Topological dynamics and classification

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Observe that $C(X) \rtimes_h \mathbb{Z}$ is never commutative when h is nontrivial.

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How do we calculate the invariants? There is a very useful exact sequence that helps in the computation of K-theory of $C(X) \rtimes_h \mathbb{Z}$. Traces are easy to describe: they are in one-to-one with invariant Borel measures on X.

Example: irrational rotation algebras

Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$, and let $h_\theta \colon S^1 \to S^1$ be $h_\theta(z) = e^{2\pi i\theta} z$ for $z \in S^1$.

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Becomes easy using classification.

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Thus $A_\theta \cong A_{\theta'}$ if and only if $\theta \pm \theta'$ is an integer.

Thank you.