Morita- and cocycle-equivalence

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November 13, 2019

Hilbert modules

Definition

Let B be a C^*-algebra. A (right) Hilbert-B-module is a right B-module with a map $\langle , \rangle_B : E \times E \to B$ (inner product) satisfying

1. $\langle \xi, \eta b + \zeta \rangle_B = \langle \xi, \eta \rangle_B b + \langle \xi, \zeta \rangle_B$
2. $\langle \xi, \eta \rangle_B^* = \langle \eta, \xi \rangle_B$
3. $\langle \xi, \xi \rangle_B \geq 0$ and $\langle \xi, \xi \rangle_B = 0 \Leftrightarrow \xi = 0$

for all $\xi, \eta, \zeta \in E$, $b \in B$ such that E is complete with respect to the norm

$$\|\xi\| := \sqrt{\|\langle \xi, \xi \rangle_B\|}$$

Examples

Remark

Similarly one defines left Hilbert B-modules and then denotes the inner product by ${}_B\langle , \rangle$

Example

- Hilbert spaces are Hilbert-$\mathbb{C}$-modules
- B is a Hilbert-B-module by right multiplication and

$$\langle a, b\rangle_B = a^* b, \quad a, b \in B$$

- If $I \subseteq B$ is a closed ideal, then I defines a Hilbert-B-module with the same formulars.

Equivariant Hilbert modules

Now let $\beta : G \to \mathrm{Aut}(B)$ be an action. The naive approach would be

Definition (WRONG)

A G-Hilbert-B-module is a Hilbert-B-module E with a B-linear unitary action $u : G \to \mathrm{Aut}(E)$, i.e.

- $\langle u_g\xi, u_g\eta\rangle_B = \langle \xi, \eta\rangle_B, \quad g \in G, \xi, \eta \in E$
- $u_g(\xi b) = u_g(\xi)b, \quad g \in G, \xi \in E, b \in B$

Example

Let $E = B$ and $u = \beta$. Then we have

- $\langle u_g(\xi), u_g(\eta)\rangle_B = \beta_g(\xi)^*\beta_g(\eta) = \beta_g(\langle \xi, \eta\rangle_B), \quad g \in G, \xi, \eta \in E$
- $u_g(\xi b) = \beta_g(\xi b) = u_g(\xi)\beta_g(b), \quad g \in G, \xi \in E, b \in B$

Equivariant Hilbert modules

Definition (correct)

Let $\beta : G \to \mathrm{Aut}(B)$ be an action. A (right) G-Hilbert-B-module (E, u) is a right Hilbert B-module E with a strongly continuous action $u : G \to \mathrm{Aut}(E)$ such that

- $\langle u_g\xi, u_g\eta\rangle_B = \beta_g(\langle \xi, \eta\rangle_B), \quad g \in G, \xi, \eta \in E$
- $u_g(\xi b) = u_g(\xi)\beta_g(b), \quad g \in G, \xi \in E, b \in B$

Remark

By $\mathrm{Aut}(E)$ we only mean invertible bounded linear operators of the Banach space E.

Adjointable Operators

Definition

Let E, F be Hilbert-B-modules. A map $T : E \to F$ is called *adjointable*, if there exists a map $T^* : F \to E$ (called adjoint) such that

$$\langle T\xi, \eta \rangle_B = \langle \xi, T^*\eta \rangle_B, \quad \xi, \eta \in E$$

Lemma

Adjointable operators are automatically B-linear and bounded with a unique adjoint. Equipped with the Operator norm, the collection $\mathcal{L}_B(E)$ of all adjointable operators $E \to E$ is a C^*-algebra.

Imprimitivity bimodules

Definition

Let A, B be G-C^*-algebras. A G-equivariant A-B-imprimitivity-bimodule is a right G-Hilbert-B-module (E, u), which is also a left G-Hilbert-A-module (with the same action u) such that

1. ${}_A\langle \xi b, \eta \rangle = {}_A\langle \xi, \eta b^* \rangle, \quad \xi, \eta \in E, b \in B$
2. $\langle a\xi, \eta \rangle_B = \langle \xi, a^*\eta \rangle_B, \quad \xi, \eta \in E, a \in A$
3. ${}_A\langle \xi, \eta \rangle \zeta = \xi \langle \eta, \zeta \rangle_B, \quad \xi, \eta, \zeta \in E$
4. $\overline{\text{span}}_A\langle E, E \rangle = A, \quad \overline{\text{span}}\langle E, E \rangle_B = B$ (Fullness)

If such an E exists, we call A and B Morita-equivalent.

Full corners

Example

Let A be a G-C^*-Algebra and $p \in M(A)$ a G-invariant full projection (i.e. $\overline{\text{span}} ApA = A$). Equip pA with the left/right-multiplication of pAp and A respectively and the restricted action of G.Define inner products by

$$
\begin{aligned}
{}_{pAp}\langle pa, pb\rangle &:= pa(pb)^*, \quad a, b \in A \\
\langle pa, pb\rangle_A &:= (pa)^* pb, \quad a, b \in A
\end{aligned}
$$

Then pA is a G-equivariant pAp-A-imprimitivity bimodule.

Example

Let $\mathcal{K}$ be the compact operators on $\ell^2(\mathbb{N})$ with the standard rank one projection $e_{11} \in \mathcal{K}$. Then $p := 1 \otimes e_{11} \in M(A \otimes \mathcal{K})$ is a G-invariant full projection with $p(A \otimes \mathcal{K})p = A$. Thus, A is Morita-equivalent to $A \otimes K$.

The dual module

Definition

Let (E, u) be a G-equivariant A-B-imprimitivity bimodule. We define a G-equivariant B-A-imprimitivity bimodule (E^*, u^*) by

- $E^* := \{\xi^* | \xi \in E\}$
- $b\xi^* := (\xi b^*)^*, \quad \xi \in E, b \in B$
- $\xi^* a := (a^* \xi)^*, \quad \xi \in E, a \in A$
- ${}_B\langle \xi^*, \eta^* \rangle := \langle \xi, \eta \rangle_B, \quad \xi, \eta \in E$
- $\langle \xi^*, \eta^* \rangle_A := {}_A\langle \xi, \eta \rangle, \quad \xi, \eta \in E$
- $u_g^*(\xi^*) := (u_g(\xi))^*, \quad g \in G, \xi \in E$

The linking algebra

Definition

Let (E, u) be a G-equivariant A-B-bimodule. We define a $*$-algebra-structure and G-action γ on $C := \begin{pmatrix} A & E \\ E^* & B \end{pmatrix}$ by

$$\begin{pmatrix} a & \xi \\ \eta^* & b \end{pmatrix} \begin{pmatrix} c & \zeta \\ \theta^* & d \end{pmatrix} := \begin{pmatrix} ac + {}_A\langle \xi, \theta \rangle & a\zeta + \xi d \\ \eta^* c + b\theta^* & \langle \eta, \zeta \rangle_B + bd \end{pmatrix}$$

$$\begin{pmatrix} a & \xi \\ \eta^* & b \end{pmatrix}^* := \begin{pmatrix} a^* & \eta \\ \xi^* & b^* \end{pmatrix}, \quad \gamma_g \begin{pmatrix} a & \xi \\ \eta & b \end{pmatrix} := \begin{pmatrix} \alpha_g(a) & u_g(\eta) \\ u_g^*(\eta^*) & \beta_g(b) \end{pmatrix}$$

$$\xi, \eta, \zeta, \theta \in E, a, c \in A, b, d \in B, g \in G$$

We let C act on the Hilbert-B-module $E \oplus B$ by

$$\begin{pmatrix} a & \xi \\ \eta^* & b \end{pmatrix} \begin{pmatrix} \zeta \\ c \end{pmatrix} := \begin{pmatrix} a\zeta + \xi c \\ \langle \eta, \zeta \rangle_B + bc \end{pmatrix}$$

Proposition

In this way, $C \subseteq \mathcal{L}_B(E \oplus B)$ becomes a G-C^*-algebra.It contains G-invariant full projections

$$p := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, q := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in M(C)$$

such that $A = pCp$ and $B = qCq$. Thus, two G-C^*-algebras A and B are Morita-equivalent, if and only if they are G-invariant full corners of a common G-C^*-algebra C.

Cocycle conjugacy

Definition

Let (A, α) be a G-C^*-algebra. An α-1-cocycle is a strictly continuous map $c : G \to UM(A)$ satisfying

$$c_{gh} = c_g \alpha_g(c_h), \quad g, h \in G$$

We call another action $\beta : G \to \mathrm{Aut}(A)$ cocycle-conjugate to α, if there is an α-1-cocycle c such that

$$\beta_g(a) = \mathrm{Ad}(c_g) \circ \alpha_g(a) := c_g \alpha_g(a) c_g^*, \quad a \in A, g \in G$$

Combes' theorem

Theorem (Combes '84)

1. *Let* A, B *be* σ*-unital, Morita-equivalent* G*-*C^**-algebras. Then there is an isomorphism* $\theta : A \otimes \mathcal{K} \to B \otimes \mathcal{K}$*, such that* $\mathrm{Ad}(\theta) \circ (\alpha \otimes \mathrm{id})$ *is cocycle-conjugate to* $\beta \otimes \mathrm{id}$*.*
2. *If in addition,* A *and* B *are stable, there is an isomorphism* $\theta : A \to B$ *such that* $\mathrm{Ad}(\theta) \circ \alpha$ *is cocycle-conjugate to* β*.*

The main ingredient is:

Theorem (Brown, Brown-Green-Rieffel '77)

1. *Let A, B be σ-unital, Morita-equivalent C^*-algebras with linking algebra C. Then there is a partial isometry $w \in M(C \otimes \mathcal{K})$ such that*

$$ww^* = \begin{pmatrix} 1_A \otimes 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad w^*w = \begin{pmatrix} 0 & 0 \\ 0 & 1_B \otimes 1 \end{pmatrix}$$

2. *If in addition A and B are stable, there is a partial isometry $w \in M(C \otimes \mathcal{K})$ such that*

$$ww^* = \begin{pmatrix} 1_A \otimes e_{11} & 0 \\ 0 & 0 \end{pmatrix}, \quad w^*w = \begin{pmatrix} 0 & 0 \\ 0 & 1_B \otimes e_{11} \end{pmatrix}$$

Proof of Combes' theorem

Proof.

Let $w \in M(C \otimes \mathcal{K})$ such that

$$ww^* = \begin{pmatrix} 1_A \otimes 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad w^*w = \begin{pmatrix} 0 & 0 \\ 0 & 1_B \otimes 1 \end{pmatrix}$$

This defines an isomorphism

$$\theta : A \otimes \mathcal{K} \to B \otimes K, \quad x \mapsto w^*xw$$

whose inverse is given by

$$\theta^{-1} : B \otimes \mathcal{K} \to A \otimes \mathcal{K}, \quad y \mapsto wyw^*$$

Write $\tilde{\alpha} = \alpha \otimes \mathrm{id}, \tilde{\beta} = \beta \otimes \mathrm{id}$. For $b \in B \otimes \mathcal{K}$ and $g \in G$, we have

$$\mathrm{Ad}(\theta) \circ \tilde{\alpha}_g(b) = w^* \tilde{\alpha}_g(wbw^*)w = w^* \tilde{\gamma}_g(w) \tilde{\beta}_g(b) \tilde{\gamma}_g(w)^* w$$

where $\tilde{\gamma}$ denotes the action on $M(C \otimes \mathcal{K})$. Thus,

$$c : G \to UM(B \otimes \mathcal{K}), \quad g \mapsto w^* \tilde{\gamma}_g(w) 1_B$$

is a $\tilde{\beta}$-1-cocycle implementing a cocycle equivalence.
The proof of the stable case is analogous, only that we use the partial isometries from the second part of the Brown-Green-Rieffel theorem and identify $A \cong A \otimes e_{11}$ and $B \cong B \otimes e_{11}$. □

Compact operators

Definition

Let E, F be Hilbert-B-modules and $\eta \in E, \xi \in F$. Then the operator

$$\Theta_{\xi,\eta} : E \to F, \quad \Theta_{\xi,\eta}(\zeta) := \xi\langle\eta,\zeta\rangle_B$$

is adjointable with $\Theta^*_{\xi,\eta} = \Theta_{\eta,\xi}$. We call

$$\mathcal{K}_B(E,F) := \overline{\mathrm{span}}\{\Theta_{\xi,\eta} : \xi \in E, \eta \in F\} \subseteq \mathcal{L}_B(E,F)$$

the compact operators.

Remark

- $\mathcal{K}_B(E) \subseteq \mathcal{L}_B(E)$ is a twosided ideal.
- If E carries an action $u : G \to \mathrm{Aut}(E)$, then G acts on $\mathcal{L}_B(E)$ and $\mathcal{K}_B(E)$ by $\mathrm{Ad}(u)_g(T) := u_g \circ T \circ u_g^{-1}$.

Lemma (Charactarization of Morita-equivalence)

1. *Let E be a G-equivariant A-B-imprimitivity bimodule. Then the left action of A defines an isomorphism of G-C^*-algebras*

$$\Phi : A \to \mathcal{K}_B(E)$$

2. *Every G-equivariant Hilbert-B-module F is a G-equivariant $\mathcal{K}_B(F)$-$\overline{\text{span}}\langle F, F\rangle_B$-imprimitivity bimodule.*

Proof.

For ${}_A\langle \xi, \eta\rangle \in {}_A\langle E, E\rangle$ and $\zeta \in E$ we have

$$\Phi({}_A\langle \xi, \eta\rangle)(\zeta) = {}_A\langle \xi, \eta\rangle\zeta = \underbrace{\Theta_{\xi,\eta}}_{\in\mathcal{K}_B(E)}(\zeta)$$

Since $\overline{\text{span}}_A\langle E, E\rangle = A$, we have that Φ is well-defined and surjective.

To prove injectivity, let $a \in A$ with $\phi(a) = 0$. For $\sum_i {}_A\langle \xi_i, \eta_i \rangle \in A$ we have

$$a \sum_i {}_A\langle \xi_i, \eta_i \rangle = \sum_i {}_A\langle \Phi(a)\xi_i, \eta_i \rangle = 0$$

Since span ${}_A\langle E, E \rangle \subseteq A$ is dense, this implies $a = 0$.
We can check equivariance for ${}_A\langle \xi, \eta \rangle \in A$: Plug in $g \in G$ and $\zeta \in E$ to get

$$\begin{aligned} \Phi(\alpha_g({}_A\langle \xi, \eta \rangle))\zeta &= {}_A\langle u_g\xi, u_g\eta \rangle \zeta = u_g(\xi)\langle u_g(\eta), \zeta \rangle_B \\ = &u_g(\xi\langle \eta, u_{g^{-1}}(\zeta) \rangle_B) = \mathrm{Ad}(u_g)(\Theta_{\xi,\eta})(\zeta) = \mathrm{Ad}(u_g)(\Phi({}_A\langle \xi, \eta \rangle))(\zeta) \end{aligned}$$

Now let F be a G-Hilbert-B-module. We equip it with the obvious left $\mathcal{K}_B(F)$-action and inner product defined by

$$_{\mathcal{K}_B(F)}\langle \xi, \eta \rangle := \Theta_{\xi,\eta}, \quad \xi, \eta \in E$$

One easily checks, that this defines a G-equivariant $\mathcal{K}_B(F)$-$\overline{\text{span}}\langle F, F \rangle_B$-imprimitivity-bimodule. □

Crossed products

Theorem

Let A, B be Morita-equivalent G-C^*-algebras. Then the crossed products $A \rtimes_{(r)} G$ and $B \rtimes_{(r)} G$ are also Morita-equivalent.

Proof for discrete G.

Let (E, u) be a G-equivariant A-B-imprimitivity bimodule. We turn $C_c(G, E)$ into a $C_c(G, A)$-$C_c(G, B)$-pre-imprimitivity-bimodule via

$$f * \xi(t) := \sum_{s \in G} f(s) u_s(\xi(s^{-1}t)), \quad f \in C_c(G, A), \xi \in C_c(G, E), t \in G$$

$$\xi * g(t) := \sum_{s \in G} \xi(s) \beta_s(g(s^{-1}t)), \quad \xi \in C_c(G, E), g \in C_c(G, B)$$

$${}_{C_c(G,A)}\langle \xi, \eta \rangle(t) := \sum_{s \in G} {}_A\langle \xi(s), u_t(\eta(s^{-1}t)) \rangle, \quad \xi, \eta \in C_c(G, E), t \in G$$

$$\langle \xi, \eta \rangle_{C_c(G,B)}(t) := \sum_{s \in G} \alpha_s(\langle \xi(s^{-1}), \eta(s^{-1}t) \rangle_B), \quad \xi, \eta \in C_c(G, E), t \in G$$

One checks, that $C_c(G, E)$ completes to an $A \rtimes G$-$B \rtimes G$-imprimitivity bimodule and an $A \rtimes_r G$-$B \rtimes_r G$-imprimitivity-bimodule respectively. To memorize the formulars, consider the $A \rtimes G$-$A \rtimes G$-imprimitivity bimodule $A \rtimes G$, write down the formulars in this case, and generalize to the above setting. For non-discrete G, insert integrals and modular functions. □

Tensor products

Definition (balanced tensor product)

Let (E, u) a G-Hilbert-A-module, (F, v) a G-Hilbert-B-module and $\phi : A \to \mathcal{L}_B(F)$ an equivariant $*$-homomorphism. Define an B-valued inner product on $E \odot_A F := {E \odot F}/{\xi a \otimes \eta \sim \xi \otimes \phi(a)\eta}$ by

$$\langle \xi \otimes \eta, \xi' \otimes \eta' \rangle_B := \langle \eta, \phi(\langle \xi, \xi' \rangle_A)\eta' \rangle_B$$

Divide $E \odot_A F$ by the nullspace of the seminorm $\|\xi\| := \sqrt{\|\langle \xi, \xi \rangle\|}$ and complete to get a G-Hilbert-B-module $(E \otimes_A F, w)$ with

$$w_g = u_g \otimes v_g, \quad g \in G$$

Induced representations

Definition

If $\phi : A \to \mathcal{L}_B(E)$ is an equivariant homomorphism, then E is also called a G-Hilbert-A-B-bimodule.

Definition

Let A, B be G-C^*-algebras, (E, u) a G-Hilbert-A-B-bimodule and (H, π, v) a covariant representation of B. Then (H, v) is canonically a G-Hilbert-B-$\mathbb{C}$ bimodule. Then

$$E\text{-}\operatorname{Ind}_B^A(\pi) : A \to \mathcal{L}_{\mathbb{C}}(E \otimes_B H)$$
$$E\text{-}\operatorname{Ind}_B^A(\pi)(a)(\xi \otimes \eta) := (a\xi) \otimes \eta, \quad a \in A, \xi \in E, \eta \in H$$

is called the induced representation on the Hilbert space $E \otimes_B H =: E\text{-}\operatorname{Ind}_B^A(H)$. A calculation shows that $(E\text{-}\operatorname{Ind}_B^A(H), E\text{-}\operatorname{Ind}_B^A(\pi), u \otimes v)$ actually is a covariant representation of A.

Morita-equivalence and Representations

Theorem

1. *Let (E, u) be a G-equivariant A-B-imprimitivity bimodule. Then*

$$\phi : E \otimes_B E^* \to A, \quad \xi \otimes \eta^* \mapsto {}_A\langle \xi, \eta \rangle$$

   is an isomorphism of G-equivariant-A-A-imprimitivity-bimodules.
2. *The functor $- \otimes_A E$ induces an equivalence of the categories of G-Hilbert-A-modules and G-Hilbert-B-modules.*
3. *The functor $E\text{-}\mathrm{Ind}_B^A$ induces an equivalence of the categories of covariant representations of B and covariant representations of A.*
4. *The above functors preserve direct sums, unitary operators and compact operators.*

Proof.

By definition, ϕ is surjective and G-equivariant. It is well-defined and isometric, since for $\xi, \eta, \zeta, \theta \in E$ we have

$$\begin{aligned} &\langle \phi(\xi \otimes \eta^*), \phi(\zeta \otimes \theta^*) \rangle_A = ({}_A\langle \xi, \eta \rangle)^* {}_A\langle \zeta, \theta \rangle \\ =&{}_A\langle \eta, \xi \rangle {}_A\langle \zeta, \theta \rangle = {}_A\langle \eta, {}_A\langle \theta, \zeta \rangle \xi \rangle = {}_A\langle \eta, \theta \langle \zeta, \xi \rangle_B \rangle \\ =&\langle \eta^*, \langle \xi, \zeta \rangle_B \theta^* \rangle_A = \langle \xi \otimes \eta^*, \zeta \otimes \theta^* \rangle_A \end{aligned}$$

The statements for the functors follow immediately and the compatibility with direct sums, and unitary/compact operators follow by computation. □

Corollary

A G-equivariant A-B-imprimitivity-bimodule E induces a KK^G-equivalence for A and B and a bijection of irreducible covariant representations of A and B.

Thank you for your attention