CLASSIFYING AMENABLE C*-ALGEBRAS

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Abstract. These are my personal lecture notes, along with some added material, from a course given by **Christopher Schafhauser** at the NSF/CBMS Regional Conference in the Mathematical Sciences held at the Texas Christian University, USA, between June 9th and 13th, 2025.

No proofreading has been done, so there are probably many typos.

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1. Introduction

The classification of simple, separable, nuclear C*-algebras has emerged as one of the central themes in the modern theory of operator algebras. Motivated by the success of Elliott's original classification of AF-algebras using (scaled, ordered) K-theory, the Elliott classification program anticipated that unital, simple, separable nuclear C*-algebras ought to be completely classified, up to isomorphism, by K-theoretic and tracial invariants. This vision has shaped a vast body of work over the past three decades, connecting deep structural properties of C*-algebras with the topology and geometry of noncommutative spaces.

A fundamental insight in the development of the program was the realization that an additional regularity property, namely $\mathcal{Z}$-stability, is essential for such classification result to hold. This regularity condition, which reflects a kind of "tameness" in the internal structure of the C*-algebras in question, now plays a decisive role in distinguishing classifiable from non-classifiable examples.

After decades of work spread over thousands of pages in the literature with at least a few dozen authors directly involved in the proof over this time, the classification program of Elliott is now a theorem:

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Theorem 1.1. *Let A and B be unital, simple, separable, nuclear C*-algebras which are $\mathcal{Z}$-stable and satisfy the UCT. Then $A \cong B$ if and only if $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$.*

While the purely infinite case (namely, when A and B have no traces) was established in the mid 90's independently by Kirchberg and Phillips, the stably finite case was completed in 2015 in work of Elliott-Gong-Lin-Niu. A few years later, a shorter and more conceptual proof was proposed by Carrión-Gabe-Schafhauser-Tikuisis-White, where the focus shifted towards obtaining first a classification of homomorphisms between the algebras in question.

At the heart of the modern approach lie a number of powerful tools: intertwining arguments; techniques from KK-theory; absorption of the Jiang-Su algebra $\mathcal{Z}$ and other regularity propertyes; the trace-kernel extension; and the study of an extended invariant which includes a variation of algebraic K_1. The lecture series reviewed the core arguments, results, and conceptual frameworks that underpin the classification of nuclear C*-algebras, with an emphasis on the themes most relevant to current research.

These notes collect the material presented in these lectures, incorporating some background at some places, both to help my understanding and for the convenience of future readers, if any will exist.

2. Classification of amenable von Neumann factors

The foundations for the advancement of the theory of von Neumann algebras were laid by Murray and von Neumann in their groundbreaking works in the early 1940's. Among other fundamental results, they showed that any von Neumann algebra decomposes as a direct integral (a generalization of a direct sum) of von Neumann algebras with trivial center (also called *factors*). Since many problems about von Neumann algebras can be reduced to the case of a factor, it is important to understand the structure of the latter. Factors can be classified into three types, with corresponding subtypes:

- **Type I**: if there is a nonzero minimal projection.
  - **Type $\mathbf{I}_n$**, for $n \in \mathbb{N}$: if the unit can be written as the sum of n minimal projections.
  - **Type $\mathbf{I}_\infty$**: otherwise.
- **Type II**: if there are no minimal projections and there is a finite projection;
  - **Type $\mathbf{II}_1$**: if the unit is a finite projection.
  - **Type $\mathbf{II}_\infty$**: if the unit is an infinite projection.
- **Type III**: if all nonzero projections are infinite;
  - **Type $\mathbf{III}_\lambda$**, for $0 \leq \lambda \leq 1$ depending on the Connes spectrum.

Factors of type I_n, for $n < \infty$, and of type II_1, always posses a unique normal tracial state.

A key notion in this area is that of *hyperfiniteness*[1]. A groundbreaking result of Connes asserts that for a separably acting von Neumann factor, hyperfiniteness is equivalent to *injectivity*, and also equivalent to *amenability*. Major breakthroughs by Murray and von Neumann, Connes, Haagerup, Krieger, and Popa culminated in the classification of hyperfinite factors:

[1] A von Neumann algebra is said to be *hyperfinite* if it contains an increasing net of finite dimensional subalgebras whose union is dense in the weak operator topology.

Theorem 2.1. There is a unique amenable, separably acting factor of type I_n, for $n \in \mathbb{N}$, I_∞, II_1, II_∞, and III_λ, for $0 < \lambda \leq 1$, while the ones of type III_0 correspond to certain ergodic flows.

Arguably the most special amenable factor is the one of type II_1: this is the hyperfinite factor $\mathcal{R}$, which can be constructed as the von-Neumann direct limit of M_{2^n}, for $n \in \mathbb{N}$, with connecting maps $a \mapsto \mathrm{diag}(a, a)$. (This is the same as the weak closure of the CAR algebra M_{2^∞} in the GNS representation associated to its unique trace.)

For a separably acting injective II_1-factor M, the following properties hold automatically:

(i) M is *hyperfinite*: the identity on M can be approximated, in the point-trace norm topology, by homomorphisms from finite-dimensional algebras.
(ii) M is *McDuff*: M is isomorphic to $M\overline{\otimes}\mathcal{R}$.
(iii) *The order on projections is determined by its (unique) trace τ_M*: for projections $p, q \in M$, we have $p \precsim_{\mathrm{MvN}} q$ if and only if $\tau_M(p) \leq \tau_M(q)$.

C*-algebraic analogs of these properties will be studied later in the context of the Toms-Winter regularity conjecture.

3. The classification theorem

It was George Elliott who first predicted that there should be a version of the classification of amenable factors for C*-algebras, and his suspicion was formalized as a conjecture which saw both great success and some reformulations to account both for major technical difficulties when dealing with K-theory (thus incorporating the assumption of the UCT, which may actually follow from nuclearity), as well as for counterexamples (thus incorporating the assumption of $\mathcal{Z}$-stability). After decades of work, this is now a theorem:

Theorem 3.1. (Classification; many authors.) Unital, simple, separable, nuclear C*-algebras which are $\mathcal{Z}$-stable and satisfy the UCT are classified by K-theory, traces, and their pairing.

More explicitly, for A and B satisfying said assumptions, we have $A \cong B$ if and only if there are group isomorphisms $\alpha_j \colon K_j(A) \to K_j(B)$, for $j = 0, 1$, and an affice homeomorphism $\gamma \colon T(B) \to T(A)$, such that $\alpha_0([1_A]) = [1_B]$ and $\gamma(\tau)(x) = \tau(\alpha_0(x))$ for all $x \in K_0(A)$. Moreover, any tuple $(\alpha_0, \alpha_1, \gamma)$ as above, is realized by some isomorphism $\varphi \colon A \to B$, in the sense that $\alpha_j = K_j(\varphi)$ and $\gamma = T(\varphi)$.

One of the primary sources of examples of amenable von Neumann factors is given by group actions on probability measure spaces:

Example 3.2. Let a countable group Γ act via probability-measure-preserving transformations on the (atomless) standard probability space (X, μ). If $\Gamma \curvearrowright (X, \mu)$ is free and ergodic, then the crossed product $L^\infty(X, \mu)\overline{\rtimes}\Gamma$ is an amenable factor of type II_1. In particular, the isomorphism type of $L^\infty(X, \mu)\overline{\rtimes}\Gamma$ does not depend on Γ or the action; it is always isomorphic to $\mathcal{R}$.

Generalizing this family of examples is its own independent line of research. We describe the motivating setup:

Example 3.3. Let a countable group Γ act via homeomorphisms on a compact metric space X. If $\Gamma \curvearrowright (X, \mu)$ is topologically free and minimal, then the crossed product $C(X) \rtimes \Gamma$ is a unital, simple, separable, nuclear C*-algebra satisfying the UCT. Very often this crossed product is $\mathcal{Z}$-stable (for example, whenever $\dim(X) < \infty$ and Γ is elementary amenable, by work of Kerr-Naryshkin), but not in generality (by examples of Giol-Kerr).

As a particularly famous concrete example, we focus on the irrational rotation algebras.

Example 3.4. Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$, and let the associated rotation algebra be denoted by A_θ. (This is the universal C*-algebra generated by unitaries u and v satisfying $uv = e^{2\pi i\theta} vu$. Equivalently, this is the crossed product of the action of $\mathbb{Z}$ on S^1 by rotation by angle $2\pi\theta$.) Then A_θ is classifiable, and its invariant can be computed. First, A_θ as a unique trace τ, induced by the normalized Lebesgue measure on S^1 (which is the unique invariant Borel probability measure on S^1). Its K-theory can be computed using the Pimsner-Voiculescu exact sequence, and yields:

$$K_0(A_\theta) = \langle 1, p\rangle \cong \mathbb{Z}^2, \quad \text{and} \quad K_1(A_\theta) = \langle u, v\rangle \cong \mathbb{Z}^2,$$

where $p \in A_\theta$ is a projection satisfying $\tau(p) = \theta$. For $\theta, \theta' \in \mathbb{R} \setminus \mathbb{Q}$, the only way to distinguish A_θ from $A_{\theta'}$ is with the pairing, which gives

$$\tau(A_\theta) = \mathbb{Z} + \theta\mathbb{Z}.$$

As a consequence, we get $A_\theta \cong A_{\theta'}$ if and only if $\theta = \pm\theta' \bmod \mathbb{Z}$, or equivalently, if and only if the associated rotations are conjugate. This is not quite a consequence of the classification theorem, since what the theorem is most useful for is to construct an isomorphism between the algebras given an isomorphism between the invariants. This is actually very easy in the case of irrational rotation algebras, and what was actually challenging in this setting was to find invariants that dinstinguish them.

Notation 3.5. For a C*-algebra A, we write $\rho_A \colon K_0(A) \times T(A) \to \mathbb{R}$ for the canonical pairing, and we will abbreviate the invariant $(K_0(A), [1_A], K_1(A), T(A), \rho_A)$ used in Theorem 3.1 as $\mathrm{KT_u}(A)$.

Remark 3.6. The classical Elliott invariant also incorporates the positive cone $K_0(A)_+ \subseteq K_0(A)$ as part of the invariant. However, for unital, simple, separable, nuclear $\mathcal{Z}$-stable C*-algebras, this is encoded in $\mathrm{KT_u}(A)$ as

$$K_0(A)_+ = \{x \in K_0(A) \colon \rho_A(x, \tau) > 0 \text{ for all } \tau \in T(A)\} \cup \{0\}.$$

The range of the invariant in Theorem 3.1 is also known: it is as large as possible:

Theorem 3.7. For any separable, unital C*-algebra B, there is a unital, simple, separable, nuclear $\mathcal{Z}$-stable C*-algebra A satisfying the UCT such that $\mathrm{KT_u}(A) \cong \mathrm{KT_u}(B)$. Moreover, A is unique up to isomorphism.

The range result also comes with concrete and tractable models constructed in the stably finite case as certain direct limits of subhomogeneous C*-algebras, and in the purely infinite case from certain ample groupoids. This has interesting consequences, such as the following two results.

Theorem 3.8. (X. Li). Every classifiable C*-algebra has a Cartan subalgebra, and thus admits a presentation as a twisted groupoid C*-algebra.

Theorem 3.9. (Spielberg, H. Li) A UCT Kirchberg algebra is weakly semiprojective if and only if its K-theory groups are direct sums of cyclic groups.

The modern approach to the classifiation theorem consists in obtaining first a classification of embeddings. Again, this is motivated by results from von Neumann algebras:

Theorem 3.10. Let M be a separably acting amenable von Neumann algebra and let N be a II_1-factor.

(!) If $\varphi, \psi \colon M \to N$ are normal and unital, and satisfy $\tau_M \circ \varphi = \tau_M \circ \psi$, then there exists a sequence $(u_n)_{n\in\mathbb{N}}$ of unitaries in N such that $u_n\varphi(a)u_n^* \to \psi(a)$ in the *trace norm* for all $a \in M$.

($\exists$) If τ_M is a normal trace on M, then there exists a unital normal homomorphism $\varphi \colon M \to N$ such that $\tau_N \circ \varphi = \tau_M$.

As it turns out, there is by now a complete C*-algebraic analog of this result. Recall that if B is a unital C*-algebra and A is a separable C*-algebra, two homomorphisms $\varphi, \psi \colon A \to B$ are said to be *approximately unitarily equivalent*, written $\varphi \approx_u \psi$, if there exist a sequence $(u_n)_{n\in\mathbb{N}}$ of unitaries in B such that $u_n\varphi(a)u_n^* \to \psi(a)$ in norm for all $a \in A$. Similarly, we say that φ and ψ are *unitarily equivalent*, written $\varphi \sim_u \psi$, if there exists a unitary in B such that $\mathrm{Ad}(u) \circ \psi = \varphi$.

Theorem 3.11. Let A be a unital, separable, nuclear C*-algebra satisfying the UCT, and let B be a unital, simple, $\mathcal{Z}$-stable C*-algebra with $QT(B) = T(B)$ (this is automatic if B is exact). Then unital embeddings $A \hookrightarrow B$ are classified up to approximate unitary equivalence by the total invariant:

$$\underline{\mathrm{K}}\mathrm{T_u}(A) = \big(K_0(A), [1_A], K_1(A), \overline{K}_1^{\mathrm{alg}}(A), K_*(\cdot, \mathbb{Z}_n), T(A), \rho_A\big).$$

The main techniques that go into the proof of Theorem 3.11 will be discussed in subsequent sections, as well as how said theorem can be used to prove Theorem 3.1.

4. Elliott's intertwining argument

In order to prove that an isomorphism between two C*-algebras exists, it is often enough to prove something weaker, namely that there exist homomorphisms in both directions that are mutual inverses up to approximate unitary equivalence. This is the content of Elliott's intertwining argument:

Theorem 4.1. (Elliott's intertwining). Let A and B be unital, separable C*-algebras. Then $A \cong B$ if and only if there exist homomorphisms $\varphi \colon A \to B$ and $\psi \colon B \to A$ such that $\psi \circ \varphi \approx_u \mathrm{id}_A$ and $\varphi \circ \psi \approx_u \mathrm{id}_B$.

In fact, given φ and ψ as above, there exists an isomorphism $\Phi \colon A \to B$ with $\Phi \approx_u \varphi$ and $\Phi^{-1} \approx_u \psi$.

Proof. Set $\varphi_1 = \varphi$, and note that $\psi \circ \varphi \approx_u \mathrm{id}_A$. For a fixed finite subset $F_1^A \subseteq A$ and $\varepsilon_1 > 0$, there exists a unitary $u_1 \in A$ such that

$$\|u_1\psi(\varphi_1(a))u_1^* - a\| < \varepsilon_1$$

for all $a \in F_1^A$. Set $\psi_1 = \mathrm{Ad}(u_1) \circ \psi$. Then

$$\varphi \circ \psi_1 = \varphi \circ \mathrm{Ad}(u_1) \circ \psi = \mathrm{Ad}(\varphi(u_1)) \circ \varphi \circ \psi \approx_u \varphi \circ \psi \approx_u \mathrm{id}_B.$$

Thus, for a fixed finite subset $F_1^B \subseteq B$, there exists a unitary $v_1 \in B$ such that

$$\|b_1\varphi(\psi_1(b))v_1^* - b\| < \varepsilon_1$$

for all $b \in F_1^B$. Set $\varphi_2 = \mathrm{Ad}(v_1) \circ \varphi_1 \colon A \to B$. Note that $\psi_1 \circ \varphi_2 \approx_u \mathrm{id}_A$. Applying a similar argument, for a finite subset $F_2^A \subseteq A$ and $\varepsilon_2 > 0$, there exists a unitary $u_2 \in A$ such that

$$\|u_2 \psi_1(\varphi_2(a)) u_2^* - a\| < \varepsilon_2$$

for all $a \in F_2^A$. Set $\psi_2 = \mathrm{Ad}(u_2) \circ \psi_1 \colon B \to A$.

Proceeding inductively, given sequences $(F_n^A)_{n\in\mathbb{N}}$ and $(F_n^B)_{n\in\mathbb{N}}$ of finite subsets of A and B, respectively, and a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive real numbers, we get corresponding homomorphisms $\varphi_n \colon A \to B$ and $\psi_n \colon B \to A$ such that $\varphi_{n+1} \approx_u \varphi_n$ and $\psi_{n+1} \approx_u \psi_n$ for all $n \in \mathbb{N}$.

$$\begin{tikzcd}
A \arrow[rr, "\mathrm{id}_A"] \arrow[dr, "\varphi_1"'] & & A \arrow[rr, "\mathrm{id}_A"] \arrow[dr, "\varphi_2"'] & & A \arrow[rr, "\mathrm{id}_A"] \arrow[dr, "\varphi_3"'] & & \cdots \arrow[r] & A \\
& B \arrow[ur, "\psi_1"] \arrow[rr, "\mathrm{id}_B"'] & & B \arrow[ur, "\psi_2"] \arrow[rr, "\mathrm{id}_B"'] & & B \arrow[ur] \arrow[r] & \cdots \arrow[r] & B.
\end{tikzcd}$$

(The triangles in the diagram do not commute exactly, only up to ε_n on F_n^A or F_n^B, as appropriate.)

If we choose the finite subsets of A and B to have dense union and satisfy $\varphi_n(F_n^A) \subseteq F_n^B$ and $\psi_n(F_n^B) \subseteq F_{n+1}^A$ for all $n \in \mathbb{N}$, and the real numbers to be summable, then one can check that $(\varphi_n(a))_{n\in\mathbb{N}}$ and $(\psi_n(b))_{n\in\mathbb{N}}$ are Cauchy for all $a \in A$ and $b \in B$. Define $\Phi \colon A \to B$ and $\Psi \colon B \to A$ by

$$\Phi(a) = \lim_{n\to\infty} \varphi_n(a) \quad \text{and} \quad \Psi(b) = \lim_{n\to\infty} \psi_n(b)$$

for all $a \in A$ and $b \in B$. Note that $\Phi \approx_u \varphi$ and $\Psi \approx_u \psi$, and that $\Psi \circ \Phi = \mathrm{id}_A$ and $\Phi \circ \Psi = \mathrm{id}_B$. This finishes the proof. □

Although Theorem 4.1 is not too difficult to prove, it leads to a template for classifying C*-algebras: it suffices to classify homomorphisms up to approximate unitary equivalence. This is the strategy taken in the classification theorem, and is explained in the next corollary.

Corollary 4.2. Let $\mathbf{C}$ be a category and let $\mathrm{F} \colon \mathbf{C}^* \to \mathbf{C}$ be a functor from the category $\mathbf{C}^*$ of C*-algebras to $\mathbf{C}$. Assume that $\mathcal{S}$ is a class of C*-algebras with the following properties:

(!) If $A, B \in \mathcal{S}$ and $\varphi, \psi \colon A \to B$ satisfy $\mathrm{F}(\varphi) = \mathrm{F}(\psi)$, then $\varphi \approx_u \psi$.
($\exists$) If $A, B \in \mathcal{S}$ and $\alpha \colon \mathrm{F}(A) \to \mathrm{F}(B)$ is a morphism in $\mathbf{C}$, then there exists a homomorphism $\varphi \colon A \to B$ with $\mathrm{F}(\varphi) = \alpha$.

Then $A \cong B$ if and only if $\mathrm{F}(A) \cong \mathrm{F}(B)$. If F is moreover invariant under approximate unitary equivalence (meanining that $\varphi \approx_u \psi$ implies $\mathrm{F}(\varphi) = \mathrm{F}(\psi)$), then any isomorphism $\mathrm{F}(A) \cong \mathrm{F}(B)$ is induced by an isomorphism $A \cong B$.

Proof. Use existence twice to find homomorphisms $\varphi \colon A \to B$ and $\psi \colon B \to A$ with $\mathrm{F}(\varphi) = \alpha$ and $\mathrm{F}(\psi) = \alpha^{-1}$. Then $\mathrm{F}(\psi \circ \varphi) = \mathrm{F}(\mathrm{id}_A)$ and $\mathrm{F}(\varphi \circ \psi) = \mathrm{F}(\mathrm{id}_B)$, so that by uniqueness we get $\psi \circ \varphi \approx_u \mathrm{id}_A$ and $\varphi \circ \psi \approx_u \mathrm{id}_B$. By Theorem 4.1, we deduce that $A \cong B$ and there exists an isomorphism $\Phi \colon A \to B$ with $\Phi \approx_u \varphi$. If F is invariant under approximate unitary equivalence, we conclude thar $\mathrm{F}(\Phi) = \mathrm{F}(\varphi) = \alpha$, as desired. □

Remark 4.3. For $\mathcal{S}$ denoting the class of unital, simple, separable, nuclear C*-algebras which are $\mathcal{Z}$-stable and satisfy the UCT and F= $\mathrm{KT_u}$[2], the existence assumption in Corollary 4.2 holds, but uniqueness fails!

The next two examples illustrate two ways in which uniqueness may fail in the setting described in the remark above.

Example 4.4. Let $A = B = \mathcal{O}_3 \otimes \mathcal{O}_3$ and let $\alpha \in \mathrm{Aut}(A)$ be the flip automorphism. By the Künneth formula, we have $K_0(A) \cong \mathbb{Z}_2$ and $K_1(A) = \{0\}$, and since $[1_A]$ generates $K_0(A)$, any automorphism of A must be trivial on K-theory (and hence it must induce the identity on $\mathrm{KT_u}(A)$). On the other hand, α is not approximately inner (that is, it is not approximately unitarily equivalent to id_A); a functorial way to detect this is using total K-theory[3].

Example 4.5. Let $\mathbb{Z}$ act on $\mathcal{Z}^{\otimes\mathbb{Z}}$ via the Bernoulli shift and set $A = \mathcal{Z}^{\otimes\mathbb{Z}} \rtimes \mathbb{Z}$. Using the Pimsner-Voiculescu exact sequence, one can easily compute that

$$K_0(A) \cong \langle [1_A] \rangle \cong \mathbb{Z} \quad \text{and} \quad K_1(A) \cong \langle u \rangle \cong \mathbb{Z},$$

and it is clear that A has a unique trace. There are only two automorphisms of this invariant: the identity, and the one that flips the sign on K_1, so that $\mathrm{Aut}(\mathrm{KT_u}(A)) \cong \mathbb{Z}_2$. On the other hand, one can show that $\mathrm{Aut}(A)/\approx_u$ is larger than $\mathbb{Z}_2$. To see this, consider the dual action $\gamma\colon \mathbb{T} \to \mathrm{Aut}(A)$. Then $\mathrm{KT_u}(\gamma_z) = \mathrm{id}_A$ for all $z \in \mathbb{T}$, essentially by connectedness of $\mathbb{T}$, while $\gamma_{z_1} \approx_u \gamma_{z_2}$ implies $z_1 = z_2$. A functorial way to prove this is by using a variant of algebraic K-theory, namely

$$\overline{K}_1^{\mathrm{alg}}(A) := \varinjlim \, \mathcal{U}_n(A) / \overline{[\mathcal{U}_n(A), \mathcal{U}_n(A)]} \cong \mathbb{T} \oplus \mathbb{Z}_2.$$

We have seen that, in order to prove that an isomorphism between C*-algebras exists, it is enough to show that there are homomorphisms between them whose compositions are approximately unitarily equivalent to the respective identities (Theorem 4.1). This reduces the isomorphism problem to the problem of constructing homomorphisms with specific properties, which is in practice easier.

Constructing homomorphisms is itself a challenging task, and in order to accomplish this we will begin by relaxing the requirement of multiplicativity.

Definition 4.6. Given C*-algebras A and B, an *approximate homomorphism* $\Phi\colon A \Rightarrow B$ between them is a sequence $\Phi = (\varphi_n)_{n\in\mathbb{N}}$ of ucp maps $\varphi_n\colon A \to B$ satisfying

$$\lim_{n\to\infty} \left\| \varphi_n(aa') - \varphi_n(a)\varphi_n(a') \right\| = 0$$

for all $a, a' \in A$.

Equivalently (at least when A is nuclear), this is a homomorphism $A \to B_\infty$.

This perspective is useful for the following (heuristic) reasons:

- Existence of homomorphisms $A \to B$ is implied by the combination of existence *and* uniqueness of approximate homomorphisms $A \Rightarrow B$.
- Uniqueness of homomorphisms $A \to B$ is implied by uniqueness of approximate homomorphisms $A \Rightarrow B$.

[2]In order to turn $\mathrm{KT_u}$ into a functor, and in particular to address the fact that K_* is covariant by T is contravariant, one may replace $T(A)$ with $\mathrm{Aff}(T(A))$.

[3]This fact also follows from the fact that the infinite tensor product of $\mathcal{O}_3$ with itself is not a strongly self-absorbing C*-algebra.

5. KK-theory and the Universal Coefficient Theorem

Kasparov's KK-theory is a formidable tool in C*-algebra theory which provides a link between (and a simultaneous generalization of) operator K-theory and the K-homology/extension theory of Atiyah and Brown–Douglas–Fillmore. Very loosely speaking, KK-theory can be viewed as an abelianization of the category of C*-algebras with homotopy classes of homomorphisms. It serves as a bridge between C*-algebras and their K-theory groups. For instance, every morphism of C*-algebras $A \to B$ induces an element of $KK(A, B)$ (see Remark 5.3), and every element of $KK(A, B)$ induces a morphism $K_*(A) \to K_*(B)$ (this is the map $\gamma_{A,B}$ described in Remark 5.11). Moreover, the computational properties of KK-theory allow for manipulations of elements of $KK(A, B)$ which are not possible at the level of homomorphisms.

The motivation for the constructionof KK-theory is to force an abelian group structure on $\mathrm{Hom}(A, B)/\approx_u$. One naive approach, which does not give the desired outcome in general, is set $V(A, B) = \mathrm{Hom}(A, B \otimes \mathcal{K})/\sim_h$ with diagonal/orthogonal addition (this is what we need the tensorial copy of $\mathcal{K}$ for). This is an abelian semigroup, and one can consider its Grothendieck enveloping group $W(A, B) = V(A, B) - V(A, B)$. Through some very basic examples, we see that this object has some pitfalls.

Example 5.1. When $A = \mathbb{C}$, we get $V(\mathbb{C}, B) = V(B) = \mathrm{Proj}(B \otimes \mathcal{K})/\sim_h$. When B is unital, we get $W(\mathbb{C}, B) = K_0(B)$, but in general this is not true. For example, for $B = C_0(\mathbb{R})$ we get $V(\mathbb{C}, B) = W(\mathbb{C}, B) = \{0\}$ although $K_0(B) \cong \mathbb{Z}$, generated by the Bott projection in $C(S^2) \cong \widetilde{B}$.

The problem with the above definition is that $B \otimes \mathcal{K}$ does not in general admit enough maps from A. The correct definition involves the use of *Cuntz pairs* (these are sometimes called quasihomomorphisms).

Definition 5.2. Given C*-algebras A and B, a *Cuntz pairs* from A to B, is a triple (φ, ψ, E) consisting of a C*-algebra E containing $B \otimes \mathcal{K}$ as an ideal, and homomorphisms $\varphi, \psi \colon A \to E$ such that $\varphi(a) - \psi(a) \in B \otimes \mathcal{K}$ for all $a \in A$. We write

$$(\varphi, \psi) \colon A \rightrightarrows E \rhd B \otimes \mathcal{K}.$$

Remark 5.3. Any homomorphism $\varphi \colon A \to B$ induces a Cuntz pair, namely $(\varphi \otimes e_{1,1}, 0) \colon A \to B \otimes \mathcal{K} \rhd B \otimes \mathcal{K}$. We will abbreviate it to simply $(\varphi, 0)$.

We will write $\mathrm{Cun}(A, B)$ for the set of all Cuntz pairs from A to B. Note that $\mathrm{Hom}(A, B) \hookrightarrow \mathrm{Cun}(A, B)$ by

Remark 5.4. Up to a suitable notion of homotopy for Cuntz pairs, we can always assume that $E = M(B \otimes \mathcal{K})$ in Definition 5.2. We will make this assumption from now on.

Remark 5.5. For any choices of A and B, there exist many homomorphisms $A \to M(B \otimes \mathcal{K})$ since $\mathcal{B}(\mathcal{H}) = M(\mathcal{K})$ embeds into $M(B \otimes \mathcal{K})$. Thus $\mathrm{Cun}(A, B)$ is a large object.

Definition 5.6. Two Cuntz pairs $(\varphi_0, \psi_0), (\varphi_1, \psi_1) \colon A \rightrightarrows M(B \otimes \mathcal{K}) \rhd B \otimes \mathcal{K}$ are said to be *homotopic* if there is a family $(\varphi_t, \psi_t) \colon A \rightrightarrows M(B \otimes \mathcal{K}) \rhd B \otimes \mathcal{K}$ of Cuntz pairs, for $t \in [0, 1]$, such that for every $a \in A$, the assignments $t \mapsto \varphi_t(a)$ and $t \mapsto \psi_t(a)$ are strictly continuous, and $t \mapsto \|\varphi_t(a) - \psi_t(a)\|$ is continuous.

Definition 5.7. Given C*-algebras A and B, we define their KK-group

$$KK(A,B) = \mathrm{Cun}(A,B)/\sim_h .$$

For a Cuntz pair $(\varphi,\psi) \in \mathrm{Cun}(A,B)$, we write $[\varphi,\psi]$ for its homotopy class in $KK(A,B)$. We endow $KK(A,B)$ with diagonal/orthogonal addition, and with inverses given by $-[\varphi,\psi] = [\psi,\varphi]$, and neutral element given by $[\varphi,\varphi]$ for any homomorphism $A \to M(B \otimes \mathcal{K})$.

What we defined above is really $KK_0(A,B)$. There is also a $KK_1(A,B)$, which is defined as $KK_0(A,SB)$. We will (almost) not need this group in these notes.

In practice one needs to assume that A is separable for the theory to work (for example, to prove the existence of the Kasparov product).

Example 5.8. Let B be a C*-algebra. Then

$$KK(\mathbb{C},B) \cong K_0(B) \quad \text{and} \quad KK(C_0(\mathbb{R}),B) \cong K_1(B).$$

Proposition 5.9. For a fixed C*-algebra A, the assignment $B \mapsto KK(A,B)$ is a covariant functor from $\mathbf{C}^*$ to the category of abelian groups, and similarly for a fixed C*-algebra B, the assignment $A \mapsto KK(A,B)$ is a contravariant functor from $\mathbf{C}^*$ to the category of abelian groups. These functors share many properties with the K-group functors: homotopy invariance, stability, direct sums, etc.

The KK-functors also interact well with respect to (certain) exact sequences. Explicitly, if $0 \to I \to E \to D \to 0$ is exact and there is a completely positive lift $D \to E$, then for every C*-algebra A there is a 6-term exact sequence

$$\begin{array}{ccccc} KK_0(A,I) & \longrightarrow & KK_0(A,E) & \longrightarrow & KK_0(A,D) \\ \uparrow & & & & \downarrow \\ KK_1(A,D) & \longleftarrow & KK_1(A,E) & \longleftarrow & KK_1(A,I) \end{array}$$

and similarly for $KK_*(E,B)$ (with arrows going backwards).

Next, we establish the existence of the Kasparov product.

Theorem 5.10. For separable C*-algebras A, B and C, there is an associative bilinear product

$$KK(A,B) \times KK(B,C) \to KK(A,C),$$

called the *Kasparov product*, which extends compositio of homomorphisms, in the sense that $[\psi] \cdot [\varphi] = [\psi \circ \varphi]$ for homomorphisms $\varphi\colon A \to B$ and $\psi\colon B \to C$.

In the picture of KK-theory using correspondences, the Kasparov product corresponds to the tensor product. (It takes a significant amount of work in this picture to show that the finite-index conditions are met for the tensor product.)

Remark 5.11. Let A and B be separable C*-algebras. Using the Kasparov product, we get

$$KK(\mathbb{C},A) \times KK(A,B) \to KK(\mathbb{C},B),$$

and by Example 5.8 this shows that any KK-class $\kappa \in KK(A,B)$ induces a group homomorphism $K_0(A) \to K_0(B)$ and similarly $K_1(A) \to K_1(B)$. This gives a well-defined group homomorphism

$$\gamma_{A,B}\colon KK(A,B) \to \mathrm{Hom}(K_*(A),K_*(B)).$$

Using six-term exact sequences, it can also be shown that any class $\kappa \in \ker(\gamma_{A,B})$ induces an extension (with a degree shift) of $K_*(A)$ by $K_{*+1}(B)$. This gives a well-defined group homomorphism

$$\varepsilon_{A,B} \colon \ker(\gamma_{A,B}) \to \mathrm{Ext}(K_*(A), K_{*+1}(B)).$$

For well-behaved C*-algebras A, the groups $KK(A,B)$ can be computed from $K_*(A)$ and $K_*(B)$. This is the contect of the *Universal Coefficient Theorem (UCT)* of Rosenberg-Schochet.

Theorem 5.12. For a separable C*-algebra A, the following are equivalent:

(1) For all separable C*-algebras B, $\gamma_{A,B}$ is surjective and $\varepsilon_{A,B}$ is an isomorphism.
(2) There exist a locally compact Hausdorff space X and a natural transformation $KK(A,\cdot) \cong KK(C_0(X),\cdot)$. (In other words, A is KK-*equivalent* to a commutative C*-algebra.)

Proof. (1) implies (2): Find X such that $K_*(A) \cong K_*(C_0(X))$; this is always possible. Using surjectivity of $\gamma_{A,C_0(X)}$, it can be shown that an isomorphism in $\mathrm{Hom}(K_*(A), K_*(C_0(X)))$ lifts to a KK-invertible class in $KK(A, C_0(X))$, which gives the desired equivalence of the functors.

(2) implies (1): For any fixed B, the functor $X \mapsto KK(C_0(X), B)$ is a generalized homologyy theory on pointed compact spaces. Algebraic topology then gives the conclusion. $\square$

Corollary 5.13. (The UCT) Let A be a separable C*-algebra. If A satisfies one of the equivalent conditions of Theorem 5.12, then for any separable C*-algebra B, there is a short exact sequence

$$0 \longrightarrow \mathrm{Ext}(K_*(A), K_{*+1}(B)) \xrightarrow{\varepsilon_{A,B}^{-1}} KK(A,B) \xrightarrow{\gamma_{A,B}} \mathrm{Hom}(K_*(A), K_*(B)) \longrightarrow 0.$$

This is the Universal Coefficient Theorem for KK-theory, and in this setting we say that A *satisfies* the UCT.

A fundamental problem is to determine which C*-algebras satisfy the UCT. Obviously abelian C*-algebras do, and there are many more examples.

Proposition 5.14. The class of C*-algebras that satisfy the UCT is closed under the following constructions:

(1) Direct limits.
(2) Two out of three in short exact sequences.
(3) Morita equivalente, and more generally KK-equivalence.
(4) Crossed products by $\mathbb{Z}$ (Pimsner-Voiculescu) and by $\mathbb{R}$ (Connes' Thom isomorphism).

Much less trivially, we also have:

Theorem 5.15. (Higson-Kasparov; Meyer-Nest) If A satisfies the UCT and G is a torsion-free amenable group, then for any action $G \curvearrowright A$, the crossed product $A \rtimes G$ satisfies the UCT.

However, not all C*-algebras satisfy the UCT:

Example 5.16. (Skandalis) If Γ is an infinite biexact group with property (T), then $C^*_\lambda(\Gamma)$ fails the UCT.

More generally, if A is any C*-algebrafor which the functor $B \mapsto A \otimes_{\min} B$ is not exact, then A does not satisfy the UCT. There are by now more examples of such C*-algebras besides the ones constructed by Skandalis, and they are never nuclear.

The following is the main open question around the UCT:

Question 5.17. Does every separable, nuclear C*-algebra satisfy the UCT?

For C*-algebras with a (twisted) groupoid model, the answer is positive:

Theorem 5.18. (Tu; Barlak-Li) Let $\mathcal{G}$ be a locally compact, Hausdorff, étale groupoid with a twist Σ. If $C^*(\mathcal{G}, \Sigma)$ is nuclear, then it satisfies the UCT.

Although most functors that we care about are invariant under approximate unitary equivalence, KK-theory is a remarkable exception: if $\varphi, \psi\colon A \to B$ satisfy $\varphi \approx_u \psi$, it may happen that $[\varphi] \neq [\psi]$ in $KK(A, B)$, although this is true if we assume the stronger statement that φ and ψ are *asymptotically* unitarily equivalent. We now proceed to describe a quotient of $KK(A, B)$ where approximately unitarily equivalent homomorphisms induce the same classes.

Definition 5.19. There is a natural topology on $KK(A, B)$ such that whenever $\varphi, \varphi_n\colon A \to B$, for $n \in \mathbb{N}$, are homomorphisms, and $\varphi_n(a) \to \varphi(a)$ for all $a \in A$, then $[\varphi_n] \to [\varphi]$ in $KK(A, B)$.

Set $\mathbb{N}^+ = \mathbb{N} \cup \{\infty\}$.

Remark 5.20. We have $\varphi_n \to \varphi$ in the point-norm topology if and only if there exists a homomorphism $\Phi\colon A \to C(\mathbb{N}^+, B)$ with $\Phi(a)(n) = \begin{cases} \varphi_n(a) & \text{if } n \in \mathbb{N} \\ \varphi(a) & \text{if } n = \infty, \end{cases}$ for all $a \in A$. In the topology on $KK(A, B)$ that we described above, we have $\kappa_n \to \kappa$ in $KK(A, B)$ if and only if there exists $\widetilde{\kappa} \in KK(A, C(\mathbb{N}^+, B))$ such that

$$\mathrm{ev}_n)_*(\widetilde{\kappa}) = \begin{cases} \kappa_n & \text{if } n \in \mathbb{N} \\ \kappa & \text{if } n = \infty. \end{cases}$$

The topology on $KK(A, B)$ is typically non-Hausdorff, and the closure of the trivial element plays an important role.

Definition 5.21. (Rørdam-Dadarlat) For separable C*-algebras A and B, we define $KL(A, B)$ to be the quotient of $KK(A, B)$ by the closuer of $\{0\}$ in $KK(A, B)$.

The following is the desired result

Theorem 5.22. Let A and B be separable C*-algebras. If $\varphi, \psi\colon A \to B$ satisfy $\varphi \approx_u \psi$, then $[\varphi] = [\psi]$ in $KL(A, B)$,

Just as for KK-theory, under the UCT assumption it is possible to compute $KL(A, B)$ from K-theoretical data. For a C*-algebra A, we write $\underline{K}(A)$ for the *total K-theory* of A, which consists of the direct sum of $K_*(A)$ and $K_*(A, \mathbb{Z}_n)$, for all $n \in \mathbb{N}$.

Theorem 5.23. (Dadarlat-Loring; Universal Multicoefficient Theorem). Let A be a separable C*-algebra satisfying the UCT. Then for every C*-algebra B there is a natural isomorphism

$$KL(A, B) \cong \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$$

respecting Kasparov products.

6. Non-stable KK-theory

The main drawback of KK-theory and extension theory is that one loses a lot of information in the stabilizations and homotopy equivalences needed to obtain computational tools. Non-stable KK-theory and non-stable extension theory focuse on removing these stabilizations and replacing the homotopies with more rigid equivalence relations (for example, approximate unitary equivalence). A landmark result in this direction is the Kirchberg- Phillips theorem which, in one form, states that for a separable nuclear C*-algebra A and simple, non-unital C*-algebra B, the group $KK(A, B)$ is naturally in bijection with asymptotic unitary equivalence classes of homomorphisms $A \to B \otimes \mathcal{O}_\infty$; see Theorem 6.1.

For the stably finite classification, the main underlying tool from non-stable KK-theory is the Dadarlat–Eilers stable uniqueness theorem; see Theorem 6.3. Roughly speaking, the theorem asserts the following: let A and B be separable C*-algebras, and let $(\varphi, \psi)\colon A \rightrightarrows M(B \otimes \mathcal{K}) \rhd B \otimes \mathcal{K}$ be a Cuntz pair. The stable uniqueness theorem states that if $[\varphi, \psi] = 0$, then there exist a homomorphism $\theta\colon A \to M(B \otimes \mathcal{K})$ and a one-parameter family of unitaries $(u_t)_{t\geq 1}$ in the minimal unitization of $M_2(B \otimes \mathcal{K})$, which in the limit conjugates $\varphi \oplus \theta$ to $\psi \oplus \theta$ pointwise in norm. The point of the theorem is that the stabilizations in KK-theory can be collected in a single map θ (and there is some freedom in choosing θ), and the homotopies can be implemented by unitaries.

Non-stable KK-theory is motivated by the following questions:

- (!) Given $(\varphi, \psi)\colon A \rightrightarrows M(B \otimes \mathcal{K}) \rhd B \otimes \mathcal{K}$ with $[\varphi, \psi] = 0$ in $KK(A, B)$ (or even two homomorphisms $\varphi, \psi\colon A \to B$ with same KK-class), how are φ and ψ related?
- (∃) Given $\kappa \in KK(A, B)$, when is κ induced by a homomorphism $A \to B$?

The most satisfactory result in this context is due to Kirchberg and Phillips:

Theorem 6.1. (Kirchberg, Phillips) Let A be a separable, nuclear C*-algebra, and let B be a simple, stable, purely infinite C*-algebra. Then there are natural identifications

$$KK(A, B) \cong \{A \hookrightarrow B\}/\sim_h \quad \text{and} \quad KK(A, B) \cong \{A \hookrightarrow B\}/\approx_u .$$

Combining the above with an intertwining argument (applied both to the functors KK and KL), one can prove the classification of purely infinite C*-algebras. A C*-algebra is said to be a *Kirchberg algebra* if it is simple, separable, nuclear and purely infinite. These are automatically $\mathcal{Z}$-stable by deep results of Kirchberg.

Theorem 6.2. (Kirchberg, Phillips) Let A and B be stable Kirchberg algebras. Then the following are equivalent:

1. $A \cong B$;
2. There is an invertible element in $KK(A, B)$ (in other words, $A \sim_{KK} B$;
3. There is an invertible element in $KL(A, B)$ (in other words, $A \sim_{KL} B$;
4. Assuming the UCT: $K_*(A) \cong K_*(B)$.

We would like to know if equiality of KK-classes can be replaced by somehting more rigid.

Theorem 6.3. (Dadarlat-Eilers' stable uniqueness theorem). Let A and B be separable C*-algebras. For a Cuntz pair $(\varphi, \psi)\colon A \rightrightarrows M(B \otimes \mathcal{K}) \rhd B \otimes \mathcal{K}$, the following are equivalent:

(1) $[\varphi, \psi] = 0$ in $KK(A, B)$;
(2) There exist a homomorphism $\theta\colon A \to M(\widetilde{B \otimes \mathcal{K}})$ and a norm-continuous path $(u_t)_{t\geq 1}$ of unitaries in $M_2(\widetilde{B \otimes \mathcal{K}})$ such that

$$\left\| u_t \begin{pmatrix} \varphi(a) & 0 \\ 0 & \theta(a) \end{pmatrix} u_t^* - \begin{pmatrix} \psi(a) & 0 \\ 0 & \theta(a) \end{pmatrix} \right\| \to 0$$

as $t \to \infty$, for all $a \in A$.

There is some amount of flexibility in the choice of θ: any homomorphism satisfying the conclusion of Voiculescu's theorem will do.

7. $\mathcal{Z}$-stability

The Jiang-Su algebra $\mathcal{Z}$ is the only classifiable C*-algebra with $\mathrm{KT_u}(\mathcal{Z}) \cong \mathrm{KT_u}(\mathbb{C})$. It has the useful property that $\mathrm{KT_u}(A \otimes \mathcal{Z}) \cong \mathrm{KT_u}(A)$ for any unital C*-algebra A.

Remark 7.1. Note that the positive cone of K_0, equivalently, the order on K_0, may change when tensoring with $\mathcal{Z}$, since for $x \in K_0(A \otimes \mathcal{Z})$ we have that $nx > 0$ for some $n \in \mathbb{N}$ implies $x \geq 0$, which this property is not true in the K_0-group of an arbitrary C*-algebra.

We will not try to define the Jiang-Su algebra very explicitly yet, and will rather focus on a more tractable C*-algebra: the CAR algebra $M_{2^\infty} = \varinjlim M_{2^n}$.

Theorem 7.2. Let A be a unital, separable C*-algebra. If $A \otimes M_{2^\infty} \cong A$, then there exist an isomorphism $\varphi\colon A \to M_2(A)$ and a sequence $(u_n)_{n\in\mathbb{N}}$ of unitaries in $M_2(A)$ such that $u_n\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)u_n^* \to \varphi(a)$ for all $a \in A$. In other words, the diagonal map $A \to M_2(A)$ is approximately unitarily equivalent to an isomorphism. The converse is also true, but it will not be necessary here.

Proof. If $A = M_{2^\infty}$, this is easy since $M_2(M_{2^\infty}) \cong M_{2^\infty}$ and any two unital homomorphisms $M_{2^\infty} \to M_{2^\infty}$ are approximately unitarily equivalent. In general, fix an isomorphism $\psi\colon A \to A \otimes M_{2^\infty}$ and choose any isoomorphism $\theta_0\colon M_{2^\infty} \to M_2(M_{2^\infty})$. Consider the following composition:

$$A \xrightarrow{\psi} A \otimes M_{2^\infty} \xrightarrow{\mathrm{id}_A \otimes \theta_0} A \otimes M_2(M_{2^\infty}) \cong M_2(A \otimes M_{2^\infty}) \xrightarrow{M_2(\psi)^{-1}} M_2(A)\ .$$

Since $\mathrm{id}_A \otimes \theta_0 \approx_u \mathrm{id}_A \otimes \mathrm{diag}_{M_{2^\infty}}$, the above composition is $\approx_u \mathrm{diag}_A$. □

This argument can be mode more general, and the relevant condition on the C*-algebra is the following:

Definition 7.3. (Toms-Winter) A C*-algebra $\mathcal{D}$ is said to be *strongly self-absorbing* if it is unital, separable, infinite dimensional, and there exists an isomorphism $\varphi\colon \mathcal{D} \to \mathcal{D} \otimes_{\min} \mathcal{D}$ which is approximately unitarily equivalent to the first factor embedding $d \mapsto d \otimes 1_{\mathcal{D}}$.

Theorem 7.4. Every strongly self-absorbing C*-algebra is simple and nuclear, and it is either purely infinite or has a unique trace (and is thus stably finite).

The Jiang-Su algebra is strongly self-absorbing, and it is in fact the "smallest" such algebra, in the following sense.

Theorem 7.5. (Winter) Every strongly self-absorbing C*-algebra is $\mathcal{Z}$-stable.

The following are all the known examples of strongly self-absorbing C*-algebras; they are in fact the only strongly self-absorbing C*-algebras that satisfy the UCT.

Examples 7.6. The following are strongly self-absorbing: UHF-algebras of infinite type; $\mathcal{Z}$, $\mathcal{O}_2$, $\mathcal{O}_\infty$ and tensor products of $\mathcal{O}_\infty$ with UHF-algebras of infinite type.

Theorem 7.7. Let A be a unital, separable C*-algebra, and let $\mathcal{D}$ be a strongly self-absorbing C*-algebra. Then the following are equivalent:

(1) $A \otimes \mathcal{D} \cong A$;
(2) There exists an isomorphism $\varphi\colon A \to A \otimes \mathcal{D}$ which is approximately unitarily equivalent to the first factor embedding;
(3) There exists a unital embedding $\mathcal{D} \to A_\infty \cap A'$.

If one has a presentation of $\mathcal{D}$ with generators and relations, then (3) above can be rephrased as a finitary statement. For example, for $\mathcal{D} = M_{2^\infty}$, condition (3) is equivalent to:

(3') For every finite subset $F \subseteq A$ and every $\varepsilon > 0$, there exist $n \in \mathbb{N}$ and matrix units $\{e_{j,k}\colon j,k = 1,\ldots,2^n\}$ in A with
$$\sum_{j=1}^{n} e_{j,j} = 1_A \quad \text{and} \quad \|e_{j,k}a - ae_{j,k}\| < \varepsilon$$
for all $a \in F$.

We now turn to the construction of $\mathcal{Z}$. Intuitively, we would like to define $\mathcal{Z}$ as the "intersection" of M_{n^∞}, for all $n \in \mathbb{N}$, or at least $M_{2^\infty} \cap M_{3^\infty}$. Although this is of course not possible in this way, we can "deform" M_{2^∞} into M_{3^∞} by setting
$$\mathcal{Z}_{2^\infty,3^\infty} = \left\{ f \in C([0,1], M_{2^\infty} \otimes M_{3^\infty} \colon f(0) \in M_{2^\infty} \otimes 1 \text{ and } f(1) \in 1 \otimes M_{3^\infty} \right\}$$
This is the generalized dimension-drop algebra of type $(2^\infty, 3^\infty)$.

Proposition 7.8. $\mathcal{Z}_{2^\infty,3^\infty}$ has no projections other than 0 and 1.

Proof. If $p \in \mathcal{Z}_{2^\infty,3^\infty}$ is a projection, then the map $\mathrm{tr} \circ \mathrm{ev}\colon [0,1] \to \mathbb{R}$ given by $t \mapsto \mathrm{tr}(p(t))$, is continuous and has range contained in $\mathbb{Z}\left[\frac{1}{6}\right]$. At 0, the value belongs to $\mathbb{Z}\left[\frac{1}{2}\right]$ and at 1 it belongs to $\mathbb{Z}\left[\frac{1}{3}\right]$. Since $\mathbb{Z}\left[\frac{1}{2}\right] \cap \mathbb{Z}\left[\frac{1}{3}\right] = \mathbb{Z}$, the result follows. $\square$

With a bit more care, one can show:

Theorem 7.9. $K_0(\mathcal{Z}_{2^\infty,3^\infty}) \cong \mathbb{Z}$ and $K_1(\mathcal{Z}_{2^\infty,3^\infty}) \cong \{0\}$.

The original construction of $\mathcal{Z}$ by Jiang and Su used the versions of $\mathcal{Z}_{2^\infty,3^\infty}$ where the UHF-algebras are replaced by matrix algebras with relatively prime dimensions. Their construction was very technical, and Rørdam-Winter, and later Schemaitat, provided a cleaner presentation (respectively, construction), which we present next.

Theorem 7.10. (Rørdam-Winter, Schemaitat) There is a unital endomorphism $\alpha\colon \mathcal{Z}_{2^\infty,3^\infty} \to \mathcal{Z}_{2^\infty,3^\infty}$ such that for all $\tau \in T(\mathcal{Z}_{2^\infty,3^\infty})$ we have $\tau \circ \alpha = \tau_{\text{Lebesgue}}$. Moreover, the stationary inductive limit of $(\mathcal{Z}_{2^\infty,3^\infty}, \alpha)$ does not depend on α, and it is the Jiang-Su algebra:
$$\mathcal{Z} \cong \varinjlim(\mathcal{Z}_{2^\infty,3^\infty}, \alpha).$$

The really challenging result around $\mathcal{Z}$ was proving that $\mathcal{Z} \otimes \mathcal{Z}$ is isomorphic to $\mathcal{Z}$ itself. The difficulty in doing this was that the most natural presentation of $\mathcal{Z} \otimes \mathcal{Z}$ as a direct limit uses two-dimensional building blocks, and there were at the time no general classification results that applied to such algebras.

One of the nice consequences of $\mathcal{Z}$-stability is the following:

Theorem 7.11. Let A be a unital, $\mathcal{Z}$-stable C*-algebra. Then there is a natural isomorphism $K_1(A) \cong \mathcal{U}(A)/\mathcal{U}_0(A)$. In particular, no matrix amplifications are needed.

8. The trace-kernel extension and classification of lifts

The trace-kernel extension lies at the heart of the new approach to the classification theorem. Let A and B be as in Theorem 3.1, and assume they have at least one trace (the traceless case is precisely Theorem 6.2). Non-stable extension theory cannot apply to A and B directly, as simple C*-algebras have no ideals. However, the sequence algebra B_∞ has a very natural ideal J_B consisting of the "tracially null sequences" in B. This is the trace-kernel ideal of B; see Definition 8.1.

The quotient $B^\infty = B_\infty/J_B$ is naturally a von Neumann factor when B has a unique trace (see Theorem 8.2; in general, it is loosely speaking a bundle of II_1-factors. Making use of the central sequences in B_∞ arising from $\mathcal{Z}$-stability, Connes' theorem can be combined with a partition of unity argument to classify morphisms $A \to B^\infty$ up to unitary equivalence by tracial data. This reduces the classification of maps $A \to B_\infty$ to classifying lifts along the trace-kernel extension.

The techniques we will present in this section have been behind the tremendous advancements in the field during the last 10 years. The ultimate goal is to show that if A and B are unital, simple, separable, nuclear $\mathcal{Z}$-stable C*-algebras satisfying the UCT (or maybe a bit less), then unital embeddings $A \hookrightarrow B$ are classified up to approximate unitary equivalence by the invariant

$$\underline{\mathrm{K}}\mathrm{T}_{\mathrm{u}} = \big(\underline{K}, \overline{K}_1^{\mathrm{alg}}, Aff(T(\cdot)), \rho\big).$$

Using intertwining arguments, it is enough to classify embeddings $A \hookrightarrow B_\infty$ (and with some care, one can also replace B_∞ with B_ω). Passing to B_∞ makes us lose a number of properties of our algebra B: the sequence algebra is generally not simple or nuclear, and it is certainly not separable. There are, however, some distinguished ideals in B_ω: given $\tau \in T(B)$, we set

$$J_\tau = \big\{ \big[(b_n)_{n\in\mathbb{N}}\big] \in B_\omega \colon \lim_{n\to\omega} \tau(b_n^* b_n)^{1/2} = 0 \big\},$$

which is an ideal in B_ω (and under very mild assumptions, this is a proper ideal).

We will assume from now on that $T(B) \neq \emptyset$, since otherwise we are in the realm of purely infinite C*-algebras, whose classification uses significantly different techniques. (For example, if B is purely infinite and simple, then B_∞ is also purely infinite and simple).

Definition 8.1. Let B be a unital, separable C*-algebra with $T(B) \neq \emptyset$. We define the *trace-kernel ideal* J_B as

$$J_B = \big\{ \big[(b_n)_{n\in\mathbb{N}}\big] \in B_\omega \colon \lim_{n\to\omega} \max_{\tau\in T(B)} \tau(b_n^* b_n)^{1/2} = 0 \big\}.$$

The quotient of B_ω by J_B, which we will denote by B^ω, turns out to be remarkably well-behaved. In the case of a unique trace, it is a von Neumann factor:

Theorem 8.2. Let B be a unital, separable C*-algebra with a unique trace. Then $B^\omega \cong \left[\pi_\tau(B)''\right]^\omega$, that is, the tracial ultrapower of $\pi_\tau(B)''$. In particular, B^ω is a finite von Neumann factor.

Proof. Note that $\pi_\tau(B)''$ is a factor because τ is an extreme trace (being the unique one on B): this follows from the fact that $\pi_\tau(B)''$ has a unique normal trace. Moreover, tracial ultrapowers of factors are again factors, so we only need to show that the isomorphism exists. The GNS map $\pi_\tau\colon B \to \pi_\tau(B)''$ induces a map $\pi_\tau^\omega\colon B^\omega \to (\pi_\tau(B)'')^\omega$, and we will show that this map is an isomorphism. Injectivity is easy, while surjectivity follows from Kaplansky's density theorem (which says that the norm-unit ball of B is trace-norm dense in the norm-unit ball of $\pi_\tau(B)''$). □

We thus obtain what we call the *trace-kernel extension*:

$$0 \longrightarrow J_B \xrightarrow{j_B} B_\omega \xrightarrow{q_B} B^\omega \longrightarrow 0.$$

This extension is vaguely reminiscent of Lin's various tracial approximations: J_B plays the role of the tracially small "wild" corner, while B^ω plays the role of the tracially large "neat" corner.

The trace-kernel extension was first used by Matui and Sato in its relative commutant version

$$0 \longrightarrow J_B \cap B' \xrightarrow{j_B} B_\omega \cap B' \xrightarrow{q_B} B^\omega \cap B' \longrightarrow 0.$$

Under nuclearity assumptions, there is a unital embedding $\mathcal{R} \to B^\omega \cap B'$, and the major problem in the remaining implication of the Toms-Winter conjecture is to show that, assuming strict comparison for B, one can product out of this a unital homomorphism $\mathcal{Z} \to B_\omega \cap B'$.

Strategy 8.3. Returning to the classification theorem, our strategy for classifying maps $A \to B_\omega$ consists of the following three steps:

Step 1: Classify embeddings $A \hookrightarrow B^\omega$.
Step 2: Prove the existence of, and classify, lifts of embeddings $A \hookrightarrow B^\omega$ along q_B, using $KK^*(A, J_B)$.
Step 3: Compute $KK^*(A, J_B)$ in terms of A and B.

As we will see, the UCT plays a major in the third step (and only there).

Remark 8.4. At this point, we see one advantage of working with ultrafilters: if B has a unique trace, then the same is true for B^ω, while for B^∞ traces are parametrized by ultrafilters on $\mathbb{N}$.

The following takes care of Step 1 in Strategy 8.3. We will assume for convenience that B has a unique trace and that it has no finite-dimensional representations, so that $\pi_\tau(B)''$ is infinite-dimensional and hence a II_1-factor.

Theorem 8.5. (Essentially Connes) Let A be a seprable, nuclear C*-algebra and let B be a separable, unital C*-algebra with a unique trace and no finite-dimensional representations. Then

(!) If $\varphi, \psi\colon A \to B^\omega$ are homomorphisms and $\tau_\omega \circ \varphi = \tau_\omega \circ \psi$, then $\varphi \approx_u \psi$.
(∃) Given $\tau \in T(A)$, there exists $\varphi\colon A \to B^\omega$ such that $\tau_\omega \circ \varphi = \tau$.

Proof. We begin by proving uniqueness. Set $\tau = \tau_\omega \circ \varphi \in T(A)$. Extend φ and ψ to $\overline{\varphi}, \overline{\psi} \colon \pi_\tau(A)'' \to B^\omega$. Then $\pi_\tau(A)''$ is hyperfinite by Connes' theorem, and it follows from the work of Murray and von Neumann that $\overline{\varphi} \approx_u \overline{\psi}$. The proof of existence is similar. $\square$

Outside of the unique trace case, one has to add $\mathcal{Z}$-stability (or at least uniform property Γ, or CPoU) for these arguments to go through. Note that $T(B_\omega) = T(B^\omega)$ in our setting. The outcome is the following:

Theorem 8.6. (Castillejos-Evington-Tikuisis-White-Winter) Let A be a seprable, nuclear C*-algebra, and let B be a unital, $\mathcal{Z}$-stable C*-algebra with $T(B) \neq \emptyset$. Then there are canonical bijections:

$$\begin{aligned}\big\{\text{unital homomorphism } A \to B^\omega\big\} &\cong \big\{\text{unital positive } \operatorname{Aff}(T(A)) \to \operatorname{Aff}(T(B_\omega))\big\} \\ &\cong \big\{\text{continuous affine } T(B_\omega) \to T(A)\big\}.\end{aligned}$$

We now move on to the second step. Recall that a homomorphism $\theta \colon A \to D$ is said to be *full* if $\theta(a)$ generates D as an ideal for all nonzero $a \in A$. Equivalently, θ is injective and $\theta(A) \cap D_0 = \{0\}$ for all proper ideals $D_0 \lhd D$. Note that θ is automatically full whenever it is nonzero and D is simple.

We will apply the following result to the trace-kernel extension.

Theorem 8.7. (Classification of lifts) Let

$$0 \longrightarrow I \longrightarrow E \xrightarrow{\ q\ } D \longrightarrow 0$$

be an exact sequence with I stable, and E unital and $\mathcal{Z}$-stable. Let A be separable and nuclear, and let $\theta \colon A \to D$ be a unital, full homomorphism. Then

(1) There exists a unital homomorphism $\psi \colon A \to E$ with $q \circ \psi = \theta$ if and only if there exists $\kappa \in KK(A, E)$ such that $[q] \cdot \kappa = [\theta]$ in $KK(A, E)$ and $\kappa_0([1_A]) = [1_E]$ in $K_0(E)$. In other words, a lift exists if and only if a KK-lift exists.

(2) For a fixed lift ψ as above, there is a canonical bijection between

$$\big\{\text{unital homomorphism } \varphi \colon A \to E \colon q \circ \varphi = \theta\big\} / \approx_{u,I}$$

and the set $\big\{\kappa \in KL(A, I) \colon \kappa_0([1_A]) = 0 \in K_0(I)\big\}$, which is given by $\varphi \mapsto [\varphi, \psi]$.

In (2) above, we have

$$\begin{tikzcd} & & & A \arrow[dl, shift right, "\psi"'] \arrow[dl, shift left, "\varphi"] \arrow[d, "\theta"] & \\ 0 \arrow[r] & I \arrow[r] & E \arrow[r, "q"'] & D \arrow[r] & 0. \end{tikzcd}$$

The two lifts for θ give rise to a Cuntz pair $(\varphi, \psi) \colon A \rightrightarrows E \rhd I$ and thus define a KK-class $[\varphi, \psi] \in KK(A, I)$. In that item, we consider the induced KL-class.

We will sketch injectivity in (2) of Theorem 8.7.

Assume that φ and ψ are lifts of θ and that $[\varphi, \psi] = 0$ in $KL(A, I)$. Consider the commutative diagram

$$\begin{tikzcd}
 & & & A \arrow[dll, shift right, "\psi"'] \arrow[dll, shift left, "\varphi"] \arrow[d, "\theta"] & \\
0 \arrow[r] & I \arrow[r] \arrow[d, "="'] & E \arrow[r, "q"] \arrow[d, "\lambda"'] & D \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I \arrow[r] & M(I) \arrow[r] & Q(I) \arrow[r] & 0.
\end{tikzcd}$$

Thus $(\lambda \circ \varphi, \lambda \circ \psi)\colon A \rightrightarrows M(I) \rhd I$ is a Cuntz pair vanishing in $KL(A, I)$. The stable uniqueness theorem of Dadarlat-Eilers tells us that in the situation described above, there are a unital representation $\mu A \to M(I)$ and a sequence of unitaries $(u_n)_{n\in\mathbb{N}}$ in $M_2(\widetilde{I})$ such that

$$u_n \begin{pmatrix} \lambda(\varphi(a)) & 0 \\ 0 & \mu(a) \end{pmatrix} u_n^* \to \begin{pmatrix} \lambda(\psi(a)) & 0 \\ 0 & \mu(a) \end{pmatrix}$$

for all $a \in A$. Their result does not just give us existence of *some* μ, but it also allows for some choice: it can be taken to be any absorbing representation (that is, any unital map $\mu\colon A \to M(I)$ such that for any other such map ν, we have $\mu \oplus \nu \approx_u \mu$). The question then becomes when a representation $A \to M(I)$ is absorbing. For this, we use the following result:

Theorem 8.8. (Elliott-Kucerovsky, Ortega-Perera-Rørdam) Let A be a separable, unital C*-algebra, let I be stable and $\mathcal{Z}$-stable, and let $\mu\colon A \to M(I)$ be a unital, full homomorphism. Then μ is absorbing.

In our case, both $\lambda \circ \varphi$ and $\lambda \circ \psi$ are absorbing by the theorem above, so in Dadarlat-Eilers we can take $\mu = \lambda \circ \varphi$. We have obtained the following: if $\varphi, \psi\colon A \to E$ are lifts of θ with $[\varphi, \psi] = 0$ in $KL(A, I)$, then

$$\varphi \oplus \varphi \approx_u \psi \oplus \psi.$$

If $E \otimes M_{2^\infty} \cong E$, then this implies that $\varphi \approx_u \psi$. In general, since we can also prove that

$$\varphi \oplus \varphi \oplus \varphi \approx_u \psi \oplus \psi \oplus \psi,$$

the rough idea is to patch these two equivalences together over $\mathcal{Z}_{2^\infty,3^\infty}$ to get $\varphi \approx_u \psi$. This finishes the sketch of injectivity in (2) of Theorem 8.7.

When we apply all of this to the trace-kernel extension, we will take $E = B_\omega$, which is too large to be $\mathcal{Z}$-stable. The correct notion in this setting, which makes everything work, is the following.

Definition 8.9. A C*-algebra E is said to be *separably $\mathcal{Z}$-stable* if for every separable subalgebra $E_0 \subseteq E$ there exists a $\mathcal{Z}$-stable subalgebra E_1 of E containing E_0.

Although $\mathcal{Z}$-sability does not pass to ultrapowers, separable $\mathcal{Z}$-stability does, and hence B_ω is separably $\mathcal{Z}$-stable if B is $\mathcal{Z}$-stable.

We have a similar issue with the ideal, which is assumed to be stable in Theorem 8.7 and this never holds for J_B. There is a similar notion of separable stability, although checking it for J_B takes a bit more work:

Theorem 8.10. Let B be a simple, unital, separable $\mathcal{Z}$-stable C*-algebra with $QT(B) = T(B)$. Then J_B is separably stable.

Proof. We will assume that B has real rank zero so that J_B does as well. In this case, a result of Hjlemborg-Rørdam asserts that J_B is (separably) stable if and only if for every projection $p \in J_B$ there exists a projection $q \in J_B$ with $pq = 0$ and $p \sim_{\mathrm{MvN}} q$. Now, given $p \in J_B$, we have

$$\tau(p) = 0 < 1 = \tau(1-p)$$

for all $\tau \in T(B_\omega)$. Using $\mathcal{Z}$-stability of B and $QT(B) = T(B)$, this gives the Cuntz subequivalence $p \precsim 1-p$. We can therefore find $q \leq 1-p$ with $q \sim_{\mathrm{MvN}} p$, as desired. □

We have now arrived at the completion of Step 1 in Strategy 8.3:

Theorem 8.11. (Classification of embeddings $A \hookrightarrow B^\omega$) Let A be a seprable, unital, nuclear C*-algebra, let B be a unital, $\mathcal{Z}$-stable. Then there is a canonical bijection

$$\big\{\text{unital } A \hookrightarrow B^\omega\big\}/\sim_u \cong \big\{\text{unital positive } \mathrm{Aff}(T(A)) \to \mathrm{Aff}(T(B_\omega))\big\}$$

given by $\theta \mapsto \mathrm{Aff}(T(\theta))$. Also, θ is full if and only if $\tau\circ\theta$ is faithful for all $\tau \in T(B_\omega)$.

We now move on to Step 2 in Strategy 8.3: classifying maps $A \to B_\omega$.

Proposition 8.12. Let A be a seprable, unital, nuclear C*-algebra, let B be a unital, $\mathcal{Z}$-stable, simple C*-algebra with $QT(B) = T(B)$. For unital homomorphisms $\varphi, \psi\colon A \to B_\omega$ with $\mathrm{Aff}(T(\varphi)) = \mathrm{Aff}(T(\psi))$, there exists a unitary $u \in \mathcal{U}(B_\omega)$ with $\mathrm{Im}(\mathrm{Ad}(u)\circ\varphi - \psi) \subseteq J_B$, and the class $\big[\mathrm{Ad}(u)\circ\varphi, \psi\big] \in KL(A, J_B)$ does not depend on u.

Proof. We begin with existence of u. Consider the diagram

$$\begin{tikzcd} & & A \arrow[d, shift right, "\psi"'] \arrow[d, shift left, "\varphi"] & & \\ 0 \arrow[r] & J_B \arrow[r] & B_\omega \arrow[r, "q_B"] & B^\omega \arrow[r] & 0, \end{tikzcd}$$

where $\mathrm{Aff}(T(\varphi)) = \mathrm{Aff}(T(\psi))$. Then $\mathrm{Aff}(T(q_B \circ \varphi)) = \mathrm{Aff}(T(q_B \circ \psi))$, and thus there exists $\overline{u} \in \mathcal{U}(B^\omega)$ with $\mathrm{Ad}(\overline{u}) \circ q_B \circ \varphi = q_B \circ \psi$. When B has a unique trace, then B^ω is a von Neumann algebra by Theorem 8.2, so that $\mathcal{U}(B^\omega)$ is connected and in this case $\overline{u}$ lifts to a unitary in B_ω. In general, using $\mathcal{Z}$-stability (via CPoU), it can be shown that $\mathcal{U}(B^\omega)$ is connected, and thus there is $u \in \mathcal{U}(B_\omega)$ with $q_B(u) = \overline{u}$ and $\mathrm{Ad}(u) \circ \varphi = q\psi$.

To show that $\big[\mathrm{Ad}(u)\circ\varphi, \psi\big] \in KL(A, J_B)$ does not depend on u, the key idea is to show that $\mathcal{U}(B^\omega \cap q_B(\varphi(A))')$ is connected. □

In view of the above, the can make the following definition.

Definition 8.13. Let A be a seprable, unital, nuclear C*-algebra, let B be a unital, $\mathcal{Z}$-stable, simple C*-algebra with $QT(B) = T(B)$. For unital homomorphisms $\varphi, \psi\colon A \to B_\omega$ with $\mathrm{Aff}(T(\varphi)) = \mathrm{Aff}(T(\psi))$, we define $\langle\varphi, \psi\rangle = \big[\mathrm{Ad}(u) \circ \varphi, \psi\big] \in KL(A, J_B)$, where $u \in \mathcal{U}(B_\omega)$ is as in the conclusion of the previous proposition.

Given C*-algebras A and B, a map $\gamma\colon \mathrm{Aff}(T(A)) \to \mathrm{Aff}(T(B))$ is said to be *faithful* if the image of the dual map $\gamma^*\colon T(B) \to T(A)$ is contained in the faithful traces.

We have arrived at the conclusion of Step 2 in Strategy 8.3:

Theorem 8.14. (Classifying maps $A \to B_\omega$) Let A be a seprable, unital, nuclear C*-algebra, let B be a unital, $\mathcal{Z}$-stable, simple C*-algebra with $QT(B) = T(B)$. Let $\gamma\colon \mathrm{Aff}(T(A)) \to \mathrm{Aff}(T(B))$ be positive, unital and faithful. Use ??? to fix a unital, full map $\theta\colon A \to B^\omega$ with $T(\theta) = \gamma$. Then

(1) There exists a unital homomorphism $\psi\colon A \to B_\omega$ with $\mathrm{Aff}(T(\psi)) = \gamma$ if and only if there is $\kappa \in KK(A, B_\omega)$ with $[q_B] \cdot \kappa = [\theta]$ in $KK(A, B^\omega)$ and $\kappa_0([1_A]) = [1_{B^\omega}]$ in $K_0(B^\omega)$.

(2) Given $\psi\colon A \to B_\omega$ as above, there is a canonical bijection
$$\big\{\varphi\colon A \to B_\omega \text{ unital}\colon \mathrm{Aff}(T(\varphi)) = \gamma\big\} \cong \big\{\kappa \in KK(A, J_B)\colon \kappa_0([1_A]) = 0\big\},$$
which is given by $\varphi \mapsto \langle \varphi, \psi\rangle$.

This is a complete classification of maps $A \to B_\omega$ using KK-theoretical and tracial data. To make its use effective, one must decide when a class $\kappa \in KK(A, B_\omega)$ as in part (1) of the theorem above exists, and also be able to compute $KL(A, J_B)$. For both of these tasks, the UCT is a crucial ingredient. Thus, in the stably finite case, the UCT remains a much more fundamental hypothesis by comparison to the purely infinite setting, where the actual classification uses KK-theory and the UCT only enters in the very last step to turn an isomorphism of K-theory into a KK-equivalence.

Without the UCT, there is still something that can be done. Note that in the next result we must assume the existence of a nice map $A \to B$.

Theorem 8.15. (Schafhauser) If A and B are unital, simple, separable, nuclear, $\mathcal{Z}$-stable C*-algebras and $\varphi\colon A \to B$ is a homomorphism such that $[\varphi] \in KK(A, B)$ and $T(\varphi)\colon T(B) \to T(A)$ are invertible, then $A \cong B$ and φ is approximately unitarily equivalent to an isomorphism.

Proving things without using the UCT (even if it turns out to be true for all separable, nuclear C*-algebras) is important if one wants to extend the classification of simple C*-algebras to more general settings, for example to obtain equivariant classification results. (For group actions, equivariant UCTs either do not exist, or are extremely hard to work with.)

Assuuming the UCT, the classification of embeddings $A \hookrightarrow B_\omega$ becomes simpler. We consider existence of lifts:

Corollary 8.16. Let A be a seprable, unital, nuclear C*-algebra satisfying the UCT, and let B be a unital, $\mathcal{Z}$-stable, simple C*-algebra with $QT(B) = T(B)$. Given a unital homomorphism $\theta\colon A \to B^\omega$, there is a unital lift $\psi\colon A \to B_\omega$ of θ if and only if there exists a group homomorphism $\alpha\colon K_0(A) \to K_0(B_\omega)$ with
$$K_0(q_B) \circ \alpha = K_0(\theta) \quad \text{and} \quad \alpha([1_A]) = [1_{B_\omega}].$$

Proof. The proof relies on the fact that $\mathcal{U}(B^\omega)$, and hence $\mathcal{U}_n(B^\omega)$ for all $n \in \mathbb{N}$, is connected. In particular, we get $K_1(B^\omega) = \{0\}$. Since projections in B^ω are classified by traces, we get $K_0(B^\omega) \cong \mathrm{Aff}(T(B_\omega))$. Combining these things with the UCT for A, we get
$$KK(A, B^\omega) \cong \mathrm{Hom}(K_0(A), K_0(B^\omega)).$$

□

Next, we give a major application:

Theorem 8.17. (Tikuisis-White-Winter; quasidiagionality theorem) Let A be a separable, nuclear C*-algebra satisfying the UCT. Then every faithful trace on A is quasidiagonal. In particular, if A is stably finite, then it is quasidiagonal.

Proof. Without loss of generality, we may assume that A is unital. Let τ_A be a faithful trace on A. By Voiculescu, if $\mathcal{Q} = \bigotimes_{n\in\mathbb{N}} M_n$ denotes the universal UHF-algebra, using nuclearity we deduce that τ_A is quasidiagonal if and only if there is a unital, trace-preserving embedding $(A, \tau_A) \hookrightarrow (\mathcal{Q}_\omega, \tau_{\mathcal{Q}})$. By Connes, there is a unital, trace-preserving embedding

$$\theta\colon (A, \tau_A) \hookrightarrow (\mathcal{R}^\omega, \tau_{\mathcal{R}}).$$

We thus are in the setting of classification of lifts and the question is whether θ can be lifted to $\mathcal{Q}_\omega$.

The original proof involved considering a cpc order zero lift and working hard to replace it by a homomorphism. Using the machinery we developed, we can turn this problem into the question of whether a certain extension splits. In our setting, by Corollary 8.16, it is enough to show that there exists a K_0-lift that preserves units:

$$\begin{tikzcd} & K_0(A) \arrow[dl, dashed] \arrow[d, "K_0(\theta)"] & \\ K_0(\mathcal{Q}_\omega) \arrow[r] & K_0(\mathcal{R}^\omega) \arrow[r] & 0. \end{tikzcd}$$

Since both $K_0(\mathcal{Q}_\omega)$ and $K_0(\mathcal{R}^\omega)$ are vector spaces over $\mathbb{Q}$, one can easily show that a lift exists using linear bases. □

Finally, we turn to Step 3 in Strategy 8.3. Since $K_1(J_B) \cong \overline{K}_1^{\mathrm{alg}}(B_\omega)$, this leads to the total invariant $\underline{\mathrm{K}}\mathrm{T}_{\mathrm{u}}$ described earlier. One gets a classification of embeddings $A \hookrightarrow B_\omega$ by $\underline{\mathrm{K}}\mathrm{T}_{\mathrm{u}}$, and then also of embeddings $A \hookrightarrow B$ (also by $\underline{\mathrm{K}}\mathrm{T}_{\mathrm{u}}$) using an intertwining argument.

The $\overline{K}_1^{\mathrm{alg}}$-group of a C*-algebra can be computed in terms of the invariant KT_{u}, since one can show that

$$\overline{K}_1^{\mathrm{alg}}(A) \cong K_1(A) \oplus \frac{\mathrm{Aff}(T(A))}{\mathrm{Im}\big(\rho_A\colon K_0(A) \to \mathrm{Aff}(T(A))\big)}.$$

This explains why, in the classification theorem up to isomorphism, it is only the invariant KT_{u} that is needed, and not $\underline{\mathrm{K}}\mathrm{T}_{\mathrm{u}}$.

9. Equivariant classification

The study of the structure of the flip automorphism of an injective factor was instrumental in Connes' proof that injectivity implies hyperfiniteness. Following his work, Jones studied finite group actions on $\mathcal{R}$, obtaining a remarkable uniqueness result for outer actions:

Theorem 9.1. (Connes, Jones) Let G be a finite group. Then any two outer actions of G on $\mathcal{R}$ are conjugate.

When trying to generalize the above results to infinite groups, conjugacy is not the right notion of equivalence. This arises naturally whenever one wants to implement an intertwining argument, as we proceed to explain. Indeed, given a discrete group G and actions $\alpha\colon G \to \mathrm{Aut}(A)$ and $\beta\colon G \to \mathrm{Aut}(B)$ on unital C*-algebras, two equivariant homomorphisms $\varphi, \psi\colon (A, \alpha) \to (B, \beta)$ are said to be *G-approximately unitarily equivalent*, written $\varphi \approx_{G,u} \psi$, if there exist a sequence $(u_n)_{n\in\mathbb{N}}$ of unitaries in B such that $u_n\varphi(a)u_n^* \to \psi(a)$ in norm for all $a \in A$ and $\|\beta_g(u_n) - u_n\| \to 0$ for all $g \in G$.

The most naive generalization of Elliott's intertwining argument (Theorem 4.1) would assert that if there exist two equivariant homomorphisms $\varphi\colon (A, \alpha) \to (B, \beta)$ and $\psi\colon (B, \beta) \to (A, \alpha)$ satisfying $\psi \circ \varphi \approx_{G,u} \mathrm{id}_A$ and $\varphi \circ \psi \approx_{G,u} \mathrm{id}_B$, then α and β are conjugate. This is, however, not true. What one *does* conclude is that α and β are cocycle conjugate, in the sense of the following definition.

Definition 9.2. Let G be a discrete group, let A be a unital C*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. A function $w\colon G \to \mathcal{U}(A)$ is said to be an *α-cocycle* if

$$w_{gh} = w_g\alpha_g(w_h)$$

for all $g, h \in G$. Given an α-cocycle w, we denote by $\alpha^w\colon G \to \mathrm{Aut}(A)$ the action given by $\alpha^w_g = \mathrm{Ad}(w_g) \circ \alpha_g$ for all $g \in G$.

We say that an action $\beta\colon G \to \mathrm{Aut}(B)$ is *cocycle conjugate* to α, written $\alpha \cong_{\mathrm{cc}} \beta$, if there exists an α-cocycle w such that α^w is conjugate to β.

With this terminology in place, we can state Ocneanu's generalization of Theorem 9.1.

Theorem 9.3. (Connes, Jones) Let G be an amenable group. Then any two outer actions of G on $\mathcal{R}$ are cocycle conjugate.

Remark 9.4. One may wonder why one does not need to consider cocycle conjugacy in Theorem 9.1, and this is because one can show that the stronger statement using conjugacy follows from cocycle conjugacy for outer actions in $\mathcal{R}$, since one can use the Rokhlin property for the model action. What is at the heart of the argument is the fact that if we consider the strengthening of G-approximate unitary equivalence where the unitaries u_n are assume to be in the fixed point algebra (and not approximately in it), then the naive version of Elliott's intertwining holds. On the other hand, when G is finite, then the condition $\|\beta_g(u_n) - u_n\| \to 0$ for all $g \in G$ can be made uniform, and one gets $\max_{g\in G} \|\beta_g(u_n) - u_n\| \to 0$. A standard argument involving functional calculus allows one to replace u_n with a G-invariant unitary in this case, thus obtaining the stronger version of G-approximate unitary equivalence.

It is useful to regard cocycle conjugacies as the isomorphisms of a certain category of C*-dynamical systems. This was conceptualized by Szabo, and we now describe these morphisms:

Definition 9.5. Let G be a discrete group and let $\alpha\colon G \to \mathrm{Aut}(A)$ and $\beta\colon G \to \mathrm{Aut}(B)$ be actions on unital C*-algebras. A *cocycle morphism* between them, written $(\varphi, \mathtt{u})\colon (A, \alpha) \to (B, \beta)$, consists of a homomorphism $A \to B$ and a β-cocycle $\mathtt{u}\colon G \to \mathrm{Aut}(B)$ satisfying

$$\mathrm{Ad}(\mathtt{u}_g) \circ \beta_g \circ \varphi = \varphi \circ \alpha_g$$

for all $g \in G$. In other words, $\varphi\colon (A,\alpha) \to (B,\beta^{\mathtt{u}})$ is an equivariant homomorphism.

Composition of cocycle morphisms is given by

$$(\varphi, \mathtt{u}) \circ (\psi, \mathtt{v}) = (\varphi \circ \psi, \varphi(\mathtt{v})\mathtt{u}).$$

Example 9.6. For an action $\beta\colon G \to \mathrm{Aut}(B)$ on a unital C*-algebra B and for $u \in \mathcal{U}(B)$, we define the associated β-cocycle $\partial_\beta u\colon G \to \mathcal{U}(B)$

$$(\partial_\beta u)_g = u\beta_g(u)^*$$

for all $g \in G$. Then $(\mathrm{Ad}(u), \partial_\beta u)\colon (B,\beta) \to (B,\beta)$ is a cocycle morphism, and we say that $(\mathrm{Ad}(u), \partial_\beta u)$ is an *inner* cocycle morphism.

We now define the appropriate notion of approximate unitary equivalence between cocycle morphisms.

Definition 9.7. Let G be a discrete group and let $\alpha\colon G \to \mathrm{Aut}(A)$ and $\beta\colon G \to \mathrm{Aut}(B)$ be actions on unital C*-algebras. We say that two cocycle morphisms $(\varphi, \mathtt{u}), (\psi, \mathtt{v})\colon (A,\alpha) \to (B,\beta)$ are *approximately unitarily equivalent*, in symbols $(\varphi, \mathtt{u}) \approx_u (\psi, \mathtt{v})$, if there is a sequence $(w_n)_{n\in\mathbb{N}}$ in $\mathcal{U}(B)$ such that $(\mathrm{Ad}(w_n), \partial_\beta w_n) \circ (\varphi, \mathtt{w}) \to (\psi, \mathtt{v})$ in the point-norm topology; equivalently

$$\|w_n\varphi(a)w_n^* - \psi(a)\| \to 0 \quad \text{and} \quad \|w_n \mathtt{u}_g \beta_g(w_n)^* - \mathtt{v}_g\| \to 0$$

for all $a \in A$ and all $g \in G$.

The following is the desired intertwining argument.

Theorem 9.8. (Szabo) Let G be a discrete group and let $\alpha\colon G \to \mathrm{Aut}(A)$ and $\beta\colon G \to \mathrm{Aut}(B)$ be actions on unital C*-algebras. Let

$$(\varphi, \mathtt{u})\colon (A,\alpha) \to (B,\beta) \quad \text{and} \quad (\psi, \mathtt{v})\colon (B,\beta) \to (A,\alpha)$$

be cocycle morphisms satisfying

$$(\psi, \mathtt{v}) \circ (\varphi, \mathtt{u}) \approx_u (\mathrm{id}_A, 1_A) \quad \text{and} \quad (\varphi, \mathtt{u}) \circ (\psi, \mathtt{v}) \approx_u (\mathrm{id}_B, 1_B).$$

Then $(A,\alpha) \cong_{\mathrm{cc}} (B,\beta)$. Moreover, there exist a β-cocycle $\mathtt{w}$ and an equivariant isomorphism $\Phi\colon (A,\alpha) \to (B,\beta^{\mathtt{w}})$ such that $(\varphi, \mathtt{u}) \approx_u (\Phi, \mathtt{w})$.

The equivariant classification in the purely infinite setting has been recently settled. The result is more general than what we state here, covering nonamenable groups (with amenable actions) and also applying to locally compact groups.

Theorem 9.9. (Gabe-Szabo) Let G be a discrete, amenable group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ and $\beta\colon G \to \mathrm{Aut}(B)$ be actions on unital Kirchberg algebras. Then we have $(A,\alpha) \cong_{\mathrm{cc}} (B,\beta)$ if and only if $(A,\alpha) \sim_{KK^G} (B,\beta)$ unitally.

Deciding if two actions are KK^G-equivalent is often much easier than computing the entire KK^G-group, particularly when G is torsion-free.

Theorem 9.10. (Higson-Kasparov; Meyer-Nest) Let G be a discrete, amenable, *torsion-free* group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ and $\beta\colon G \to \mathrm{Aut}(B)$ be actions. Then a class in $KK^G\big((A,\alpha),(B,\beta)\big)$ is invertible if and only if it is invertible in $KK(A,B)$.

10. Actions on $\mathcal{O}_2$

We now specialize to a particular case, namely actions on the Cuntz algebra $\mathcal{O}_2$. Our goal will be to show that any two $\mathcal{O}_2$-stable outer actions of an amenable group are cocycle conjugate. For convenience, we will assume that G is infinite. First, we point out $\mathcal{O}_2$-stability is automatic when G is torsion-free, essentially by the Baum-Connes conjecture.

Theorem 10.1. Let G be a countable, amenable, torsion-free group. Then any outer action of G on $\mathcal{O}_2$ absorbs $\mathrm{id}_{\mathcal{O}_2}$ up to cocycle conjuacy.

Thus, for torsion-free, amenable groups, it will follow that there is a unique outer action on $\mathcal{O}_2$.

We will need some preparation.

Definition 10.2. Let G be an infinite, countable group. Label the canonical generators of $\mathcal{O}_\infty$ as s_g, for $g \in G$, and let $\gamma^G\colon G \to \mathrm{Aut}(\mathcal{O}_\infty)$ be given by $\gamma^G_g(s_h) = s_{gh}$ for all $g, h \in G$.

The following is essentially due to Kishimoto, and follows from the fact that $B_\omega \cap B'$ is purely infinite simple (due to Kirchberg), and that β_ω is outer.

Theorem 10.3. Let G be an infinite, countable group and let $\beta\colon G \to \mathrm{Aut}(B)$ be an outer action on a Kirchberg algebra B. Then $(B, \beta) \otimes (\mathcal{O}_\infty, \gamma^G) \cong_{\mathrm{cc}} (B, \beta)$. Equivalently, there exists a unital, equivariant homomorphism

$$(\mathcal{O}_\infty, \gamma^G) \to (B_\omega \cap B', \beta_\omega).$$

The property that (B, β) absorbs $(\mathcal{O}_\infty, \gamma^G)$ up to cocycle conjugacy is called *isometric shift-absorption* in the work of Gabe-Szabo.

The main step in showing that there is a unique outer, $\mathcal{O}_2$-stable action on $\mathcal{O}_2$ is the following.

Theorem 10.4. Let G be a countable, amenable group, let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action on a unital, separable, nuclear C*-algebra A, and let $\beta\colon G \to \mathrm{Aut}(\mathcal{O}_2)$ be outer and $\mathcal{O}_2$-stable. Then there exists a cocycle morphism

$$(\varphi, \mathtt{w})\colon (A, \alpha) \to (\mathcal{O}_2, \beta),$$

and any two such morphisms are approximately unitarily equivalent.

Proof. We begin with existence. Note that $A \rtimes_\alpha G$ is a separable, unital, nuclear C*-algebra. By Kirchberg's embedding theorem, it follows that there exists a unital embedding $\widetilde{\varphi}\colon A \rtimes_\alpha G \to \mathcal{O}_2$. For $g \in G$, denote by $u_g \in A \rtimes_\alpha G$ the canonical unitary. Set $\varphi = \widetilde{\varphi}|_A$ and $\mathtt{w}_g = \varphi(u_g)$ for all $g \in G$. Then $(\varphi, \mathtt{w})$ is a cocycle morphism $(A, \alpha) \to (\mathcal{O}_2, \mathrm{id}_{\mathcal{O}_2})$. The desired cocycle morphism is then the following composition:

$$(A, \alpha) \xrightarrow{(\varphi, \mathtt{w})} (\mathcal{O}_2, \mathrm{id}_{\mathcal{O}_2}) \hookrightarrow (\mathcal{O}_2, \mathrm{id}_{\mathcal{O}_2}) \otimes (\mathcal{O}_2, \beta) \xrightarrow{\cong_{\mathrm{cc}}} (\mathcal{O}_2, \beta).$$

We now turn to uniqueness. For convenience, we will consider equivariant homomorphisms $\varphi, \psi\colon (A, \alpha) \to (\mathcal{O}_2, \beta)$ and we will show that $\psi \sim_{G,u} \varphi$. By non-equivariant classification, there exists a unitary $w_0 \in (\mathcal{O}_2)_\omega$ with $\mathrm{Ad}(w_0^*) \circ \varphi = \psi$. Using Theorem 10.3 with $B = \mathcal{O}_2$, we fix Cuntz isometries $s_g \in (\mathcal{O}_2)_\omega \cap \mathcal{O}_2'$ with $\beta_g(s_h) = s_{gh}$ for all $g, h \in G$.

Given a finite subset $F \subseteq G$, define

$$w = \frac{1}{|F|^{1/2}} \sum_{g \in G} s_g \beta_g(w_0) \in (\mathcal{O}_2)_\omega.$$

Using centrality of the Cuntz isometries at the second step, using that φ is equivariant at the third step, and using that $\mathrm{Ad}(w_0^*) \circ \varphi = \psi$ at the fourth step, we get

$$\begin{aligned} w^*\varphi(a)w &= \frac{1}{|F|} \sum_{g,h \in G} \beta_g(w_0)^* s_g^* \varphi(a) s_h \beta_h(w_0) \\ &= \frac{1}{|F|} \sum_{g \in G} \beta_g(w_0)^* \varphi(a) \beta_g(w_0) \\ &= \frac{1}{|F|} \sum_{g \in G} \beta_g(w_0^* \varphi(\alpha_{g^{-1}}(a) w_0) \\ &= \psi(a) \end{aligned}$$

for all $a \in A$. Moreover,

$$\beta_g(w) = \frac{1}{|F|^{1/2}} \sum_{h \in G} s_{gh} \beta_{gh}(w_0),$$

and thus $\|\beta_g(w) - w\| \leq \frac{|gF \triangle F}{|F|}$, which is small if F is a Følner set. Finally, note that $w^*w = 1$ (but $ww^* \neq 1$).

Using a standard diagonal argument in the ultrapower, we deduce that there is an isometry $v \in (\mathcal{O}_2)_\omega$ with $\beta_g(v) = v$ and $\mathrm{Ad}(v^*) \circ \varphi = \psi$. We will denote this by $\psi \precsim_G \varphi$. By symmetry, we also get $\varphi \precsim_G \psi$. Our goal is to show that this implies $\varphi \sim_{G,u} \varphi$ in $(\mathcal{O}_2)_\omega$. For this, we will use Connes' 2-by-2 matrix trick: set $\pi = \begin{pmatrix} \varphi & 0 \\ 0 & \psi \end{pmatrix} : A \to M_2(\mathcal{O}_2)$, and set $p = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $q = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. It is a standard fact that if $p \sim_{\mathrm{MvN}} q$ in $\big(M_2(\mathcal{O}_2)_\omega \cap \pi(A)'\big)^G$, then $\varphi \approx_{u,G} \psi$ in $\mathcal{O}_2$. This fixed point algebra is separably $\mathcal{O}_2$-stable and p and q are properly infinite, full projections in it (they are full because they generate the same ideal and $p + q = 1$, so the ideal they generate is everything). A result of Cuntz says that $p \sim_{\mathrm{MvN}} q$ in $\big(M_2(\mathcal{O}_2)_\omega \cap \pi(A)'\big)^G$ if and only if

$$[p]_0 = [q]_0 \text{ in } K_0\left(\big(M_2(\mathcal{O}_2)_\omega \cap \pi(A)'\big)^G\right).$$

Since the above K_0-group is zero by separable $\mathcal{O}_2$-stability, this show that $p \sim_{\mathrm{MvN}} q$ and thus $\varphi \approx_{G,u} \psi$, as desired. $\square$

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