THE CLASSIFICATION OF KIRCHBERG ALGEBRAS

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Abstract. These are lecture notes of a course given by **Chris Phillips** at the *Thematic Program on Abstract Harmonic Analysis, Banach and Operator Algebras* at the Fields Institute, Toronto, June 9-13, 2014. The last section contains notes from a lecture by **Eberhard Kirchberg** on the same topic.

Warning: little proofreading has been done.

Contents

1. Introduction

This lecture series outlines a proof of the classification of Kirchberg algebras satisfying the Universal Coefficient Theorem (UCT) of Schochet. The theorem was proved independently by Eberhard Kirchberg ([5]) and by Christopher Phillips ([8]) in the late 90's. The two approaches have only a few lemmas in common, but are otherwise quite different. This course focuses on Phillips' proof. The main references are [6] and [8].

Here is the definition of a Kirchberg algebra.

Definition 1.1. A *Kirchberg algebra* is a purely infinite, simple, nuclear, separable C^*-algebra (as far as we know, not necessarily satisfying the UCT).

We give a very short outline of the lecture series:

(1) Exactness implies $\mathcal{O}_2$-embeddable.

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(2) Kirchberg's $\mathcal{O}_2$-absorption theorem: $A \otimes \mathcal{O}_2 \cong \mathcal{O}_2$ for A separable, simple, nuclear and unital.
(3) Kirchberg's $\mathcal{O}_\infty$-absorption theorem: $A \otimes \mathcal{O}_\infty \cong A$ for A separable, simple, nuclear and purely infinite.
(4) Asymptotic morphisms between unital Kirchberg algebras are asymptotically unitarily equivalent if and only if they are homotopic.
(5) An asymptotic morphism between unital Kirchberg algebras is essentially a homomorphism.
(6) Unit respecting KK-equivalence between unital Kirchberg algebras implies isomorphism.

2. Embedding of exact C^*-algebras in $\mathcal{O}_2$

Recall that if A is a unital C^*-algebra, a unital, completely positive linear map $V\colon A \to A$ is said to be *nuclear* if for every $\varepsilon > 0$ and every finite subset $F \subseteq A$, there exist n in $\mathbb{N}$ and unital completely positive maps $T\colon A \to M_n$ and $S\colon M_n \to A$ such that the diagram

$$\begin{tikzcd} A \arrow[rr, "V"] \arrow[dr, "T"'] & & A \\ & M_n \arrow[ur, "S"'] & \end{tikzcd}$$

is approximately commutative, up to an error of ε, on the finite set F. In short, $\|(S \circ T)(a) - V(a)\| < \varepsilon$ for all a in F.

Proposition 2.1. Let A be a unital, purely infinite, simple C^*-algebra and let $V\colon A \to A$ be a unital, completely positive, nuclear map. Given $\varepsilon > 0$ and a finite subset $F \subseteq A$, there exists a proper isometry s in A such that

$$\|s^*as - V(a)\| < \varepsilon$$

for all a in F.

Proof. We can assume that there is an exact factorization

$$\begin{tikzcd} A \arrow[rr, "V"] \arrow[dr, "T"'] & & A \\ & M_n \arrow[ur, "S"'] & \end{tikzcd} .$$

We proceed by proving special cases of increasing degree of generality.

Case 1: $n = 1$.

There exists a state $\omega\colon A \to \mathbb{C}$ such that $V(a) = \omega(a)1$ for all a in A. Then ω is a weak-* limit of pure states, by a result in Diximier's book [3] (it only needs A to be prime and without type I ideals). Such states can be excised (Akerman-Anderson-Petersen): given a finite subset $F \subseteq A$, there is b in A_+ with $\|b\| = 1$ such that

$$\|bab - \omega(a)b^2\| < \frac{\varepsilon}{2}$$

for all a in F. Use real rank zero for A to find a nonzero projection p in A such that $\|pb - p\|$ is small enough so that replacing $\|pbabp - p\omega(a)b^2p\|$ by $\|pap - \omega(a)p\|$

induces an error of at most $\frac{\varepsilon}{2}$. Using pure infiniteness of A, find a nonzero projection $q \neq 1$ with $q \leq p$ such that q is Murray-von Neumann equivalent to 1. The (proper) isometry s inducing this equivalence is the desired isometry.

Case 2: n is arbitrary, and $S\colon M_n \to A$ is a homomorphism (but not necessarily unital).

One reduces to the case $n = 1$ using lots of manipulations with matrices. Among others, one uses the state on $M_n(A)$ given by

$$\tau\left(\sum_{j.k=1}^{n} e_{j,k} \otimes a_{j,k}\right) = \frac{1}{n}\sum_{j.k=1}^{n} \langle T(a_{j,k})\delta_k, \delta_j\rangle,$$

where $\delta_1, \ldots, \delta_n$ are the standard basis vectors in $\mathbb{C}^n$.

General case.

Use methods of Kasparov's generalization of Stinespring's dilation theorem to Hilbert modules, to dilate S to a homomorphism to $M_n(A)$. Since A is purely infinite, this can be stuck back into A. □

For the following proposition, the characterization of exactness that is used is the following: if A is a C^*-algebra, then it is exact if and only if there exist a Hilbert space $\mathcal{H}$ and an injective nuclear map $A \to \mathcal{B}(\mathcal{H})$.

Recall that if A is a unital C^*-algebra and E is a linear subspace of A, then we say that E is an *operator system* if E is self-adjoint and contains the unit of A.

Proposition 2.2. Let A be a separable, unital, exact C^*-algebra, let $E \subseteq A$ be an operator system and let $\varepsilon_0 > 0$. Then for every $\varepsilon > 0$ with $\varepsilon < \varepsilon_0$, there exists n in $\mathbb{N}$ such that whenever B_1 and B_2 are separable unital C^*-algebras with B_2 nuclear, and $V\colon E \to B_1$ and $W\colon E \to B_2$ are unital completely positive maps with V injective and $\|\mathrm{id}_{M_n} \otimes V^{-1}\| < 1 + \varepsilon$, then there exists a unital completely positive map $T\colon B_1 \to B_2$ such that $\|T \circ V - W\| < \varepsilon_0$.

Theorem 2.3. Let A be a separable, unital, exact C^*-algebra, let B be a unital Kirchberg algebra, and let $\varphi, \psi\colon A \to B$ be unital homomorphisms (necessarily injective). Then

$$1_{\mathcal{O}_2} \otimes \varphi, 1_{\mathcal{O}_2} \otimes \psi\colon A \to \mathcal{O}_2 \otimes B$$

are approximately unitarily equivalent.

Before presenting its proof, we give a useful application.

Corollary 2.4. Let A be a separable, unital, exact C^*-algebra, and let $\varphi, \psi\colon A \to \mathcal{O}_2$ be unital homomorphisms. Then φ and ψ are approximately unitarily equivalent.

Proof. The proof is immediate from Elliott's theorem $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$ (see [4]), together with the fact that the isomorphism μ can be chosen so that $a \mapsto \mu(a \otimes 1_{\mathcal{O}_2})$ is approximately unitarily equivalent to $\mathrm{id}_{\mathcal{O}_2}$. □

The proof of Theorem 2.3 uses Proposition 2.1 and Proposition 2.2, as well as the following lemma.

Lemma 2.5. Let A be a unital C^*-algebra, let s and t be proper isometries in A. Suppose that u and v are unitaries in A such that

$$\|s^*us - v\| \quad \text{and} \quad \|t^*vt - u\|$$

are small. Then there is a unitary z in $\mathcal{O}_2 \otimes A$ such that $z(1 \otimes u)z^*$ is close to $1 \otimes v$. Moreover, the unitary z depends on s, t and things in $\mathcal{O}_2 \otimes 1_A$; and in particular does not depend on u or v.

The assumptions of the lemma imply that $\|[ss^*, u]\|$ and $\|[tt^*, v]\|$ are small. Roughly speaking, u looks like a piece of v, and v looks like a piece of u. One would like to conclude that u looks like v, but in general there may be K-theoretic obstructions. Tensoring with $\mathcal{O}_2$ eliminates them, and this is enough. One takes z in $C^*(\mathcal{O}_2 \otimes 1_A, 1_{\mathcal{O}_2} \otimes s, 1_{\mathcal{O}_2} \otimes t)$ to chop up $1 \otimes u$ and rearrange to get $1 \otimes v$.

Proof. (of Theorem 2.3.) Need to show approximately unitary equivalence on finite dimensional operator systems of the form

$$E = \operatorname{span}\{1, u_1, u_1^*, \ldots, u_n, u_n^*\} \subseteq A.$$

Apply Proposition 2.2 with $V = \varphi|_E$ and $W = \psi|_E$ to get a unital completely positive map $S\colon B \to B$ such that $S|_{\psi(E)} \approx (\varphi|_E) \circ (\psi|_E)^{-1}$. Do it again with φ and ψ exchanged to get $T\colon B \to B$ such that $T|_{\varphi(E)} \approx (\psi|_E) \circ (\varphi|_E)^{-1}$.

Use Proposition 2.1 to find proper isometries s and t in B such that

$$\|s^*\varphi(u_j)s - \psi(u_j)\| \quad \text{and} \quad \|t^*\psi(u_j)t - \varphi(u_j)\|$$

are small. Then the unitary z from Lemma 2.5 gives

$$\|z\varphi(u_j)z^* - \psi(u_j)\|$$

small. This is approximate unitary equivalence. □

Proposition 2.6. Let A be a separable, unital, exact C^*-algebra. If there is a unital injective homomorphism

$$\varphi\colon A \to (\mathcal{O}_2)^\infty$$

that lifts to a unital completely positive map $V\colon A \to \ell^\infty(\mathbb{N}, \mathcal{O}_2)$, then there is a unital injective homomorphism $\psi\colon A \to \mathcal{O}_2$.

Proof. Write $V = (V_1, V_2, \ldots)\colon A \to \ell^2(\mathbb{N}, \mathcal{O}_2)$. Choose unitaries u_n for n in $\mathbb{N}$ with dense span in A. For n in $\mathbb{N}$, set

$$E_n = \operatorname{span}\{1, u_1, u_1^*, \ldots, u_n, u_n^*\}.$$

Then E_n is a finite dimensional operator system in A. One needs to "speed up" the multiplicativity of the unital completely positive maps V_n, and one way of doing so is by choosing an sufficiently rapidly increasing sequence $(k(n))_{n\in\mathbb{N}}$, and grouping the maps

$$V_1(a), \ldots, V_{k(1)}(a) \in \bigoplus_{j=1}^{k(1)} \mathcal{O}_2 \hookrightarrow \mathcal{O}_2 \qquad V_{k(1)+1}(a), \ldots, V_{k(2)}(a) \in \bigoplus_{j=1}^{k(2)-k(1)} \mathcal{O}_2 \hookrightarrow \mathcal{O}_2$$

and so on. This "speeding up" trick is needed to apply Proposition 2.2. For fast enough convergence, one will get unital completely positive maps $S_m, T_m\colon \mathcal{O}_2 \to \mathcal{O}_2$ such that

$$S_m \approx V_m \circ V_{m+1}^{-1} \ \text{ on } V_{m+1}(E_m) \quad \text{and} \quad T_m \approx V_{m+1} \circ V_m^{-1} \ \text{ on } V_m(E_m)$$

for all m in $\mathbb{N}$.

Fix m in $\mathbb{N}$. Then for $\ell = 1, \ldots, m$, we have

$$S_m(V_{m+1}(u_\ell)) \approx V_m(u_\ell)$$

and both $V_{m+1}(u_\ell)$ and $V_m(u_\ell)$ are approximately unitaries since V_m and V_{m+1} are almost multiplicative. Use Proposition 2.1 to get a proper isometry s_m such that

$$s_m^* V_{m+1}(u_\ell) s_m \approx V_m(u_\ell)$$

for all $\ell = 1, \ldots, m$. Similarly, find a proper isometry t_m such that

$$t_m^* V_m(u_\ell) t_m \approx V_{m+1}(u_\ell)$$

for all $\ell = 1, \ldots, m$. Replace $V_m(u_\ell)$ and $V_{m+1}(u_\ell)$ with nearby unitaries and use Lemma 2.5 to get a unitary z_m in $\mathcal{O}_2 \otimes \mathcal{O}_2$ such that

$$z_m \left(1_{\mathcal{O}_2} \otimes V_{m+1}(u_\ell)\right) z_m^* \approx 1_{\mathcal{O}_2} \otimes V_m(u_\ell)$$

for all $\ell = 1, \ldots, m$. One can arrange to use an arbitrary sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive real numbers to get

$$\|z_n \left(1_{\mathcal{O}_2} \otimes V_{n+1}|_{E_n}\right) z_n^* - 1_{\mathcal{O}_2} \otimes V_n|_{E_n}\| < \varepsilon_n$$

for all n in $\mathbb{N}$. Define a sequence $(R_n)_{n\in\mathbb{N}}$ of unital completely positive maps $A \to \mathcal{O}_2 \otimes \mathcal{O}_2$ by

$$R_1 = V_1, \quad R_2 = \mathrm{Ad}(z_1) \circ (1_{\mathcal{O}_2} \otimes V_2), \ \ldots, R_n = \mathrm{Ad}(z_1 \cdots z_{n-1}) \circ (1_{\mathcal{O}_2} \otimes V_n).$$

Since the sequence $(V_n)_{n\in\mathbb{N}}$ is asymptotically a homomorphism, it follows that $(R_n)_{n\in\mathbb{N}}$ also is asymptotically a homomorphism. Now, using that $\sum\limits_{n\in\mathbb{N}} \varepsilon_n < \infty$, one shows that for each a in E_m, the sequence $(R_n(a))_{n\geq m}$ is Cauchy in $\mathcal{O}_2 \otimes \mathcal{O}_2$. We conclude that the maps R_n converge pointwise to a map, which is necessarily a homomorphism $A \to \mathcal{O}_2 \otimes \mathcal{O}_2$. This map is easily seen to be unital and injective, using that so is $V\colon A \to \ell^\infty(\mathbb{N}, \mathcal{O}_2)$. By Elliott's theorem ([4]), there is an isomorphism $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$, and this finishes the proof. $\square$

We recall a useful definition.

Definition 2.7. A separable C^*-algebra A is said to be *quasidiagonal* if there exist a separable Hilbert space $\mathcal{H}$, a faithful representation $\pi\colon A \to \mathcal{B}(\mathcal{H})$ and an increasing sequence $(p_n)_{n\in\mathbb{N}}$ of finite rank projections on $\mathcal{H}$ with $p_n \to 1_{\mathcal{H}}$ in the strong operator topology and such that $\lim\limits_{n\to\infty} \|[p_n, \pi(a)]\| = 0$ for all a in A.

What we have done so far allows us to show that exact *quasidiagonal* algebras embed into $\mathcal{O}_2$, as we show below. Notice that Kirchberg algebras are never quasidiagonal. Nevertheless, proving this is needed to show that *every* exact C^*-algebra embeds into $\mathcal{O}_2$, since we will reduce the general statement to the quasidiagonal case. See Theorem 2.9.

Corollary 2.8. Let A be a separable, unital, exact, quasidiagonal C^*-algebra. Then there is a unital embedding of A into $\mathcal{O}_2$.

Proof. We will use Proposition 2.6, so we will produce a unital injective homomorphism $\varphi\colon A \to (\mathcal{O}_2)^\infty$ and a unital completely positive lift $V\colon A \to \ell^\infty(\mathbb{N}, \mathcal{O}_2)$.

Fix a separable Hilbert space $\mathcal{H}$, a faithful unital representation $\pi\colon A \to \mathcal{B}(\mathcal{H})$

and an increasing sequence $(p_n)_{n\in\mathbb{N}}$ of finite rank projections on $\mathcal{H}$ as in Definition 2.7. For n in $\mathbb{N}$, denote $d(n) = \dim(p_n\mathcal{H})$ and let

$$\sigma_n \colon M_{d(n)} \cong p_n\mathcal{B}(\mathcal{H})p_n \to \mathcal{O}_2$$

be a unital embedding. Define a unital completely positive map $V_n \colon A \to \mathcal{O}_2$ by $V_n(a) = \sigma_n(p_n\pi(a)p_n)$ for all a in A. Then

$$V = (V_1, V_2, \ldots) \colon A \to \ell^\infty(\mathbb{N}, \mathcal{O}_2)$$

is clearly a unital completely positive lift of the map $\varphi = \kappa_\infty \circ V \colon A \to \ell^\infty(\mathbb{N}, \mathcal{O}_2) \to (\mathcal{O}_2)^\infty$ it induces. Moreover, φ is a homomorphism since $\lim\limits_{n\to\infty} \|[p_n, \pi(a)]\| = 0$ for all a in A, and it is injective and unital because so is π. The result now follows from Proposition 2.6. $\square$

We are now ready to show that exactness implies $\mathcal{O}_2$-embeddability in full generality.

Theorem 2.9. Let A be a separable, unital, exact C^*-algebra. Then there is a unital embedding of A into $\mathcal{O}_2$.

Proof. Since the classification of Kirchberg algebras will only use this result for nuclear C^*-algebras, we will at some point assume that A is nuclear, since it leads to a significant simplification of the argument.

Set

$$B_0 = C_0([-\infty, \infty), A)^+,$$

which is a unital, exact, separable, quasidiagonal C^*-algebra. Denote by $\tau \colon \mathbb{Z} \to \mathrm{Aut}(B_0$ the action of translation, this is, $\tau_n(f)(t) = f(t-n)$ for n in $\mathbb{N}$, for $f \in B_0$, and for t in $[-\infty, \infty)$. Then $B = B_0 \rtimes_\tau \mathbb{Z}$ contains

$$C_0((-\infty, \infty, A) \rtimes_\tau \mathbb{Z} \cong \mathcal{K} \otimes C(S^1) \otimes A$$

as a subalgebra. If one can embed B into $\mathcal{O}_2$, then

$$A \hookrightarrow \mathcal{K} \otimes C(S^1) \otimes A \hookrightarrow B \hookrightarrow \mathcal{O}_2$$

gives an embedding into $\mathcal{O}_2$, and cutting down by the image p of 1_A in $\mathcal{O}_2$ gives a unital embedding into $p\mathcal{O}_2p \cong \mathcal{O}_2$.

Since B_0 is quasidiagonal, Corollary 2.8 gives a unital injective homomorphism $\varphi_0 \colon B_0 \to \mathcal{O}_2$. Also, $\varphi_0 \circ \tau$ is approximately unitarily equivalent to φ_0 by Theorem 2.3. If we had $\varphi_0 \circ \tau = \mathrm{Ad}(u) \circ \varphi_0$ for some unitary u in $\mathcal{O}_2$, we would then conclude that φ_0 is equivariant with respect to τ and the inner action $\mathrm{Ad}(u)$, and thus

$$B = B_0 \rtimes_\tau \mathbb{Z} \to \mathcal{O}_2 \rtimes_{\mathrm{Ad}(u)} \mathbb{Z} \cong C(S^1) \otimes \mathcal{O}_2 \hookrightarrow \mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$$

would be the desired embedding.

However, we cannot expect such a unitary u to exist in general. Instead, there is a sequence $(u_n)_{n\in\mathbb{N}}$ of unitaries in $\mathcal{O}_2$ such that $\mathrm{Ad}(u_n) \circ \varphi_0$ converges to $\varphi_0 \circ \tau$ pointwise. We *do* get conjugacy in $(\mathcal{O}_2)^\infty$ by considering the unitary $\kappa\left((u_n)_{n\in\mathbb{N}}\right)$. One therefore gets an injective homomorphism

$$B \hookrightarrow C(S^1) \otimes (\mathcal{O}_2)^\infty \hookrightarrow (C(S^1) \otimes \mathcal{O}_2)^\infty \hookrightarrow (\mathcal{O}_2 \otimes \mathcal{O}_2)^\infty \cong (\mathcal{O}_2)^\infty.$$

If we could lift $B \to (\mathcal{O}_2)^\infty$ to a unital completely positive map $B \to \ell^\infty(\mathbb{N}, A)$, then the result would follow from Proposition 2.6. If A is assumed to be nuclear (instead of exact), then B is also nuclear, and hence the unital completely positive lift exists by Choi-Effros lifting result (Corollary 3.11 in [2]).

In the general case when A is exact, considerable additional work is needed, and the proof ultimately relies on a lifting theorem of Effros and Haagerup. $\square$

3. Kirchberg's absorption theorem $A \otimes \mathcal{O}_2 \cong \mathcal{O}_2$

This section and the next one will make extensive use of central sequence algebras, so we begin by defining these algebras.

Let D be a C^*-algebra and let $\mathcal{U} \in \beta\mathbb{N} \setminus \mathbb{N}$ be a free ultrafilter. We set

$$c_0(\mathcal{U}, D) = \left\{ f \in \ell^\infty(\mathbb{N}, D) \colon \lim_{n \to \mathcal{U}} f(n) = 0 \right\}$$

and denote by $D^{\mathcal{U}}$ the quotient $D^{\mathcal{U}} = \ell^\infty(\mathbb{N}, D)/c_0(\mathcal{U}, D)$. There is an embedding $D \hookrightarrow D^{\mathcal{U}}$ as constant sequences. We write $D_{\mathcal{U}}$ for the relative commutant $D_{\mathcal{U}} = D^{\mathcal{U}} \cap D'$.

In the conclusion of the following lemma, it is actually important to know that the isometry one gets is not a unitary, since it will later imply that a certain projection (its range projection) is not the unit; see the claim in the proof of Theorem 3.2. It will essentially imply that $A_{\mathcal{U}}$ is not the complex numbers.

Lemma 3.1. Let A be a unital Kirchberg algebra and let $\mathcal{U} \in \beta\mathbb{N} \setminus \mathbb{N}$ be a free ultrafilter. If a and b are self-adjoint elements in $A_{\mathcal{U}}$ with $\mathrm{sp}(b) \subseteq \mathrm{sp}(a)$, then there exists a nonunitary isometry s in $A_{\mathcal{U}}$ such that $s^*as = b$ and ss^* commutes with a.

Proof. We will give enough details to explain where we need $\mathcal{U}$ to be a *free* ultrafilter. Indeed, the conclusion is false for ∞ in place of $\mathcal{U}$, since $A^\infty \cap A'$ is not simple.

Scale both a and b to get $\|a\|, \|b\| \leq \frac{\pi}{2}$; this can be done without affecting the condition that $\mathrm{sp}(b) \subseteq \mathrm{sp}(a)$. Now set $u = e^{ia}$ and $v = e^{ib}$, which are unitaries in $A_{\mathcal{U}}$. In order to prove the lemma, we need to find a nonunitrary isometry s in $A_{\mathcal{U}}$ with $sus^* = v$ and such that ss^* commutes with u. Set $X = \mathrm{sp}(u)$, which is a closed subset of S^1. Note that $\mathrm{sp}(v) \subseteq X$. Let $z \in C(X)$ be the canonical inclusion $X \hookrightarrow \mathbb{C}$. We get homomorphisms

$$\varphi, \psi \colon C(X, A) \to A \hookrightarrow A^{\mathcal{U}}$$

given by $\varphi(f \otimes a) = f(u)a$ and $\psi(f \otimes a) = f(v)a$ for all f in $C(X)$ and all a in A.

We claim that φ is injective. Since A is simple, $\ker(\varphi)$ must be of the form $C_0(U) \otimes A$ for some open set U in X. However, if U were not the empty set, we would conclude that the spectrum of u is smaller than X, which is a contradiction. Hence φ is injective.

Let

$$V = (V_n)_{n \in \mathbb{N}}, W = (W_n)_{n \in \mathbb{N}} \colon C(X) \otimes A \to \ell^\infty(\mathbb{N}, A)$$

be unital completely positive lifts of φ and ψ, respectively. Let $(u_n)_{n \in \mathbb{N}}$ be a sequence of unitaries in A with dense linear span, and consider the finite dimensional operator systems

$$E_n = \mathrm{span}\{1, z \otimes 1_A, z^* \otimes 1_A, u_1 \otimes 1_{C(X)}, u_1^* \otimes 1_{C(X)}, \ldots, u_n \otimes 1_{C(X)}, u_n^* \otimes 1_{C(X)}\} \subseteq C(X) \otimes A.$$

Fix n and k in $\mathbb{N}$, and fix $\varepsilon > 0$. We claim that there exists m such that $\mathrm{id}_{M_k} \otimes V_m$ is injective on E_n and

$$\left\| \mathrm{id}_{M_k} \otimes \left[V_m|_{V_m^{-1}(E_n)} \right]^{-1} \right\| < 1 + \varepsilon.$$

We have

$$\limsup_{m\to\mathcal{U}} \|(\mathrm{id}_{M_k}\otimes V_m)(x)\| = \|x\|$$

for all $x\in M_k\otimes E_n\subseteq M_k\otimes A$. This also works for ∞ in place of $\mathcal{U}$, but we would like to have

$$\lim_{m\to\mathcal{U}} \|(\mathrm{id}_{M_k}\otimes V_m)(x)\| = \|x\|$$

to prove the claim. This is why we need $\mathcal{U}$ instead of ∞: the key point is that $\lim\limits_{m\to\mathcal{U}} \|(\mathrm{id}_{M_k}\otimes V_m)(x)\|$ *exists*, and therefore it equals its $\limsup$.

Most of the remaining work is a careful choice of tolerances, and it is omitted. $\square$

The following technical result is crucial in the proof of the absorption theorems. This kind of outcome is rare.

Theorem 3.2. Let A be a unital Kirchberg algebra and let $\mathcal{U}\in\beta\mathbb{N}\setminus\mathbb{N}$ be a free ultrafilter. Then $A_{\mathcal{U}}$ is a unital, purely infinite simple C^*-algebra.

Proof. We claim that if B is a nonzero hereditary subalgebra of $A_{\mathcal{U}}$, then there exists a projection e in B with $e\neq 1$ and such that e is Murray-von Neumann equivalent to 1.

Once the claim has been proved, we will have concluded that every nonzero hereditary subalgebra of $A_{\mathcal{U}}$ contains an infinite projection. Moreover, $A_{\mathcal{U}}$ is simple because it contains a projection equivalent to its unit. This implies that $A_{\mathcal{U}}$ is purely infinite.

We prove the claim. Choose $a\in B_{sa}$ with $1\in\mathrm{sp}(a)$, and take $b=1$. Use Lemma 3.1 to find a nonuitary isometry s in $A_{\mathcal{U}}$ as in the lemma, and set $e=ss^*$. Then e is a projection and it commutes with a. One can show that e belongs to B, and it is not the unit because s is not a unitary. $\square$

We recall Elliott's intertwining argument.

Theorem 3.3. Let A and B be separable, unital C^*-algebras, and let $\varphi\colon A\to B$ and $\psi\colon B\to A$ be unital homomorphisms such that $\varphi\circ\psi$ is approximately unitarily equivalent to id_B and $\psi\circ\varphi$ is approximately unitarily equivalent to id_A. Then A is isomorphic to B.

Definition 3.4. Let A and B be separable, unital C^*-algebras. An *approximately central embedding* of A into B is a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of injective unital homomorphisms $\varphi_n\colon A\to B$ such that

$$\lim_{n\to\infty}\|\varphi_n(a)b-b\varphi_n(a)\|=0$$

for all a in A and all b in B.

In the definition above, such an approximately central embedding of A into B gives an injective unital homomorphism $A\to B_\infty = B^\infty\cap B'$.

Lemma 3.5. Let A and B be separable, simple unital C^*-algebras. If there is an approximately central embedding of A into B and A is purely infinite, then B is also purely infinite.

Proof. We know that B is infinite since it has a unital injective homomorphism from an infinite C^*-algebra. Moreover, B has an approximately central embedding of $M_2\oplus M_3$, so results of Blackadar-Kumjian-Rørdam in [1] imply that B is purely infinite. $\square$

Lemma 3.6. Let A be unital, separable, nuclear and simple, and suppose that A has an approximately central embedding of $\mathcal{O}_2$. Then any two unital endomorphisms of A are approximately unitarily equivalent.

Proof. Fix an endomorphism $\gamma\colon A \to A$. We claim that γ is approximately unitarily equivalent to id_A.

By Lemma 3.5, A is purely infinite. We can apply Proposition 2.1 to γ because it is unital and completely positive (since it is a homomorphism) and nuclear (because A is nuclear). We thus get a sequence $(v_n)_{n\in\mathbb{N}}$ of isometries in A such that

$$\lim_{n\to\infty} \|v_n^*\gamma(a)v_n - a\| = 0$$

for all a in A. Let $v = \kappa_{\mathcal{U}}((v_n)_{n\in\mathbb{N}})$, which is an element in $A^{\mathcal{U}}$. Let $e = vv^*$, which is a projection in $A^{\mathcal{U}}$. Note that $v\gamma(a)v^* = a$ for all a in A. In particular, if a is a unitary in A, then so is $\gamma(a)$, and thus $v^*\gamma(a)v$ is a unitary. This implies that vv^* is a unitary (one has to check this, but one cannot cut down a unitary "badly" and get a unitary again). Thus $vv^* = e$ commutes with a. Since a is an arbitrary unitary in A, it follows that

$$e \in A^{\mathcal{U}} \cap A'$$

and $A^{\mathcal{U}} \cap A'$ is purely infinite and simple. The unital homomorphism $\mathcal{O}_2 \to A^{\mathcal{U}} \cap A'$ can be chosen to moreover commute with e (and more generally, with any separable subalgebra of $A^{\mathcal{U}} \cap A'$, using a standard approximation argument with central sequences). One uses this homomorphism to show that $2[e] = [e]$ in $K_0(A^{\mathcal{U}} \cap A')$, so $[e] = 0$. Since $[1] = 0$ as well, we conclude that $[e] = [1]$ in $K_0(A^{\mathcal{U}} \cap A')$. By pure infiniteness, there exists w in $A^{\mathcal{U}} \cap A'$ such that $w^*w = 1$ and $ww^* = e$. Now, $u = w^*v$ is a unitary in $A^{\mathcal{U}}$ and

$$u^*au = v^*waw^*v = v^*av = \gamma(a)$$

for all a in A. With a bit more work, one gets an approximate unitary equivalence between γ and id_A. □

If in the lemma above one replaces nuclearity of A with exactness, one can probably conclude that any two unital *nuclear* endomorphisms of A are approximately unitarily equivalent.

Theorem 3.7. Let A be a simple, nuclear, unital, separable C^*-algebra, and suppose that A has an approximately central embedding of $\mathcal{O}_2$. Then $A \cong \mathcal{O}_2$.

Proof. Since A is exact, there is a unital embedding $A \hookrightarrow \mathcal{O}_2$ by Theorem 2.9. By semiprojectivity of $\mathcal{O}_2$, there is a unital homomorphism $\mathcal{O}_2 \to A$. The composition

$$\varphi\colon \mathcal{O}_2 \to A \to \mathcal{O}_2$$

is a unital injective endomorphism of $\mathcal{O}_2$, so it is approximately unitarily equivalent to $\mathrm{id}_{\mathcal{O}_2}$. Moreover, the composition

$$\psi\colon A \to \mathcal{O}_2 \to A$$

is approximately unitarily equivalent to id_A by Lemma 3.6. Now use Elliott's intertwining argument (Theorem 3.3) to conclude that $A \cong \mathcal{O}_2$. □

Theorem 3.8. Let A be a simple, nuclear, unital, separable C^*-algebra. Then $A \otimes \mathcal{O}_2 \cong \mathcal{O}_2$.

Proof. Set $B = \bigotimes_{n=1}^{\infty} \mathcal{O}_2$. Then B has an approximately central embedding of $\mathcal{O}_2$ (simply as tensor factors), so $B \otimes A$ also has an approximately central embedding of $\mathcal{O}_2$. Theorem 3.7 gives $B \cong \mathcal{O}_2$ and $\mathcal{O}_2 \otimes A \cong B \otimes A \cong \mathcal{O}_2$. □

4. Kirchberg's absorption theorem $A \otimes \mathcal{O}_\infty \cong A$

The proofs in this section will be simplified with respect to what appears in the paper [6]. We recall an earlier result.

Theorem 4.1. (Theorem 3.3 in [7]) Let D be a unital, purely infinite simple C^*-algebra. Then any two unital homomorphisms $\varphi, \psi \colon \mathcal{O}_\infty \to D$ are approximately unitarily equivalent (and actually asymptotically unitarily equivalent).

Theorem 4.2. Let A be a Kirchberg algebra. Then $A \otimes \mathcal{O}_\infty \cong A$.

Proof. We will only prove the unital case. Recall that if $\mathcal{U}$ is a free ultrafilter, then $A^{\mathcal{U}} \cap A'$ is purely infinite and simple, so there there is a unital homomorphism $\beta_0 \colon \mathcal{O}_\infty \to A_{\mathcal{U}}$.

Claim 1: There is a unital homomorphism $\beta \colon \mathcal{O}_\infty \otimes A \to A$ such that $a \mapsto \beta(1 \otimes a)$ is approximately unitarily equivalent to id_A.

Using nuclearity of $\mathcal{O}_\infty$ and Choi-Effros lifting theorem ([2]), lift $\beta_0 \colon \mathcal{O}_\infty \to A_{\mathcal{U}}$ to a unital completely positive map $Q = (Q_1, Q_2, \ldots) \colon \mathcal{O}_\infty \to \ell^\infty(\mathbb{N}, A)$. Choose increasing sequences $(F_n)_{n\in\mathbb{N}}$ and $(G_n)_{n\in\mathbb{N}}$ of finite subsets of A and $\mathcal{O}_\infty$, respectively, with dense unions. Set $\delta_k = \frac{1}{2^k}$ (really only need $\sum_{k\in\mathbb{N}} \delta_k < \infty$).

Claim 1.1: Fix k in $\mathbb{N}$. Then there exist a finite subset $G_k' \subseteq \mathcal{O}_\infty$ and a positive real number $\varepsilon_k > 0$ such that whenever $S, T \colon \mathcal{O}_\infty \to A$ are unital completely positive maps which are ε_k-multiplicative on G_k', with ε_k-commuting ranges on G_k', and whose ranges ε_k commute with F_k on G_k', then there exists a unitary v in A such that

$$\|vS(c)v^* - T(c)\| < \frac{2}{2^k}$$

for al c in G_k, and $\|vav^* - a\| < \frac{1}{2^k}$ for all a in F_k. As a first step in the proof, assume that all relations for S and T are exact and that $G_k' = \mathcal{O}_\infty$. Then $\eta = S \otimes T \colon \mathcal{O}_\infty \otimes \mathcal{O}_\infty \to A$ is a homomorphism whose range commutes with F_k. We then take $v = \eta(v_k)$, and this unitary would yield $\|vS(c)v^* - T(c)\| < \frac{1}{2^k}$ for all c in G_k' and $va = av$ for all a in F_k.

To prove Claim 1.1, assume the conclusion is false. Let $\{x_1, x_2, \ldots\}$ be an enumeration of a dense subset of $\mathcal{O}_\infty$. For each n in $\mathbb{N}$, the choices $G_k' = \{x_1, \ldots, x_n\}$ and $\varepsilon_k = \frac{1}{n}$ yield unital completely positive maps S_n and T_n that do not satisfy the conclusion of the claim. One can assemble these to get unital completely positive maps

$$S, T \colon \mathcal{O}_\infty \to \ell^\infty(\mathbb{N}, A)$$

such that $\kappa_\infty \circ S$ and $\kappa_\infty \circ T$ are homomorphisms $\mathcal{O}_\infty \to A^\infty$. (Note that the ranges of these maps commute with F_k.) Use the exact case $F_k \subseteq A \hookrightarrow A^\infty$ to get $w = \eta(v_k)$. Lift w to a sequence in $\ell^\infty(\mathbb{N}, A)$ consisting of unitaries (one uses semiprojectivity of $C(S^1)$ and a standard perturbation argument). For n large enough, since we mod out by $c_0(\mathbb{N}, A)$, we will have contradicted the assumption that $\{x_1, \ldots, x_n\}$ and $\frac{1}{n}$ yields a counterexample. This proves Claim 1.1.

Recall that $Q = (Q_1, Q_2, \ldots)\colon \mathcal{O}_\infty \to \ell^\infty(\mathbb{N}, A)$ is a unital completely positive lift of $\beta_0\colon \mathcal{O}_\infty \to A_{\mathcal{U}}$. Find a subsequence $(n(k))_{k\in\mathbb{N}}$ so that $Q_{n(k)}$ commutes withim ε_k with $Q_{n(k+1)}$ on G'_{k+1}, as well as with F_k, for all k in $\mathbb{N}$. This gives $Q_{n(k+1)}$ approximately unitarily equivalent, within $\frac{1}{2^k}$ on G_k, to $Q_{n(k)}$. Find unitaries w_k, for k in $\mathbb{N}$, satisfying

$$w_k Q_{n(k+1)}(x) w_k^* \approx Q_{n(k)}(x)$$

for all x in G_k. Consider

$$T_1 = Q_{n(1)}, \quad T_2 = \mathrm{Ad}(w_1) \circ Q_{n(2)}, \quad \ldots, \quad T_k = \mathrm{Ad}(w_1 \cdots w_{k-1}) \circ Q_{n(k)}, \ldots$$

Using that $\sum\limits_{k\in\mathbb{N}} \frac{1}{2^k} < \infty$ and that $G_k \subseteq G'_k \subseteq G'_{k+1}$ for all k in $\mathbb{N}$, one shows that $(T_k)_{k\in\mathbb{N}}$ converges pointwise to a homomorphism $\gamma\colon \mathcal{O}_\infty \to A$. One also gets a homomorphism $\rho\colon A \to A$ given by

$$\rho(a) = \lim_{k\to\infty} \mathrm{Ad}(w_k \cdots w_1)(a)$$

for all a in A. Then the range of ρ commutes with the range of γ. Then $\beta = \gamma \otimes \rho\colon \mathcal{O}_\infty \otimes A \to A$ is the desired unital homomorphism. This finishes the proof of Claim 1.

Denote by $\alpha\colon A \to \mathcal{O}_\infty \otimes A$ the map $a \mapsto 1_{\mathcal{O}_\infty} \otimes a$. By construction, the composition $\beta \circ \alpha$ is approximately unitarily equivalent to id_A. We need the reverse composition to also be equivalent to the identity, and we show how arrange this in the following claim.

Claim 2: The composition $\alpha\circ\beta$ is approximately unitarily equivalent to $\mathrm{id}_{\mathcal{O}_\infty\otimes A}$
It follows from Theorem 4.1 that the maps $\mathcal{O}_\infty \to D = \mathcal{O}_\infty \otimes \mathcal{O}_\infty$ given by $x \mapsto 1_{\mathcal{O}_\infty} \otimes x$ and $x \mapsto x \otimes 1_{\mathcal{O}_\infty}$ are approximately unitarily equivalent, so there is a sequence $(v_k)_{k\in\mathbb{N}}$ of unitaries in $\mathcal{O}_\infty \otimes \mathcal{O}_\infty$ such that

$$\|v_k(x \otimes 1_{\mathcal{O}_\infty})v_k^* - 1_{\mathcal{O}_\infty} \otimes x\| < \frac{1}{2^k}$$

for all x in G_k. We will use the unitaries v_n to "twist" β so as to be able to get an approximate unitary equivalence between $\alpha \circ \beta$ and $\mathrm{id}_{\mathcal{O}_\infty\otimes A}$ from $\beta \circ \alpha \sim \mathrm{id}_A$.

Find a sequence $(w_n)_{n\in\mathbb{N}}$ of unitaries in A (not to be confused with the unitaries w_n used before to prove a previous claim – those unitaries will not be used anymore) implementing the approximate unitary equivalence betwen $\beta \circ \alpha$ and id_A, this is,

$$\lim_{n\to\infty} \|w_n \beta(1 \otimes a) w_n^* - a\| = 0$$

for all a in A.

Claim 2.1 (without proof): There exists a unital homomorphism $\sigma\colon \mathcal{O}_\infty \otimes \mathcal{O}_\infty \to \mathcal{O}_\infty \otimes A$ such that

$$\sigma(x \otimes 1_{\mathcal{O}_\infty}) = 1 \otimes \beta(x \otimes 1_A) \quad \text{and} \quad \sigma(1_{\mathcal{O}_\infty} \otimes x) = x \otimes 1_{\mathcal{O}_\infty}$$

for all x in $\mathcal{O}_\infty$.

Note that $\sigma(c)$ commutes with $1 \otimes \beta(a \otimes 1_{\mathcal{O}_\infty})$ for all c in $\mathcal{O}_\infty \otimes \mathcal{O}_\infty$ and all a in A. For n in $\mathbb{N}$, set

$$u_n = (1 \otimes w_n)\sigma(v_n),$$

which is a unitary in $\mathcal{O}_\infty \otimes A$.

Claim 3.1: One has
$$u_n(\alpha \circ \beta)(c)u_n^* \to c$$
for all c in $\mathcal{O}_\infty \otimes A$.

It is enough to do it for simple tensors of the form $x \otimes 1_A$ and $1_{\mathcal{O}_\infty} \otimes a$. For x in $\mathcal{O}_\infty$, we have
$$\begin{aligned} u_n(1 \otimes \beta(x \otimes 1))u_n^* &= (1 \otimes w_n)\sigma(v_n)\sigma(x \otimes 1)\sigma(v_n^*)(1 \otimes w_n^*) \\ &= (1 \otimes w_n)\sigma(v_n(x \otimes 1)v_n^*)(1 \otimes w_n^*) \\ &\approx (1 \otimes w_n)\sigma(1 \otimes x)(1 \otimes w_n^*) \\ &= (1 \otimes w_n)(x \otimes 1)(1 \otimes w_n^*) \\ &= x \otimes 1. \end{aligned}$$
For a in A, we have

All the claims have been proved now, so we are ready to prove the theorem. The maps $\alpha\colon A \to \mathcal{O}_\infty \otimes A$ and $\beta\colon \mathcal{O}_\infty \otimes A \to A$ satisfy the hypotheses of Elliott's intertwining argument (Theorem 3.3), so we conclude that $\mathcal{O}_\infty \otimes A \cong A$. □

The following alternative approach (based on more recent ideas related to strongly self-absorbing C^*-algebras) would also yield the result. It follows from Theorem 4.1 that the maps $\mathcal{O}_\infty \to \mathcal{O}_\infty \otimes \mathcal{O}_\infty$ given by $a \mapsto 1_{\mathcal{O}_\infty} \otimes a$ and $a \mapsto a \otimes 1_{\mathcal{O}_\infty}$ are approximately unitarily equivalent. It follows that $\mathcal{O}_\infty^{\otimes\infty}$ is strongly self-absorbing. Since there is a unital embedding $\mathcal{O}_\infty \hookrightarrow A_{\mathcal{U}}$, there is also a unital embedding $\mathcal{O}_\infty^{\otimes\infty} \hookrightarrow A_{\mathcal{U}}$. One uses general arguments to show that $A \otimes \mathcal{O}_\infty^{\otimes\infty} \cong A$, and in particular $A \otimes \mathcal{O}_\infty \cong A$ too.

5. Asymptotic morphisms

The main technical difference will be replacing $\ell^\infty(\mathbb{N}, A)$ with $C_b([0,\infty), A)$, and similarly $c_0(\mathbb{N}, A)$ with $C_0([0,\infty), A)$.

Definition 5.1. Let A and B be separable C^*-algebras. An *asymptotic morphism* from A to B, is a family $\varphi = (\varphi_t)_{t\in[0,\infty)}$ of maps $\varphi_t\colon A \to B$, satisfying the following conditions:

(1) For every a in A, the map $[0,\infty) \to B$ given by $t \mapsto \psi_t(a)$ is continuous.
(2) For every λ in $\mathbb{C}$ and every a and b in A, we have
$$\lim_{t\to\infty} \|\varphi_t(\lambda a + b) - \lambda\varphi_t(a) - \varphi_t(b)\| = 0,$$
$$\lim_{t\to\infty} \|\varphi_t(ab) - \varphi_t(a)\varphi_t(b)\| = 0, \quad \text{and} \quad \lim_{t\to\infty} \|\varphi_t(a^*) - \varphi_t(a)^*\| = 0.$$

An asymptotic morphism $\varphi\colon A \to B$ is equivalent to a homomorphism $A \to C_b([0,\infty), B)/C_0([0,\infty), B)$. When A is nuclear, there is always a completely positive contractive lift $A \to C_b([0,\infty), B)$, so the only thing that can fail in general is asymptotic multiplicativity.

Definition 5.2. Two asymptotic morphisms $\varphi^{(0)}, \varphi^{(1)}\colon A \to B$ are said to be *homotopic* if ther are restrictions to endpoints of an asymptotic morphism $s \mapsto \varphi^{(s)}\colon A \to C([0,1], B)$.

We write $[[\varphi]]$ for the homotopy class of an asymptotic morphism $\varphi\colon A \to B$

and we denote by $[[A,B]]$ the set of all homotopy classes of asymptotic morphisms $A \to B$.

Remark 5.3. The algebra

$$C_b([0,\infty), C([0,1]) \otimes B)/C_0([0,\infty), C([0,1]) \otimes B)$$

is much bigger than

$$C([0,1]) \otimes (C_b([0,\infty), B)/C_0([0,\infty), B))\,.$$

We will take as a fact that for A nuclear and B unital, there is a natural isomorphism

$$KK^0(A,B) \cong [[SA, \mathcal{K} \otimes SB]].$$

Taking suspensions really does make a difference. Indeed, if $\varphi\colon A \to B$ is an asymptotic morphism and p is a projection in A, then $\varphi_t(p)$ is approximately a projection for t large enough. Hence $p \mapsto [\varphi_t(p)] \in K_0(B)$ is well-defined (this is, independent of t) for t large enough. We thus get a well defined group homomorphism $K_*(A) \to K_*(B)$. When $A = \mathbb{C}$ and $B = C_0(\mathbb{R}^2)$, there are no projections in B, so the conclusion is that every asymptotic morphism $\mathbb{C} \to C_0(\mathbb{R}^2)$ is asymptotically zero and $[[\mathbb{C}, C_0(\mathbb{R}^2)]] = 0$. On the other hand,

$$KK^0(\mathbb{C}, C_0(\mathbb{R}^2)) \cong \mathbb{Z}$$

and the Bott element is a generator. In particular, $[[\mathbb{C}, C_0(\mathbb{R}^2)]]$ and $[[S\mathbb{C}, SC_0(\mathbb{R}^2)]]$ do not agree.

We will nevertheless see that for Kirchberg algebras, one can really use unsuspended E-theory and still get KK-theory.

Definition 5.4. If φ and ψ are asymptotic morphisms from A to B with B unital, we say that they are *asymptotically unitarily equivalent* if there exists a continuous path $(u_t)_{t\in[0,\infty)}$ of unitaries in B such that

$$\lim_{t\to\infty} \|u_t\varphi_t(a)u_t^* - \psi_t(a)\| = 0$$

for all a in A.

When A and B are unital C^*-algebras, the set $[[SA, \mathcal{K}\otimes SB]]$ is an abelian group with direct sum as addition. Moreover, there is a well defined product

$$[[A,B]] \times [[B,C]] \to [[A,C]]$$

which involves using reparametrizations of asymptotic morphisms (simply taking $t \mapsto \psi_t \circ \varphi_t$ will not work in general).

For a C^*-algebra D, denote by D^+ its minimal unitization (which equals D if it already has a unit) and set

$$D^\sharp = \mathcal{K} \otimes \mathcal{O}_\infty \otimes D.$$

Definition 5.5. Let A and B be C^*-algebras with A unital. An asymptotic morphism $\varphi\colon A \to B$ is said to be *full* if the tail projection $\varphi_t(1)$, for t large enough, is a full projection in B. (When B is simple, this amounts $\varphi_t(1)$ being nonzero.)

For a simple, nuclear, separable unital C^*-algebra A, and a C^*-algebra D, consider the set

$$\big[\big[A, D^\sharp\big]\big]_+$$

of *full* asymptotic morphisms $A \to D^\sharp$. The following is the main result of this section, and uses Kirchberg's stability theorems repeatedly and in very crucial ways. Its proof will be presented in the next section.

Theorem 5.6. Let A be a simple, separable, nuclear, unital C^*-algebra, and let D be a unital C^*-algebra. If $\varphi, \psi \colon A \to \mathcal{K} \otimes \mathcal{O}_\infty \otimes D$ are full asymptotic morphisms, then they are homotopic if and only if they are asymptotically unitarily equivalent.

We will postpone the proof until the next section, and will give some consequences here.

Corollary 5.7. Let A be a simple, separable, nuclear, unital C^*-algebra, and let D be a unital C^*-algebra. Then every full asymptotic morphism $\varphi \colon A \to \mathcal{K} \otimes \mathcal{O}_\infty \otimes D$ is asymptotically unitarily equivalent to a homomorphism.

Proof. It is easy to see that every asymptotic morphism (full or not) is homotopic to any of its reparametrizations. Therefore φ is asymptotically unitarily equivalent to its reparametrizations, by Theorem 5.6. One uses this to find an increasing sequence $(t_n)_{n\in\mathbb{N}}$ of real numbers satisfying $\lim\limits_{n\to\infty} t_n = \infty$, and unitaries u_n in $M(D^\sharp)$ such that

$$\|u_n \varphi_{t_{n+1}}(a) u_n^* - \varphi_{t_n}(a)\| < \varepsilon_n$$

for all a in a finite subset F_n of A, with $F_1 \subseteq F_2 \subseteq \cdots \subseteq A$ and $\bigcup\limits_{n\in\mathbb{N}} F_n$ dense in A, and $\sum\limits_{n\in\mathbb{N}} \varepsilon_n < \infty$. Using an argument similar to the one used in the proof of Theorem 4.2, one gets a homomorphism $\psi \colon A \to D^\sharp$ given by

$$\psi(a) = \lim_{n\to\infty} (\mathrm{Ad}(u_1 \cdots u_{n-1}) \circ \varphi_{t_n})(a)$$

for a in A. With a bit more work, one can show that ψ is asymptotically unitarily equivalent to φ. $\square$

Fix a simple, separable, nuclear, unital C^*-algebra A. For a unital C^*-algebra D, define

$$E_A(D) = [[A, \mathcal{K} \otimes \mathcal{O}_\infty \otimes D]]_+.$$

Then E_A is functorial for unital homomorphisms $D_1 \to D_2$. For general D, define $E_A(D)$ as

$$E_A(D) = \ker\left(E_A(D^+) \to E_A(\mathbb{C})\right),$$

where the unital map $D^+ \to \mathbb{C}$ is the canonical one. One shows that this definition agrees with the one given before when D has a unit. One can show that there is a natural isomorphism

$$E_A(D) \cong KK^0(A, D)$$

and that E_A is functorial for arbitrary homomorphisms $D_1 \to D_2$.

For fixed A as above, one shows that E_A is the zero group of a homotopy theory on C^*-algebras which is stable, has long exact sequences, and is homotopy invariant. One concludes that E_A itself is middle exact. The proof is not too different from similar arguments that show the analogous statement for K_0.

One needs at some point that $\mathcal{K} \otimes \mathcal{O}_\infty$ has a continuously parametrized approximate identity $t \mapsto e_t$ consisting of projections. One can in general not require that $s \leq t$ imply $e_s \leq e_t$. However, one *can* require that $s + 1 \leq t$ imply $e_s \leq e_t$. (Warning: the published paper has a misprint around this part.)

If one has such a description of the functor E_A, Higson's characterization of KK-theory shows that there exists a C^*-algebra B such that

$$E_A = KK(B, \cdot).$$

With some work, one can show that one can tale $A = B$.

Use Higsons's results to get that E_A gives a functor $\widehat{E_A}$ on the category whose objects are separable C^*-algebras, and with morphisms between C_1 and C_2 given by elements in $KK^0(C_1, C_2)$. We also need natural transformations of E_A, and that well-behaved functors extend to natural transformations of $\widehat{E_A}$.

For unital D (can be gotten for general D, but it involves more technicalities), we define natural transformations

$$\alpha_D \colon KK^0(A, D) \to E_A(D) \quad \text{and} \quad \beta_D \colon E_A(D) \to KK^0(A, D)$$

as follows. The transformation β_D is given by the following composition:

$$\beta_D \colon E_A(D) = [[A, \mathcal{K} \otimes \mathcal{O}_\infty \otimes D]]_+ \hookrightarrow [[A, \mathcal{K} \otimes \mathcal{O}_\infty \otimes D]] \longrightarrow [[SA, S(\mathcal{K} \otimes \mathcal{O}_\infty \otimes D)]] \longrightarrow$$

$$\xrightarrow{\cong} KK^0(A, \mathcal{K} \otimes \mathcal{O}_\infty \otimes D) \xrightarrow{\cong} KK(A, D),$$

where the second arrow sends an asymptotic morphism $(\varphi_t)_{t \in [0,\infty)} \colon A \to \mathcal{K} \otimes \mathcal{O}_\infty \otimes D$, to the asymptotic morphism

$$(\mathrm{id}_{C_0(\mathbb{R})} \otimes \varphi_t)_{t \in [0,\infty)} \colon SA \to S(\mathcal{K} \otimes \mathcal{O}_\infty \otimes D).$$

Also, the last isomorphism comes from the fact that $\mathcal{K} \otimes \mathcal{O}_\infty$ is KK-equivalent to $\mathbb{C}$.

Intuitively speaking, the first arrow in the definition of the map β_D picks up some things (precisely, the non-full asymptotic morphisms), while the second arrow collapses some classes. The overall result turns out to be an isomorphism.

To get α_D, consider the inclusion $\iota \colon A \hookrightarrow \mathcal{K} \otimes \mathcal{O}_\infty \otimes A$. This given an element $[[\iota]]$ in $E_A(A)$. Given ηin $KK^0(A, D)$, we get a map

$$\widehat{E_A} \colon E_A(A) \to E_A(D).$$

Finally, we define $\alpha_D \colon KK^0(A, D) \to E_A(D)$ by

$$\alpha_D(\eta) = \widehat{E_A}(\eta)([[\iota]])$$

for all η in $KK^0(A, D)$.

To prove that α_D and β_D are natural inverse transformations, one needs to use more category theory. One looks at where special classes such as $[[\mathrm{id}_A]]$ go, and uses manipulations of several forms. It is convenient, but not necessary, to assume that A is a Kirchberg algebra. This gives us an isomorphism $A \otimes \mathcal{O}_\infty \cong \mathcal{O}_\infty$ by Theorem 4.2, and the fact that classes in $E_A(D)$ come from homomorphisms by Corollary 5.7.

6. Proof of homotopy implies asymptotic unitary equivalence for asymptotic morphisms

Recall the following result of Rørdam.

Theorem 6.1. (Need reference) Let D be a unital C^*-algebra. Then any two unital homomorphisms $\mathcal{O}_2 \to D \otimes \mathcal{O}_\infty$ are approximately unitarily equivalent.

Recall that a unital, separable C^*-algebra D is said to be *strongly self-absorbing* (this is modern terminology, but the notion was around at the time these theorems were proved) if there exists an isomorphism $\varphi\colon D \to D \otimes_{\min} D$ such that $d \mapsto \varphi(1_D \otimes d)$ is approximately unitarily equivalent to id_D.

Corollary 6.2. The Cuntz algebras $\mathcal{O}_2$ and $\mathcal{O}_\infty$ are strongly self-absorbing.

Proof. We prove it for $\mathcal{O}_2$ first. Use Elliott's theorem (see [4]) to choose any isomorphism $\mu\colon \mathcal{O}_2 \otimes \mathcal{O}_2 \to \mathcal{O}_2$. The composition

$$\mathcal{O}_2 \xrightarrow{1_{\mathcal{O}_2} \otimes \mathrm{id}_{\mathcal{O}_2}} \mathcal{O}_2 \otimes \mathcal{O}_2 \xrightarrow{\mu} \mathcal{O}_2$$

is approximately unitarily equivalent to $\mathrm{id}_{\mathcal{O}_2}$ by Theorem 4.1, so the result follows.

For $\mathcal{O}_\infty$, one uses any isomorphism $\mathcal{O}_\infty \otimes \mathcal{O}_\infty \cong \mathcal{O}_\infty$ given by Theorem 4.2, and the corresponding composition is approximately unitarily equivalent to $\mathrm{id}_{\mathcal{O}_\infty}$ by Theorem 6.1. $\square$

Recall the following well-known facts about $D^\sharp = \mathcal{K} \otimes \mathcal{O}_\infty \otimes D$:

(1) $K_0(D^\sharp)$ is the set of Murray-von Neumann equivalence classes of full projections in $D^\sharp$, with addition given by direct sum (or addition of orthogonal representatives).
(2) Any corner of $D^\sharp$ is $\mathcal{O}_\infty$-stable.
(3) The canonical map
$$\mathcal{U}(D^\sharp)/\mathcal{U}_0(D^\sharp) \to K_1(D^\sharp)$$
is an isomorphism.

Proposition 6.3. Any two full asymptotic morphisms $\varphi, \psi\colon \mathcal{O}_2 \to \mathcal{O}_\infty \otimes D$ are asymptotically unitarily equivalent. Similarly, any two full asymptotic morphisms $\varphi, \psi\colon \mathcal{O}_\infty \to \mathcal{O}_\infty \otimes D$ are asymptotically unitarily equivalent.

Proof. We just give a rough idea of the proof. We do it for $\mathcal{O}_2$; the argument for $\mathcal{O}_\infty$ is analogous. One can use semiprojectivity of $\mathcal{O}_2$ to reduce the statement to continuous paths of unital homomorphisms, similarly for $\mathcal{O}_\infty$. Use Lin-Phillips Theorem 4.1, or Rørdam's Theorem 6.1 in the case of $\mathcal{O}_\infty$, on the pair of homomorphisms

$$(\varphi_t)_{t\in[n,n+1]}, (\psi_t)_{t\in[n,n+1]}\colon \mathcal{O}_2 \to C([0,1], \mathcal{O}_\infty \otimes D)$$

with tolerances going to zero as $n \to \infty$. One has to be careful with the gluing of these homomorphisms, so as to make thing match at the integers. We omit the details. $\square$

For the purpose of the following definition, Q will be a separable, unital, nuclear C^*-algebra; usually either $\mathcal{O}_2$ or $\mathcal{O}_\infty$.

Definition 6.4. An asymptotic morphism $A \to B$ is said to have a *standard factorization through* $Q \otimes A$ if it is asymptotically unitarily equivalent to one of the form

$$A \xrightarrow{1_Q \otimes \mathrm{id}_A} Q \otimes A \xrightarrow{(\rho_t)_{t\in[0,\infty)}} B.$$

When $Q = \mathcal{O}_2$, we will call this a *trivializing factorization.*

Lemma 6.5. Any two full asymptotic morphisms $\varphi, \psi\colon A \to D^\sharp$ with trivializing factorizations are asymptotically unitarily equivalent.

Proof. Find trivializing factorizations

$$\varphi\colon\ A \xrightarrow{1_{\mathcal{O}_2}\otimes \mathrm{id}_A} \mathcal{O}_2 \otimes A \xrightarrow{(\rho_t)_{t\in[0,\infty)}} D^\sharp$$
$$\psi\colon\ A \xrightarrow{1_{\mathcal{O}_2}\otimes \mathrm{id}_A} \mathcal{O}_2 \otimes A \xrightarrow{(\sigma_t)_{t\in[0,\infty)}} D^\sharp\ .$$

One can assume that φ and ψ are both unital by cutting down by their tail projections and cutting down to the respective corners, since these are again $\mathcal{O}_\infty$-stable. We use Theorem 3.8 to find an isomorphism $A\otimes\mathcal{O}_2 \cong \mathcal{O}_2$, and use Proposition 6.3 to conclude that $\rho = (\rho_t)_{t\in[0,\infty)}$ and $\sigma = (\sigma_t)_{t\in[0,\infty)}$ are asymptotically unitarily equivalent. Since the first maps in the factorizations of φ and ψ are the same, we conclude that φ and ψ are themselves asymptotically unitarily equivalent. $\square$

We will assume from now on that the tail projection of every (not necessarily full) asymptotic morphism from A is an honest projection (and constant in t). One can always arrange this up to asymptotic unitary equivalence. With this in mind, one can define addition of asymptotic morphisms to $D^\sharp = \mathcal{K}\otimes\mathcal{O}_\infty\otimes D$ by moving tail projections in $\mathcal{K}$ to be orthogonal (and using that $M_2\otimes\mathcal{K}\cong\mathcal{K}$).

Lemma 6.6. Let $\varphi, \psi\colon A \to D^\sharp$ be full asymptotic morphisms. Then $\varphi\oplus\psi$ is full. If moreover they both have trivializing factorizations, then so does $\varphi\oplus\psi$.

Proof. The assumption that both asymptotic morphisms be full is crucial in defining a trivializing factorization for their sum. One gets two tail projections p and q that are Murray-von Neumann equivalent, so that $pD^\sharp p$ is isomorphic to $qD^\sharp q$. The trivializing factorization for $\varphi\oplus\psi$ is the following extension:

$$\begin{tikzcd}
A \arrow[r] & (\mathcal{O}_2\oplus\mathcal{O}_2)\otimes A \arrow[r, "(\rho_t)_{t\in[0,\infty)}\oplus(\sigma_t)_{t\in[0,\infty)}"] \arrow[d, hook] & (p+q)D^\sharp(p+q) \arrow[d, hook] \\
A \arrow[r] & M_2(\mathcal{O}_2)\otimes A \arrow[r] & M_2(D^\sharp)\cong D^\sharp.
\end{tikzcd}$$

$\square$

Proposition 6.7. If A is a Kirchberg algebra, then any full asymptotic morphism $\varphi\colon A\to D^\sharp$ has a standard factorization through $\mathcal{O}_\infty\otimes A$.

Proof. Use Theorem 4.2 to choose an isomorphism $\mu\colon \mathcal{O}_\infty\otimes A\to A$. Consider the composition

$$A \xrightarrow{1_{\mathcal{O}_\infty}\otimes\mathrm{id}_A} \mathcal{O}_\infty\otimes A \xrightarrow{\mu} \xrightarrow{(\varphi_t)_{t\in[0,\infty)}} D^\sharp.$$

We need to show that

$$\mu\circ(1_{\mathcal{O}_\infty}\otimes\mathrm{id}_A)\colon A\to\mathcal{O}_\infty\otimes A\to A$$

is asymptotically unitarily equivalent to id_A. This follows from the fact that the composition

$$\mathcal{O}_\infty \xrightarrow{1_{\mathcal{O}_\infty}\otimes\mathrm{id}_{\mathcal{O}_\infty}} \mathcal{O}_\infty\otimes\mathcal{O}_\infty$$

is not just approximately unitarily equivalent to $\mathrm{id}_{\mathcal{O}_\infty}$, but also asymptotically unitarily equivalent to $\mathrm{id}_{\mathcal{O}_\infty}$, by Theorem 4.1. $\square$

A bit more work in the proof of the above proposition shows that up to asymptotic unitary equivalence, one always has a factorization of the form

$$\varphi\colon\ A \xrightarrow{1_{\mathcal{O}_\infty}\otimes \mathrm{id}_A} \mathcal{O}_\infty \otimes A \xrightarrow{(\mathrm{id}_{\mathcal{O}_\infty}\otimes \rho_t)_{t\in[0,\infty)}} \mathcal{O}_\infty \otimes D^\sharp \xrightarrow{\cong} D^\sharp.$$

Lemma 6.8. If $\nu\colon A \to D^\sharp$ is a full asymptotic morphism with a trivializing factorization, then $\varphi \oplus \nu$ is asymptotically unitarily equivalent to φ for any full asymptotic morphism $\varphi\colon A \to D^\sharp$.

Proof. Denote by e the tail projection of φ, and by f the tail projection of ν. Then $[e] = 1$ and $[f] = 0$ in $K_0(OI)$ and $e + f = 1$. One has the factorization

$$A \to \mathcal{O}_\infty \otimes A \to D^\sharp$$

and $e\mathcal{O}_\infty e \oplus f\mathcal{O}_\infty f \hookrightarrow \mathcal{O}_\infty$ unitally. Note that $\mathcal{O}_2$ embeds unitally into $f\mathcal{O}_\infty f$. Hence φ is the direct sum of a full asymptotic morphism ψ with another full asymptotic morphism λ that has a trivializing factorization. By Lemma 6.6, $\lambda \oplus \nu$ has a trivializing factorization, and by Lemma 6.5, $\lambda \oplus \nu$ and λ are asymptotically unitarily equivalent. With $\sim$ denoting asymptotic unitary equivalence below, we have

$$\varphi \oplus \nu = \psi \oplus \lambda \oplus \nu \sim \psi \oplus \lambda = \varphi,$$

as desired. □

We are now ready to prove Theorem 5.6.

Proof. (of Theorem 5.6) We first describe how homotopy implies approximate unitary equivalence for homomorphisms. Consider a homotopy $s \mapsto \psi^{(s)}$ for $s \in [0, 1]$. Up to asymptotic unitary equivalence, we can find a standard factorization

$$\psi^{(s)}\colon\ A \longrightarrow \mathcal{O}_\infty \otimes A \xrightarrow{\mathrm{id}_{\mathcal{O}_\infty}\otimes \varphi^{(s)}} \mathcal{O}_\infty \otimes D^\sharp \xrightarrow{\cong} D^\sharp.$$

Let F be a finite subset of A and let $\varepsilon > 0$. Choose N in $\mathbb{N}$ and

$$0 = s_0 < s_1 < \cdots < s_N = 1$$

such that $\left\|\varphi^{(s_j)}(a) - \varphi^{(s_{j-1})}(a)\right\| < \varepsilon$ forall a in F and all $j = 1, \ldots, N$.

Choose nonzero projections $e_0, \ldots, e_N, f_1, \ldots, f_N$ in $\mathcal{O}_\infty$ with $\sum\limits_{j=0}^{N} e_j + \sum\limits_{k=1}^{N} f_k = 1_{\mathcal{O}_\infty}$ and such that $[e_j] = 1$ and $[f_k] = -1$ in $K_0(\mathcal{O}_\infty)$ for all $j = 0, \ldots, N$ and all $k = 1, \ldots, N$.

For $j = 0, \ldots, N$ and for $k = 1, \ldots, N$, denote by

$$\iota_j\colon e_j\mathcal{O}_\infty e_j \hookrightarrow \mathcal{O}_\infty \quad \text{and} \quad \gamma_k\colon f_k\mathcal{O}_\infty f_k \hookrightarrow \mathcal{O}_\infty$$

the canonical inclusions. Consider the direct sum of the asymptotic morphisms

$$A \longrightarrow (e_0\mathcal{O}_\infty e_0)\otimes A \xrightarrow{\iota_0\otimes\varphi^{(s_0)}} \mathcal{O}_\infty\otimes D^\sharp \longrightarrow D^\sharp$$

$$A \longrightarrow (f_1\mathcal{O}_\infty f_1)\otimes A \xrightarrow{\gamma_1\otimes\varphi^{(s_1)}} \mathcal{O}_\infty\otimes D^\sharp \longrightarrow D^\sharp$$

$$A \longrightarrow (e_1\mathcal{O}_\infty e_1)\otimes A \xrightarrow{\iota_1\otimes\varphi^{(s_1)}} \mathcal{O}_\infty\otimes D^\sharp \longrightarrow D^\sharp$$

$$\vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \qquad\qquad \vdots$$

$$A \longrightarrow (f_N\mathcal{O}_\infty f_N)\otimes A \xrightarrow{\gamma_N\otimes\varphi^{(s_N)}} \mathcal{O}_\infty\otimes D^\sharp \longrightarrow D^\sharp$$

$$A \longrightarrow (e_N\mathcal{O}_\infty e_N)\otimes A \xrightarrow{\iota_N\otimes\varphi^{(s_N)}} \mathcal{O}_\infty\otimes D^\sharp \longrightarrow D^\sharp.$$

Take the direct sum of the second and third line above, fourth and fifth, and so on. For each of them, we find a unital embedding $\mathcal{O}_2 \hookrightarrow (f_j+e_j)\mathcal{O}_\infty(f_j+e_j)$, which exists since $[f_j+e_j]=0$ in $K_0(\mathcal{O}_\infty)$. We thus get a trivializing factorization for each of these sums, so they all get absorbed by the first line which is the first homomorphism in the homotopy $\psi^{(s_0)}=\psi^{(0)}$. The conclusion is that the whole direct sum is asymptotically unitarily equivalent to $\psi^{(0)}$.

We now consider the direct sum of the asymptotic morphisms

$$A \longrightarrow (e_0\mathcal{O}_\infty e_0)\otimes A \xrightarrow{\iota_0\otimes\varphi^{(s_0)}} \mathcal{O}_\infty\otimes D^\sharp \longrightarrow D^\sharp$$

$$A \longrightarrow (f_1\mathcal{O}_\infty f_1)\otimes A \xrightarrow{\gamma_1\otimes\varphi^{(s_0)}} \mathcal{O}_\infty\otimes D^\sharp \longrightarrow D^\sharp$$

$$\vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \qquad\qquad \vdots$$

$$A \longrightarrow (e_1\mathcal{O}_\infty e_1)\otimes A \xrightarrow{\iota_1\otimes\varphi^{(s_{N-1})}} \mathcal{O}_\infty\otimes D^\sharp \longrightarrow D^\sharp$$

$$A \longrightarrow (f_N\mathcal{O}_\infty f_N)\otimes A \xrightarrow{\gamma_N\otimes\varphi^{(s_{N-1})}} \mathcal{O}_\infty\otimes D^\sharp \longrightarrow D^\sharp$$

$$A \longrightarrow (e_N\mathcal{O}_\infty e_N)\otimes A \xrightarrow{\iota_N\otimes\varphi^{(s_N)}} \mathcal{O}_\infty\otimes D^\sharp \longrightarrow D^\sharp.$$

These changes product an error of at most ε on F. With a similar argument as we used for the other direct sum, we conclude that this one is asymptotically unitarily equivalent to $\psi^{(1)}$.

The conclusion is that the two homomorphisms are asymptotically unitarily equivalent to homomorphisms that are arbitrarily close on a given finite set. This implies approximate unitary equivalence.

Unfortunately, this argument is discrete, so we briefly explain how to modify it to get asymptotic unitary equivalence.

Let $\varepsilon \to 0$ and let F increase to A. Also use $t \mapsto \varphi_t^{(s)}$ and lengths of paths $s \mapsto \varphi_t^{(s)}(a)$ possibly going to ∞. These choices force N to increase.

One splits the projections e_N and f_N and uses homotopies between these in $\mathcal{O}_\infty$. It is crucial that any two projections in a Kirchberg algebra with same class on K_0 have the same "size". We omit the details. $\square$

7. Classification of Kirchberg algebras using KK-theory

This section contains the main results of this lecture series.

Theorem 7.1. Let A and B be unital Kirchberg algebras and let η be an invertible element in $KK^0(A, B)$ such that

$$\eta \times [1_A] = [1_B].$$

Then there is an isomorphism $\theta\colon A \to B$ such that $KK(\theta) = \eta$.

Proof. Recall that there are isomorphisms $A \otimes \mathcal{O}_\infty \cong A$ and $B \otimes \mathcal{O}_\infty \cong B$ by Theorem 4.2. Find full asymptotic morphisms $\varphi\colon A \to B$ and $\psi\colon B \to A$ such that $[[\varphi]] = \eta$ and $[[\psi]] = \eta^{-1}$. By Theorem 5.6 and Corollary **??**, one can assume that φ and ψ are homomorphisms by Corollary 5.7. Then $\varphi \circ \psi\colon B \to A$ is asymptotically unitarily equivalent to id_B, and $\psi \circ \varphi$ is asymptotically unitarily equivalent to id_A. In particular, we have approximate unitary equivalence, and Elliott's intertwining argument (Theorem 3.3) gives us the desired isomorphism $\theta\colon A \to B$. $\square$

There is also a non-unital version of the classification theorem.

Theorem 7.2. Let A and B be stable Kirchberg algebras and let η be an invertible element in $KK^0(A, B)$. Then there is an isomorphism $\theta\colon A \to B$ such that $KK(\theta) = \eta$.

In the presence of the UCT, one can use K-theory instead of KK-theory.

Corollary 7.3. Let A and B be Kirchberg algebras satisfying the UCT. Suppose there is a $\mathbb{Z}_2$-graded group isomorphism

$$\varphi_*\colon K_*(A) \to K_*(B)$$

(with $\varphi_0([1_A]) = [1_B]$ if A and B are unital). Then there is an isomorphism $\theta\colon A \to B$ such that $K_*(\theta) = \varphi$.

Proof. The isomorphism φ_* is an invertible element in $\mathrm{Hom}(K_*(A), K_*(B))$. Using the UCT for the pair (A, B), one can lift it to an element η in $KK^0(A, B)$, which represents φ_* on K-theory. By results of Skandalis, η is invertible. In the unital case, one checks that η respects the classes of the identities. The result then follows from Theorem 7.1 in the unital case, and from Theorem 7.2 in the non-unital case. $\square$

The following question remains open, for more than 25 years now:

Question 7.4. Does every separable, nuclear C^*-algebra satisfy the UCT?

There are known examples of *non-nuclear* C^*-algebras not satisfying the UCT.

8. Outlook: an attempt of unifying the classification of finite and infinite C^*-algebras

Definition 8.1. Let A be a simple, unital, separable C^*-algebra. We say that A is *tracially approximately* $\mathcal{O}_2$, $\mathrm{TA}\mathcal{O}_2$ for short, if for every $\varepsilon > 0$, for every finite subset F of A, and for every positive element x in A with $\|x\| = 1$, there exist a projection p in A and a unital homomorphism

$$\varphi\colon \mathcal{O}_2 \to pAp$$

such that

(a) $\|pa - ap\| < \varepsilon$ for all a in F;
(b) $\mathrm{dist}(pap, \varphi(\mathcal{O}_2)) < \varepsilon$ for all a in F;
(c) $1 - p \precsim x$ (this is vacuous in the purely infinite case);
(d) $\|pxp\| > 1 - \varepsilon$.

Nothing has been done with this definition. One should check the following (and if most come out to be false, the definition should be changed):

(1) Is every $\mathrm{TA}\mathcal{O}_2$ algebra purely infinite?
(2) Is every Kirchberg algebra $\mathrm{TA}\mathcal{O}_2$?
(3) Do $\mathrm{TA}\mathcal{O}_2$ algebras absorb $\mathcal{O}_\infty$?
(4) Do $\mathrm{TA}\mathcal{O}_2$ algebras absorb $\mathcal{Z}$?
(5) Can one use the general machinery of TA classification of Lin (TAF, TAI) to classify $\mathrm{TA}\mathcal{O}_2$ algebras?
(6) For a finite group action on a $\mathrm{TA}\mathcal{O}_2$ algebra, is the tracial Rokhlin property equivalent to pointwise outerness? (This seems to be the case for Kirchberg algebras.)

9. A different point of view using Rørdam groups

This section contains notes from the lecture given by Eberhard Kirchberg.

Theorem 2.9 leads to some sort of unsuspended and "liftable" E-theory. For a separable, exact, stable C^*-algebra A and a σ-unital, stable, strongly purely infinite C^*-algebra B, define the *Rørdam semigroup* $SR(A, B)$ using asymptotic morphisms $V = (V_t)_{t\in[0,\infty)}\colon A \to B$ with V_t nuclear, completely positive and contractive. Equivalence is given by asymptotic unitary equivalence with unitaries taken in the multiplier algebra $M(B)$ of B.

We describe the zero element of $SR(A, B)$. Starting with such A, one finds a nuclear embedding $\varphi_0\colon A \otimes \mathcal{O}_2 \to B$. Then the zero element h_0 of $SR(A, B)$ is defined to be the class of the map $A \to B$ given by $a \mapsto \varphi_0(a \otimes 1_{\mathcal{O}_2})$.

The Grothendieck group $R(A, B)$ of $SR(A, B)$ is called the *Rørdam group* associated to the pair (A, B).

Since this is a special case of E-theory, there is an embedding

$$R(A, B) \to \mathrm{Ext}(A, SB) \cong KK^0(A, B).$$

Denote by $\mathrm{Hom}_{\mathrm{nuc}}(A, B)$ the group of *nuclear* homomorphisms $A \to B$, and by $[\mathrm{Hom}_{\mathrm{nuc}}(A, B)]$ the set of (stabilized) asymptotic unitary equivalence classes of nuclear homomorphisms $A \to B$. The point is to prove that the natural map

$$[\mathrm{Hom}_{\mathrm{nuc}}(A, B)] \to R(A, B)$$

is onto, and that the map $R(A, B) \to \mathrm{Ext}(A, SB)$ is injective. The first statement roughly says that every asymptotic morphism of the kind considered when defining $R(A, B)$ is asymptotically unitarily equivalent to a nuclear homomorphism (compare with Corollary 5.7), and the second statement roughlt says that homotopy implies asymptotic unitary equivalence for such asymptotic morphisms (compare with Theorem 5.6).

We describe an application to classification. For Kirchberg algebras A and B, we have natural isomorphisms

$$[\mathrm{Hom}_{\mathrm{nuc}}(A, B)] \cong [\mathrm{Hom}(A, B)] \cong R(A, B) \cong KK(A, B) \cong \mathrm{Ext}(A, SB).$$

If A and B satisfy the UCT, there is a short exact sequence

$$0 \to \mathrm{Ext}(K_*(A), K_*(B)) \to KK(A, B) \to \mathrm{Hom}(K_*(A), K_*(B)) \to 0$$

and by results of Skandalis, every invertible element in $\mathrm{Hom}(K_*(A), K_*(B))$ lifts to an invertible KK-class in $KK(A, B)$. This class comes from an invertible element in $[\mathrm{Hom}(A, B)]$, so there are homomorphisms $\varphi\colon A \to B$ and $\psi\colon B \to A$ such that $\varphi \circ \psi\colon B \to A$ is asymptotically unitarily equivalent to id_B, and $\psi \circ \varphi$ is asymptotically unitarily equivalent to id_A. In particular, we have approximate unitary equivalence, and Elliott's intertwining argument (Theorem 3.3) gives us an isomorphism $A \cong B$.

Here is another application.

Theorem 9.1. Let A be a separable, exact C^*-algebra. Then there exists a subalgebra B of the CAR algebra M_{2^∞} that contains a hereditary subalgebra D of M_{2^∞} as an ideal, such that $A \cong B/D$.

Now, there is a canonical unital embedding $M_{2^\infty} \hookrightarrow \mathcal{O}_2$, so we may consider $E = D\mathcal{O}_2 D$, which is given by projections. If E contains the unit of M_{2^∞}, then we are done since

$$E + B \subseteq M_{2^\infty} \subseteq \mathcal{O}_2 \quad \text{and} \quad E + B/E \cong A.$$

Otherwise, we have an increasing sequence $(p_n)_{n\in\mathbb{N}}$ of projections and using this one can show that $E \cong \mathcal{O}_2 \otimes \mathcal{K}$. One gets a nuclear injection into the corona

$$A \hookrightarrow M(\mathcal{O}_2 \otimes \mathcal{K})/(\mathcal{O}_2 \otimes \mathcal{K}).$$

The extension defined by this Busby invariant is $E + B$. We really only need that $\mathrm{Ext}(\mathcal{O}_2, \cdot)$ is trivial.

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