrd, dashed, "\psi_\mu"'] & A_\lambda \arrow[r, "\varphi_{\infty,\lambda}"] \arrow[rd, dashed, "\psi_\lambda"] & A \arrow[d, dashed, "\psi"] \\
& & B.
\end{tikzcd}
$$

We will usually take $D = \mathbb{Z}_{>0}$, and express the directed system as a pair of sequences $((A_n)_{n \in \mathbb{Z}_{>0}}, (\varphi_n)_{n \in \mathbb{Z}_{>0}})$, where $\varphi_n \colon A_n \to A_{n+1}$ is a morphism. For $n \geq m$, we set $\varphi_{n,m} = \varphi_{n-1} \circ \cdots \circ \varphi_m$.

Direct limits, if they exist, are unique up to unique isomorphism.

Theorem 11.2. Direct limits exist in the categories of abelian groups, groups, R-modules for any ring R, algebras over $\mathbb{C}$ and C^*-algebras.

Proof. We only sketch the proof. Given a directed system (A_λ), define $A = \sqcup_{\lambda \in D} A_\lambda / \sim$, where if $x \in A_\lambda$ and $y \in A_\mu$, we say that $x \sim y$ if there is $\nu \geq \lambda, \mu$ such that $\varphi_{\nu,\lambda}(x) = \varphi_{\nu,\mu}(y)$.

We briefly describe how to define the relevant operations on A. Given $a, b \in A$, choose $\lambda, \mu \in D$ and $x \in A_\lambda, y \in A_\mu$ such that $a = [x]$ and $b = [y]$. Define $a + b = [\varphi_{\nu,\lambda}(x) + \varphi_{\nu,\mu}(y)]$. The operation is well-defined, is associative, has an identity, inverses and is commutative. One defines the other operations analogously.

For $\lambda \in D$, define $\varphi_{\infty,\lambda} \colon A_\lambda \to A$ by $\varphi_{\infty,\lambda}(a) = [a]$. To check the universal property, let B and $(\psi_\lambda)_{\lambda \in D}$ be as in the definition, and define $\psi(\varphi_{\infty,\lambda}(a)) = \psi_\lambda(a)$ for $a \in A_\lambda$. One checks that ψ has the desired property and that it is unique. This finishes the proof for all categories mentioned except for C^*-algebras.

We describe the proof for the category of C^*-algebras. We will assume that $(A_\lambda)_{\lambda \in D}$ is a directed system of C^*-algebras with injective maps. Let $A^{(0)}$ be the algebraic direct limits as $*$-algebras. Define $\|\varphi_{\infty,\lambda}(a)\| = \|a\|$ for $\lambda \in D$ and $a \in A_\lambda$. Since all maps are isometric, this is well-defined. Algebraic arguments show that this really defines a norm and that $(A^{(0)}, \|\cdot\|)$ is a normed $*$-algebra with $\|a^*a\| = \|a\|^2$ for all $a \in A^{(0)}$. Take $\varinjlim A_\lambda = A$ to be the completion of $A(0)$. It remains to check the universal property, but this follows from the corresponding universal property for $A^{(0)}$.

For the general case (this is, when the connecting maps are not necessarily injective), one defines the norm by $\|\varphi_{\infty,\lambda}(a)\| = \lim_{\mu \geq \lambda} \|\varphi_{\mu,\lambda}(a)\|$ and then one completes a suitable quotient of $A^{(0)}$. $\square$

Remark 11.3. In the category of C^*-algebras, $\varinjlim A_\lambda = \overline{\cup_{\lambda \in D} \varphi_{\infty,\lambda}(A_\lambda)}$.

The closure really enlargens a lot. For instance, $M_{2^\infty} = \varinjlim M_{2^n}$ is a Banach space, so it can't have a countable (algebraic) basis, by the Baire category theorem. However, $\cup_{n \in \mathbb{Z}_{>0}} M_{2^n}$ certainly has a countable basis.

Example 11.4. Suppose that A is a C^*-algebra and $A_0 \subseteq A_1 \subseteq \cdots \subseteq A$ are subalgebras such that $\overline{\cup_{n \in \mathbb{Z}_{>0}} A_n} = A$. Then $A \cong \varinjlim A_n$, where the connecting maps are the inclusions.

Theorem 11.5. Let $(A_\lambda)_{\lambda \in D}$ be a directed system of C^*-algebras with injective maps. Let $A = \varinjlim A_\lambda$ and let I be an ideal of A. Then

$$I = \varinjlim \varphi_\lambda^{-1}(I) = \overline{\bigcup_{\lambda \in D} \varphi_{\infty,\lambda}(A) \cap I)}.$$

Proof. We give a rough sketch of the proof. Let $J = \overline{\cup_{\lambda \in D} I \cap \varphi_{\infty,\lambda}(A_\lambda)}$. It suffices to show that $J = I$. Notice that J is an ideal (closure of an algebraic ideal in the direct limit) and that $J \subseteq I$. The canonical map $\psi\colon A/I \to A/J$ is isometric and has dense range, hence is an isomorphism, and it follows from the definition of ψ that in this case, $I = J$. $\square$

Corollary 11.6. If A_λ is simple for all $\lambda \in D$, then $A = \varinjlim A_\lambda$ is simple.

Example 11.7. $\mathbb{C} \to M_2(\mathbb{C}) \to M_4(\mathbb{C}) \to \cdots$ is a directed system, where the map $M_{2^n} \to M_{2^{n+1}}$ is given by $a \mapsto \operatorname{diag}(a, a)$. The direct limit, denoted M_{2^∞}, is called the 2^∞ UHF-algebra, or the CAR algebra. Let's construct a representation on a Hilbert space.

Set $\mathcal{H} = L^2\left(\prod_{n \in \mathbb{Z}_{>0}} \{0,1\}\right)$ with the product of the normalized counting measures. Take the matrix units in M_{2^n} to be the obvious partial isometries from

$$L^2\left(\{x\} \times \prod_{n+1}^{\infty} \{0,1\}\right)$$

to

$$L^2\left(\{y\} \times \prod_{n+1}^{\infty} \{0,1\}\right)$$

for $x, y \in \prod_1^n \{0,1\}$. In this way we may regard M_{2^∞} as a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$.

This is a particular case of a more general construction that we study in the following section.

12. UHF-algebras and their classification.

Definition 12.1. Let $(d_n)_{n \in \mathbb{Z}_{>0}}$ be a sequence in $\mathbb{Z}_{>0}$. The UHF-algebra associated with it is the direct limit of the sequence

$$\mathbb{C} \to M_{d_1} \to M_{d_1 d_2} \to \cdots,$$

where the map $M_{d_1 \cdots d_n} \to M_{d_1 \cdots d_n d_{n+1}}$ is given by $a \mapsto \operatorname{diag}(a, \ldots, a)$ (there are d_{n+1} copies of a) The choice of the homomorphism, as long as it is unital, is irrelevant, as we will see.

These direct limit C^*-algebras are called UHF-algebras of type $d = (d_n)_{n \in \mathbb{Z}_{>0}}$. They are simple and have a unique tracial state. UHF-algebras can be classified up to isomorphism in terms of the sequence $(d_n)_{n \in \mathbb{Z}_{>0}}$ for any choice of unital homomorphisms in the system. The most convenient expression uses K-theory.

Notation 12.2. If $n \in \mathbb{Z}_{>0}$, we denote by M_{n^∞} the UHF-algebra $\varinjlim_m M_{n^m}$.

Lemma 12.3. Let n and m be natural numbers. Then M_{n^∞} and M_{m^∞} are non-isomorphic if $(n, m) = 1$. In fact, there are no non-zero homomorphisms $M_{n^\infty} \to M_{m^\infty}$ whenever n doesn't divide m.

Proof. We just sketch the proof.

$$\begin{tikzcd}
M_{n^r} \arrow[r] \arrow[d, dashed] \arrow[drr, dashed] & \cdots \arrow[r] & M_{n^\infty} \arrow[d, dashed] \\
M_{m^s} \arrow[r] & \cdots \arrow[r] & M_{m^\infty}
\end{tikzcd}$$

Start with a unital map $M_{n^\infty} \to M_{m^\infty}$. Choose any $r \in \mathbb{Z}_{>0}$ and consider the associated unital map $M_{n^s} \to M_{m^\infty}$. Using semiprojectivity of M_{n^s}, lift this map to a unital map $M_{n^s} \to M_{m^r}$ for some $r \in \mathbb{Z}_{>0}$. If n doesn't divide m, then such a map doesn't exist. $\square$

Definition 12.4. Denote by $\mathcal{P}$ the set of all prime numbers. A *supernatural number* if a function $S\colon \mathcal{P} \to \mathbb{Z}_{>0} \cup \{\infty\}$. We call if infinite if $\sum_{p \in \mathcal{P}} S(p) = \infty$, and finite otherwise.

If S is finite, we associate to it the natural number $n = \prod_{p \in \mathcal{P}} p^{S(p)}$.

Definition 12.5. Given supernatural numbers S_1 and S_2, we say that S_1 divides S_2, written S_1/S_2, if $S_1(p) \leq S_2(p)$ for all prime numbers p. Notice that this extends the usual notion of divisibility in $\mathbb{Z}_{>0}$.

Remark 12.6. S_1/S_2 if and only if for all finite supernatural number S such that S/S_1, it follows that S/S_2.

Definition 12.7. Let S be a supernatural number. We say that a UHF-algebra is *of type* S if $A \cong \varinjlim A_n$ with $A_n \cong M_{r(n)}$ for some $r(n) \in \mathbb{Z}_{>0}$ with unital maps, and $S = \sup_n S_n$, with $N_{S_n} = r(n)$ for all $n \in \mathbb{Z}_{>0}$.

Notice that such a system exists if and only if $r(n)/r(n+1)$ for all $n \in \mathbb{Z}_{>0}$.

Remark 12.8. To each sequence $d = (d_n)_{n\in\mathbb{Z}_{>0}}$, associate the function $S_d\colon \mathcal{P} \to \mathbb{Z}_{>0} \cup \{\infty\}$ given by

$$S_d(p) = \sup\{\ell\colon \text{ there is } n \in \mathbb{Z}_{>0} \text{ with } p^\ell/d_1 \cdots d_n\}.$$

Then the UHF-algebra associated to d has type S_d. Notice that it isn't clear that every UHF-algebra has a unique type.

We need several intermediate results.

Lemma 12.9. Let $m, n \in \mathbb{Z}_{>0}$. Then there exists a unital homomorphism$M_n \to M_m$ if and only if n/m.

Proof. If n/m, set $a \mapsto \mathrm{diag}(a, \ldots, a)$. Conversely, assume that $\varphi\colon M_n \to M_m$ is a unital homomorphism. Set $r = \mathrm{rank}(\varphi(e_{1,1}))$. Then $n = \mathrm{rank}(1) = m \cdot r$, so n/m. □

Lemma 12.10. Suppose n/m. Then any two unital homomorphisms $\varphi, \psi\colon M_n \to M_m$ are unitarily equivalent.

Proof. Let $(e_{j,k})_{1\leq j,k\leq n}$ be a system of matrix units for M_n. Then $\varphi(e_{1,1})$ and $\psi(e_{1,1})$ are both projections in M_m with rank m/n, so there exists a partial isometry $s \in M_m$ with $s^*s = \varphi(e_{1,1})$ and $ss^* = \psi(e_{1,1})$. Take $u = \sum_{j=1}^n \psi(e_{j,1})s\varphi(e_{1,j})$. One checks that u is a unitary in M_m, and that $u\varphi(e_{k,l}) = \psi(e_{k,l})u$ for all $1 \leq j, k \leq n$. □

Lemma 12.11. Let A be an UHF-algebra of type S, let $n \in \mathbb{Z}_{>0}$ and let $\varphi\colon M_n \to A$ be a unital homomorphism. Then n/S.

Proof. Lift $M_n \to A$ to a finite stage in the S-decomposition of A. □

Theorem 12.12. Let A and B be UHF-algebras of type S_A and S_B respectively. Then $A \cong B$ if and only if $S_A = S_B$.

In particular, every UHF-algebra has a unique type.

Proof. Suppose $A \cong B$. Then for every $n \in \mathbb{Z}_{>0}$, we have that n/S_A if and only if n/S_B, by the Lemma above. Hence $S_A = S_B$, and in particular the type of a UHF-algebra is well-defined.

Conversely, suppose that A and B have the same type. Write $A = \varinjlim A_n$ and $B = \varinjlim B_n$ with $A_n \cong M_{r(n)}$ and $B_n \cong M_{s(n)}$ satisfying $r(n)/r(n+1)$ and $s(n)/s(n+1)$ for all $n \in \mathbb{Z}_{>0}$. Denote by φ_n and ψ_n the connecting maps. Having the same type means that for all n there exists m such that $r(n)/s(m)$, and that for every m there exists n such that $s(m)/r(n)$.

Construct by induction sequences $(n_k)_{k\in\mathbb{Z}_{>0}}$ and $(m_k)_{k\in\mathbb{Z}_{>0}}$ and homomorphisms $\alpha_k\colon A_{n_k} \to B_{m_k}$ and $\beta_k\colon B_{m_k} \to A_{n_{k+1}}$ such that $\beta_k \circ \alpha_k = \varphi_{n_{k+1},n_k}$ and $\alpha_{k+1} \circ \beta_k = \psi_{m_{k+1},m_k}$ for all $k \in \mathbb{Z}_{>0}$.

$$\begin{tikzcd}
A_{n_1} \arrow[r] \arrow[d, "\alpha_1"'] & A_{n_2} \arrow[r] \arrow[d, "\alpha_2"'] & A_{n_3} \arrow[r] \arrow[d, "\alpha_3"] & \cdots \arrow[r] & A \arrow[d, shift left, "\alpha"] \\
B_{m_1} \arrow[r] \arrow[ur, "\beta_1"] & B_{m_2} \arrow[r] \arrow[ur, "\beta_2"] & B_{m_3} \arrow[r] & \cdots \arrow[r] & B. \arrow[u, shift left, "\beta"]
\end{tikzcd}$$

Using the universal property of direct limits, this will imply that $\alpha \circ \beta = \mathrm{id}_B$ and $\beta \circ \alpha = \mathrm{id}_A$, so A and B are isomorphic.

Take $n_1 = 1$. Find m_1 such that $r(n_1)/s(m_1)$. Then there exists a unital homomorphism $\alpha_1\colon A_{n_1} \to B_{m_1}$. Find n_2 such that $s(m_1)/r(n_2)$. Then there exists a unital homomorphism $\beta_1^{(0)}\colon B_{m_1} \to A_{n_2}$. Now, $\beta_1^{(0)} \circ \alpha_1$ and φ_{n_2,n_1} are unital maps $A_{n_1} \to A_{n_2}$, and by a previous Lemma, they are unitarily equivalent. Let $u_2 \in M_{n_2}$ be the unitary that implements the equivalence. Replace $\beta_1^{(0)}$ by $\beta_1 = \mathrm{Ad}(u) \circ \beta_1^{(0)}$. This gives $\beta_1 \circ \alpha_1 = \varphi_{n_2,n_1}$. Construct the other maps analogously using correcting unitaries. □

Proposition 12.13. If A has supernatural number S and B has supernatural number T, then there exists a unital homomorphism $A \to B$ if and only if S divides T. Moreover, any two such maps are approximately unitarily equivalent.

13. Real rank and stable rank.

13.1. C^*-algebras of lower rank.

Almost all of the known classification results apply to classes of C^*-algebras for which every algebra has either real rank zero or stable rank one. This section contains the definitions of these properties as well as some of its basic properties.

Definition 13.1. Let A be a unital C^*-algebra. Se say that

1. The *real rank of A is zero*, written $RR(A) = 0$, if $\mathrm{Inv}(A)_{sa}$ is dense in A_{sa}.
2. The *topological stable rank of A is one*, written $tsr(A) = 1$, if $\mathrm{Inv}(A)$ is dense in A.

If A is not unital, then the real (topological stable) rank of A is zero (one) if so is that of A^+.

Theorem 13.2. Let A be a C^*-algebra, and suppose that $RR(A) = 0$ (or $tsr(A) = 1$). Then

(1) For every ideal I of A, both I and A/I have real rank zero (topological stable rank one).
(2) For every $n \in \mathbb{Z}_{>0}$, the algebra $M_n(A)$ has real rank zero (topological stable rank one).
(3) Every hereditary subalgebra of A has real rank zero (topological stable rank one).
(4) In particular, every corner of A has real rank zero (topological stable rank one).

Moreover, A is an arbitrary C^*-algebra and p is a projection in $M(A)$ such that both pAp and $(1-p)A(1-p)$ have real rank zero (topological stable rank one), then A itself has real rank zero (topological stable rank one).

Proof. We only sketch the proof.

(a) Is straightforward.

(b) Follows from the fact that if pAp and $(1-p)A(1-p)$ have real rank zero (stable rank one), then so does A. Indeed, for $n = 2$, if we let $p = e_{1,1}$, then $pM_2(A)p$ and $(1-p)M_2(A)(1-p)$ are isomorphic to A. For general n, use induction: $M_n(A)$ has complementary corners A and $M_{n-1}(A)$.

(c) This proof is omitted.

(d) Follows from (c).

The last claim uses the following fact. Write $x \in A$ as $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for $a \in pAp$ and $d \in (1-p)A(1-p)$. If d is invertible in $(1-p)A(1-p)$, then x is invertible in A if and only if $a - db^{-1}c$ is invertible. Moreover, if a is self-adjoint, then d and $a - db^{-1}c$ are self-adjoint as well. Using this, one approximates the diagonals by invertible elements and works out the estimates. ☐

Proposition 13.3. Direct limits of C^*-algebras with real rank zero (topological stable rank one) have real rank zero (topological stable rank one).

Example 13.4. AF-algebras have real rank zero and topological stable rank one.

Topological stable rank was introduced by Rieffel with the purpose of showing that the irrational rotation algebras have cancelation of projections. In fact, topological stable rank is closely related to this property. Indeed, we have the following:

Lemma 13.5. If $tsr(A) = 1$, then A has cancelation of projections: $p \oplus q \sim p \oplus q'$ implies $q \sim q'$, for $p, q, q' \in M_\infty(A)$.

Lemma 13.6. If A is a unital C^*-algebra such that $tsr(A) = 1$, then $\mathcal{U}(A)/\mathcal{U}_0(A) \to K_1(A)$ is surjective. If A is moreover commutative, then it is an isomorphism.

Theorem 13.7. Let A be a unital C^*-algebra. Then the following are equivalent:

(1) $RR(A) = 0$;
(2) Every hereditary subalgebra of A has an increasing countable approximate identity consisting of projections;
(3) The elements of A_{sa} of finite spectrum are dense in A_{sa}.

Theorem 13.8. Let A be a unital C^*-algebra. Then the following are equivalent:

(1) $tsr(A) = 1$;
(2) The elements of $\mathcal{U}_0(A)$ of finite spectrum are dense in $\mathcal{U}_0(A)$.

13.2. **Higher values of real rank and stable rank.**

For every $n \in \mathbb{Z}_{>0} \cup \{\infty\}$, there are definitions of real rank n and topological stable rank n, which we define below. Real rank is supposed to match dimension theory for topological spaces, and real rank is supposed to match what is called the Bass stable rank for unital rings.

Definition 13.9. (Higson) Given a unital C^*-algebra A and given $n \in \mathbb{Z}_{>0}$, we say that A has *real rank no more than n*, written $RR(A) \leq n$, if for every $\varepsilon > 0$ and for all $x_0, \ldots, x_n \in A_{sa}$, there exist $y_0, \ldots, y_n \in A_{sa}$ such that $\sum_{j=0}^n \|x_j - y_j\| < \varepsilon$ and $\sum_{j=0}^n y_j^2$ is invertible.

Moreover, we say that the *real rank of A is n*, written $RR(A) = n$, if $RR(A) \leq n$ and $RR(A) \nleq n-1$. Finally, the real rank of A is infinity, written $RR(A) = \infty$, if $RR(A) \nleq n$ for all $n \in \mathbb{Z}_{>0}$.

Remark 13.10. In the commutative case, the real rank and the dimension theory of the underlying space coincide.

Let X be a compact Hausdorff space and interpret $x_0, \ldots, x_n \in C(X)_{sa}$ as a continuous function $f\colon X \to \mathbb{R}^{n+1}$, and $y_0, \ldots, y_n \in C(X)_{sa}$ with $\sum_{j=0}^n y_j^2$ invertible to be a function $g\colon X \to \mathbb{R}^{n+1}$ that misses 0. Hence, $RR(C(X)) \leq n$ is the statement that any function $X \to \mathbb{R}^{n+1}$ can be perturbed to miss the origin. One eventually gets that $RR(C(X)) = \dim X$, where dim denotes the covering dimension of X.

Example 13.11. We claim that $RR(C([-1,1])) \neq 0$. Take $f(t) = t$ for $t \in [-1,1]$. If g is a real valued continuous function and $\|f-g\| < 1$, then the Intermediate Value Theorem implies that there exists $t_0 \in (-1,1)$ such that $g(t_0) = 0$. It is intuitively clear that $RR(C([-1,1])) \leq 1$, and hence $RR(C([-1,1])) = 1$.

Remark 13.12. If X is a compact Hausdorff space, then $RR(C(X)) = 0$ if and only if X is totally disconnected.

Definition 13.13. (Rieffel) The topological stable rank of a unital C^*-algebra A, denoted $tsr(A)$, is the smallest n such that for every $a_1, \ldots, a_n \in A$ and for every $\varepsilon > 0$, there exist $b_1, \ldots, b_n \in A$ such that $\|b_j - a_j\| < \varepsilon$ and there exist $c_1, \ldots, c_n \in A$ such that $\sum_{j=1}^n c_j b_j = 1$ (equivalently, $\sum_{j=1}^n b_j^* b_j$ is invertible).

There is an entirely algebraic condition for n being the "Bass stable range" of a unital ring R. Moreover, the *Bass stable rank* of a unital ring R, denoted $Bsr(R)$, is the least $n \in \mathbb{Z}_{>0}$ such that m is in the Bass stable range of R for all $m \geq n$.

Theorem 13.14. (Rieffel) For any Banach algebra A, we have $Bsr(A) \leq tsr(A)$.

Theorem 13.15. (Vaserstein) For any unital C^*-algebra A, we have $Bsr(A) = tsr(A)$.

Remark 13.16. In the commutative case, the real rank and the dimension theory of the underlying space are related.

Let X be a compact Hausdorff space and interpret $x_1, \ldots, x_n \in C(X)$ as continuous functions $f\colon X \to \mathbb{C}^n \cong \mathbb{R}^{2n}$, and $y_1, \ldots, y_n \in C(X)$ with $\sum_{j=0}^n y_j^* y_j$ invertible to be a function $g\colon X \to \mathbb{C}^n \cong \mathbb{R}^{2n}$ that misses 0. Hence, $tsr(C(X)) \leq n$ is the statement that any function $X \to \mathbb{R}^{2n}$ can be perturbed to miss the origin. One gets that $2n > \dim X$, and the stable rank of $C(X)$ is the smallest such integer.

In particular, if X is a compact Hausdorff space, then

$$RR(C(X)) = 2tsr(C(X)) - 1.$$

For example, $tsr(C([-1,1])) = 1$.

Proposition 13.17. In general, one has that $RR(A) \leq 2tsr(A) - 1$.

The inequality can be strict.

Example 13.18. Let $\mathcal{H}$ be an infinite dimensional Hilbert space. Then $RR(\mathcal{B}(\mathcal{H})) = 0$ using bounded Borel functional calculus. However, $tsr(\mathcal{B}(\mathcal{H})) = \infty$, which follows from the fact that if s is a non-unitary isometry and $x \in \mathcal{B}(\mathcal{H})$ is such that $\|s - x\| < 1$, then x is not invertible. `How does this help?`.

Theorem 13.19. Let A be a unital C^*-algebra and let $n \in \mathbb{Z}_{>0}$. Then

$$tsr(M_n(A)) = \left\lceil \frac{tsr(A) - 1}{n} \right\rceil + 1.$$

In particular:

(1) If $tsr(A) \geq 2$, then $tsr(M_n(A)) = 2$ for n large enough.
(2) $tsr(A) = 1$ if and only if $tsr(M_n(A)) = 1$ for all n if and only if $tsr(A \otimes \mathcal{K}) = 1$.
(3) $tsr(A) = \infty$ if and only if $tsr(M_n(A)) = \infty$ for all n, although tsr$(A) \geq 2$ implies tsr$(A \otimes \mathcal{K}) = 2$!

There is no formula for the real rank of $M_n(A)$ in terms of the real rank of A and n, except when A is commutative in which case we have

$$RR(M_n(C(X))) = \left\lceil \frac{RRC(X) - 1}{n} \right\rceil + 1.$$

It is conjectured that this formula holds for noncommutative unital C^*-algebras.

On the other hand, $RR(A \otimes \mathcal{K}) = 1$ as long as $RR(A) \geq 1$, and $RR(A) = 0$ if and only if $RR(M_n(A)) = 0$ for all n if and only if $RR(A \otimes \mathcal{K}) = 0$.

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