# ON $m$-COVERING FAMILIES OF BEATTY SEQUENCES WITH IRRATIONAL MODULI

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ABSTRACT. We generalise Uspensky's theorem characterising eventual exact (e.e.) covers of the positive integers by homogeneous Beatty sequences, to e.e. $m$-covers, for any $m \in \mathbb{N}$, by homogeneous sequences with irrational moduli. We also consider inhomogeneous sequences, again with irrational moduli, and obtain a purely arithmetical characterisation of e.e. $m$-covers. This generalises a result of Graham for $m = 1$, but when $m > 1$ the arithmetical description is more complicated. Finally we speculate on how one might make sense of the notion of an exact $m$-cover when $m$ is not an integer, and present a 'fractional version' of Beatty's theorem.

## 1. INTRODUCTION AND STATEMENT OF RESULTS

Throughout this paper, the integer and fractional parts of a real number $x$ will be denoted by $\lfloor x \rfloor$ and $\{x\}$ respectively. Hence

$$\{x\} = x - \lfloor x \rfloor. \tag{1.1}$$

We trust that no confusion will arise from using the same notation for sets as for fractional parts of numbers.

Next, we define the terms in the title of the article.

**Definition 1.1.** Let $\alpha, \beta \in \mathbb{R}$ with $\alpha > 0$. Denote

$$S(\alpha, \beta) := \{\lfloor n\alpha + \beta \rfloor : n \in \mathbb{N}\}. \tag{1.2}$$

We wish to think of $S(\alpha, \beta)$ as a multiset of integers : in other words, if some integer appears more than once (which will be the case whenever $\alpha < 1$), then we take account of the number of times it appears. The multiset $S(\alpha, \beta)$ is called a *Beatty sequence*. The number $\alpha$ is called the *modulus* of the sequence. If $\beta = 0$ we say that the Beatty sequence is *homogeneous*, otherwise *inhomogeneous*. Note that, if $\alpha \in \mathbb{N}$, then $S(\alpha, \beta)$ is an arithmetic progression (AP).

**Definition 1.2.** Let $m$ be a positive integer, $I$ a finite index set and $\{S_i : i \in I\}$ a family of multisets of integers. The family is said to be an *$m$-cover* if every integer appears at least $m$ times in the union of the $S_i$, counting multiplicities. If every integer appears exactly $m$ times, we say that the $m$-cover is *exact*. A little more generally, if every sufficiently large positive integer appears at least (resp. exactly) $m$ times, we speak of an *eventual (exact) $m$-cover*. Eventual exact $m$-covers are the primary objects

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of study in this paper, and we shall henceforth use the acronym $m$-EEC for these. In addition, we shall always drop the prefix when $m = 1$.

**Remark 1.3.** It is not hard to see that an eventual (exact) $m$-covering family of APs is in fact an (exact) $m$-cover. However, the same need not be true of more general Beatty sequences.

**Definition 1.4.** Let $m > 1$ and $\{S_i : i \in I\}$ be an $m$-EEC. We say that this covering family is *reducible* if there exist positive integers $m_1, m_2$ satisfying $m_1 + m_2 = m$ and a partition $I = J \sqcup K$, such that $\{S_i : i \in J\}$ is an $m_1$-EEC and $\{S_i : i \in K\}$ is an $m_2$-EEC. Otherwise, the cover is called *irreducible*.

The basic problem of interest is to characterise all $m$-EEC's consisting of Beatty sequences. The main new results of this paper provide such characterisations for all $m \in \mathbb{N}$, when the moduli of the sequences are all irrational.

We begin with a brief survey of the existing literature. Henceforth, it is to be understood that 'cover' always refers to a covering family of Beatty sequences. It is clear that a necessary condition for the family $\{S(\alpha_i, \beta_i) : i = 1, ..., k\}$ to be an $m$-EEC is that

$$\sum_{i=1}^{k} \frac{1}{\alpha_i} = m. \tag{1.3}$$

There is a considerable literature on the case $m = 1$ - for a recent overview and a much more exhaustive list of references than those given here, see Section 10 of [F09]. In the case of homogeneous sequences, there is a classical result :

**Theorem 1.5.** *Let $\alpha_1, ..., \alpha_k$ be positive real numbers. Then $\{S(\alpha_1, 0), ..., S(\alpha_k, 0)\}$ is an EEC if and only if (1.3) holds and either*
*(i) $k = 1$ and $\alpha_1 = 1$, or*
*(ii) $k = 2$ and $\alpha_1 \notin \mathbb{Q}$.*

The sufficiency of condition (i) is trivial, that of (ii) is known as Beatty's theorem, though it was first discovered by Lord Rayleigh[1]. That $k \leq 2$ is necessary was first proven by Uspensky [U], using Kronecker's approximation theorem. A more elementary proof was later provided by Graham [Gr63].

When one allows inhomogeneous sequences, there is no such simple classification. However, a certain amount is known. In the case of two sequences with irrational moduli, there is the following generalisation of Beatty's theorem :

**Theorem 1.6. (Skolem** [S]**, Fraenkel** [F69]**)** *Let $\alpha_1, \beta_1, \alpha_2, \beta_2$ be real numbers, with $\alpha_1, \alpha_2$ positive, irrational and satisfying (1.3). Then $\{S(\alpha_1, \beta_1), S(\alpha_2, \beta_2)\}$ is an EEC if and only if*

$$\frac{\beta_1}{\alpha_1} + \frac{\beta_2}{\alpha_2} \in \mathbb{Z}. \tag{1.4}$$

[1]Condition (ii) guarantees that *every* positive integer occurs exactly once in the multiset $S(\alpha_1, 0) \cup S(\alpha_2, 0)$, and Beatty's theorem is usually stated in this form.

Let $\{S(\alpha_1, \beta_1), S(\alpha_2, \beta_2)\}$ be an EEC and suppose $\{S(a_i, \phi_i) : i = 1, ..., \mu\}$ and $\{S(c_j, \psi_j) : j = 1, ..., \nu\}$ are exact covering families of arithmetic progressions. Then, clearly,

$$\left\{\bigcup_{i=1}^{\mu} S(\alpha_1 a_i, \alpha_1 \phi_i + \beta_1)\right\} \cup \left\{\bigcup_{j=1}^{\nu} S(\alpha_2 c_j, \alpha_2 \psi_j + \beta_2)\right\} \tag{1.5}$$

is also an EEC. Graham [Gr73] proved that any EEC in which at least one of the moduli is irrational must have the form (1.5). In particular, this implies that the moduli in an EEC are either all rational or all irrational. It also reduces the classification of EEC's with irrational moduli to that of EEC's with integer moduli, that is, of exact covering families of APs. The latter problem has a long history but remains inadequately resolved. For an introduction to known results and open problems concerning covers and exact covers by APs, see Problems F13-14 in [Gu]. One noteworthy fact is that the moduli in a covering family of APs cannot all be distinct. A beautiful proof of this, using generating functions, can be found in [E]. Graham's 1973 result implies that the same is true of EEC's with irrational moduli. An important open problem in this field concerns EEC's with distinct rational moduli. Fraenkel [F73] conjectured the following :

**Fraenkel's Tiling Conjecture.** *Let $k \geq 3$, let $0 < \alpha_1 < \cdots < \alpha_k$ and let $\beta_1, ..., \beta_k$ be any real numbers. Then the family $\{S(\alpha_i, \beta_i) : i = 1, ..., k\}$ is an EEC if and only if*

$$\alpha_i = \frac{2^k - 1}{2^{k-i}}, \quad i = 1, ..., k. \tag{1.6}$$

So let us turn to $m > 1$. Now one is interested in characterising irreducible $m$-EECs. In the case of APs, the existence of irreducible exact $m$-covers, for every $m > 1$, was first demonstrated by Zhang Ming-Zhi [Z]. Graham and O'Bryant [GrOB] studied $m$-EEC's with rational moduli, and proposed a generalisation of Fraenkel's Tiling Conjecture. The remainder of this paper is concerned with irrational moduli. The only result we could find in the literature is the following generalisation of Beatty's theorem :

**Theorem 1.7.** *Let $m \in \mathbb{N}$ and $\alpha_1, \alpha_2$ be positive irrational numbers satisfying $1/\alpha_1 + 1/\alpha_2 = m$. Then every positive integer appears exactly $m$ times in the multiset union $S(\alpha_1, 0) \cup S(\alpha_2, 0)$.*

This result seems to first appear in [OB]. The proof given there is not difficult, but employs generating functions. A completely elementary proof was given by Larsson [L], whose motivation for studying $m$-covers came from combinatorial games. Note that Theorem 1.7 implies that irreducible, homogeneous $m$-EEC's with irrational moduli do exist for every $m > 1$. It turns out, however, that Theorem 1.7 describes all of them. The first main result of this paper is the following :

**Theorem 1.8.** *Let $m \in \mathbb{N}$. Let $\alpha_1, ..., \alpha_k$ be positive irrational numbers satisfying (1.3). Then $\{S(\alpha_1, 0), ..., S(\alpha_k, 0)\}$ is an $m$-EEC if and only if $k$ is even, $k = 2l$ say,*

*and the $\alpha_i$ can be re-ordered so that*

$$\frac{1}{\alpha_{2i-1}} + \frac{1}{\alpha_{2i}} \in \mathbb{Z}, \quad i = 1, ..., l. \tag{1.7}$$

From this we shall deduce the following generalisation of Theorem 1.5 :

**Theorem 1.9.** *Let $m \in \mathbb{N}$ and $\alpha_1, ..., \alpha_k$ be positive real numbers, not all rational. Then $\{S(\alpha_1, 0), ..., S(\alpha_k, 0)\}$ is an irreducible $m$-EEC if and only if $k = 2$ and (1.3) holds.*

Turning to the inhomogeneous case, Theorem 1.6 generalises verbatim to $m > 1$. Since we could not find this fact stated explicitly anywhere in the literature, and our proof of it follows a different approach from that in [F69], we state it as a separate result :

**Theorem 1.10.** *Let $m \in \mathbb{N}$. Let $\alpha_1, \beta_1, \alpha_2, \beta_2$ be real numbers, with $\alpha_1, \alpha_2$ positive, irrational and satisfying (1.3). Then $\{S(\alpha_1, \beta_1), S(\alpha_2, \beta_2)\}$ is an $m$-EEC if and only if (1.4) holds.*

Given the previous three results, it is natural to ask if Graham's 1973 result generalises as follows :

**Question 1.11.** *Let $m \in \mathbb{N}$. Is it true that every $m$-EEC with irrational moduli has the form*

$$\bigcup_{k=1}^{t} \left\{ \left\{ \bigcup_{i=1}^{\mu_k} S(\alpha_{2k-1}a_{i,k}, \alpha_{2k-1}\phi_{i,k} + \beta_{2k-1}) \right\} \cup \left\{ \bigcup_{j=1}^{\nu_k} S(\alpha_{2k}c_{j,k}, \alpha_{2k}\psi_{j,k} + \beta_{2k}) \right\} \right\} \tag{1.8}$$

*where there are positive integers $m_1, ..., m_t, d_1, ..., d_t$ satisfying $m_1d_1 + \cdots + m_td_t = m$, and, for each $k = 1, ..., t$, one has*

*(i)*

$$\frac{1}{\alpha_{2k-1}} + \frac{1}{\alpha_{2k}} = m_k, \tag{1.9}$$

*(ii) $\{S(a_{i,k}, \phi_{i,k}) : i = 1, ..., \mu_k\}$ and $\{S(c_{j,k}, \psi_{j,k}) : j = 1, ..., \nu_k\}$ are exact $d_k$-covering families of APs*

*(iii)*

$$\frac{\beta_{2k-1}}{\alpha_{2k-1}} + \frac{\beta_{2k}}{\alpha_{2k}} \in \mathbb{Z} \, ? \tag{1.10}$$

The second main result of our paper is a negative answer to this question. We shall give explicit counterexamples and provide a description, in terms of APs, of the most general possible form of an $m$-EEC with irrational moduli (see Section 4 below). While this provides a 'purely arithmetical/combinatorial' characterisation of such $m$-EECs, generalising [Gr73], we do not find our result satisfactory and feel that a simpler and more explicit description may be possible. This point will be discussed again later on.

The rest of the paper is organised as follows. In Section 2, we give prerequisite notation, terminology and background results. As well as extending known theorems, a secondary purpose of our paper is to provide a uniform treatment of this material, something which we have found lacking in the existing literature. Our approach is based on Weyl's equidistribution theorem, and is thus most similar in spirit to that followed by Uspensky [U]. However, he only employed a weaker equidistribution result (Kronecker's theorem), and we also make more explicit the formula for the representation function $r(N)$, which counts the number of occurrences of the integer $N$ in a covering family, in terms of sums of fractional parts (see eq. (2.12) below). Already in Section 2, we will prove Theorem 1.10 - this proof is extremely simple and provides the reader with a quick glimpse of our method. Section 3 deals with homogeneous Beatty sequences and the proof of Theorems 1.8 and 1.9. This section is the heart of the paper. In Section 4, we turn to the inhomogeneous case and the issue of how to properly generalise [Gr73]. In Section 5, we briefly broach the question of how one might make sense of the notion of $m$-cover, when $m$ is not an integer. What we will actually prove is a fractional version of Beatty's theorem. This follows a suggestion of Fraenkel, who was also interested in possible connections to combinatorial games. Further development of this line of investigation is left for future work, the possibilities for which we summarise in Section 6.

## 2. PRELIMINARIES AND PROOF OF THEOREM 1.10

Our approach is based on standard results concerning equidistribution of sequences. We have chosen the following formulation as it seems the most natural one, even if we could get away with something less (see Remark 2.2 below) :

**Lemma 2.1.** *Let $\tau_1 = \frac{p_1}{q_1}, ..., \tau_k = \frac{p_k}{q_k}$ be rational numbers written in lowest terms, whose denominators are co-prime, i.e.: GCD$(p_i, q_i)$ $= 1$ for $i = 1, .., k$ and GCD$(q_i, q_j) = 1$ for $i \neq j$. Let $\theta_1, ..., \theta_l$ be irrational numbers which are affine independent over $\mathbb{Q}$, i.e.: the equation*

$$c_0 + c_1\theta_1 + \cdots + c_l\theta_l = 0, \quad c_0, c_1, ..., c_l \in \mathbb{Q}, \tag{2.1}$$

*has only the trivial solution $c_0 = c_1 = \cdots = c_l = 0$.*

*For each $i = 1, ..., k$, let $\mu_i$ be the measure on $[0, 1)$ which gives measure $1/q_i$ to each point mass $u/q_i$, $u = 0, 1, ..., q_i - 1$. Let $\mu_0$ be Lebesgue measure and let $\mu$ be the measure on $[0, 1)^{k+l}$ given by the product*

$$\mu = \mu_1 \times \cdots \times \mu_k \times \mu_0^l. \tag{2.2}$$

*Then, as $n$ ranges over the natural numbers, the $(k + l)$-tuple*

$$(\{n\tau_1\}, ..., \{n\tau_k\}, \{n\theta_1\}, ..., \{n\theta_l\}) \tag{2.3}$$

*is equidstributed on $[0, 1)^{k+l}$ with respect to $\mu$.*

*Proof.* When all the moduli are rational $(l = 0)$, this is just the Chinese Remainder Theorem. For general $l > 0$, the lemma thus asserts that the $l$-tuple

$$(\{n\theta_1\}, ..., \{n\theta_l\}) \tag{2.4}$$

is equidistributed on $[0,1)^l$, when $n$ runs through any infinite arithmetic progression. This fact can be immediately deduced from the multi-dimensional Weyl criterion - see [KN], for example. $\square$

**Remark 2.2.** As previously noted, we will not be needing the full force of the lemma. What we will use is the consequence that, for any subintervals $I_1, ..., I_l$ of $[0,1)$ and any arithmetic progression $S(a,b)$, there are arbitrarily large $n \in S(a,b)$ for which the $l$-tuple (2.4) lies in $I_1 \times \cdots \times I_l$.

Fix $m, k \in \mathbb{N}$. Let real numbers $\alpha_i, \beta_i$, $i = 1, ..., k$, be given with the $\alpha_i$ positive, irrational and satisfying (1.3). To simplify notation, put

$$\theta_i := \frac{1}{\alpha_i}, \quad \gamma_i := -\frac{\beta_i}{\alpha_i}, \quad i = 1, ..., k. \tag{2.5}$$

Hence,

$$\sum_{i=1}^{k} \theta_i = m. \tag{2.6}$$

For $N \in \mathbb{N}$ and $i \in \{1, ..., k\}$, set

$$r_i(N) := \#\{n \in \mathbb{N} : \lfloor n\alpha_i + \beta \rfloor = N\}. \tag{2.7}$$

Setting

$$r(N) := \sum_{i=1}^{k} r_i(N), \tag{2.8}$$

we note that the family $\{S(\alpha_i, \beta_i) : i = 1, ..., k\}$ is an $m$-EEC if and only if $r(N) = m$ for all $N \gg 0$. The function $r(\cdot)$ will be called the *representation function* associated to the family $\{S(\alpha_i, \beta_i) : i = 1, ..., k\}$.

For each $i$, since $\alpha_i$ is irrational, there is at most one integer $n_i$ such that $n_i\alpha_i + \beta_i \in \mathbb{Z}$. Hence $n\alpha_i + \beta_i \notin \mathbb{Z}$ for all $n \gg 0$ and all $i$. It follows that, for $N \gg 0$,

$$r_i(N) = \#\{n \in \mathbb{N} : N < n\alpha_i + \beta_i < N + 1\}, \tag{2.9}$$

the point being that both inequalities are strict. One easily deduces that

$$r_i(N) = \lfloor (N+1)\theta_i + \gamma_i \rfloor - \lfloor N\theta_i + \gamma_i \rfloor. \tag{2.10}$$

Define the function $\epsilon : \mathbb{Z} \to \mathbb{R}$ by

$$\epsilon(N) := \sum_{i=1}^{k} \{N\theta_i + \gamma_i\}. \tag{2.11}$$

From (1.1), (2.6) and (2.10) one easily deduces that

$$r(N) = m + (\epsilon(N) - \epsilon(N+1)). \tag{2.12}$$

Hence, the Beatty sequences form an $m$-EEC if and only if the function $\epsilon(N)$ is constant for all $N \gg 0$. We can already quickly deduce Theorem 1.10. For if we have only two sequences, then since $\theta_1 + \theta_2 \in \mathbb{Z}$ one has

$$\epsilon(N) = \begin{cases} \{\gamma_1 + \gamma_2\}, & \text{if } \{N\theta_1 + \gamma_1\} < \{\gamma_1 + \gamma_2\}, \\ 1 + \{\gamma_1 + \gamma_2\}, & \text{otherwise.} \end{cases} \tag{2.13}$$

It follows that, if $\gamma_1+\gamma_2 \in \mathbb{Z}$, then $\epsilon(N) = 1$ for all $N \in \mathbb{Z}$, whereas if $\gamma_1+\gamma_2 \notin \mathbb{Z}$ then, since $\theta_1 \notin \mathbb{Q}$, a very weak form of Lemma 2.1 (already known to Dirichlet) implies that $\{N\theta_1 + \gamma_1\} - \{\gamma_1 + \gamma_2\}$ will be both positive and negative for arbitrarily large $N$.

## 3. THE HOMOGENEOUS CASE - PROOFS OF THEOREMS 1.8 AND 1.9.

The proof of Theorem 1.8 will exhibit the main ideas of this paper, so we will present it in detail, which will allow us to be more brief with all subsequent proofs. So let's now assume that all our sequences are homogeneous. Hence $\beta_i = \gamma_i = 0$ for $i = 1, ..., k$ and

$$\epsilon(N) = \sum_{i=1}^{k} \{N\theta_i\}. \tag{3.1}$$

Let $V$ be the vector space over $\mathbb{Q}$ spanned by $1, \theta_1, ..., \theta_k$. Since the $\theta_i$ are irrational, we know that $\dim(V) > 1$. Let $\dim(V) := d+1$ and, without loss of generality, assume that $1, \theta_1, ..., \theta_d$ form a basis for $V$. Hence there exist rational numbers $q_{j,i}$, $0 \le j \le d$, $1 \le i \le k$ such that

$$\theta_i = q_{0,i} + \sum_{j=1}^{d} q_{j,i}\theta_j, \quad i = 1, ..., k, \tag{3.2}$$

where

$$1 \le i \le d \;\Rightarrow\; q_{j,i} = \begin{cases} 1, & \text{if } j = i, \\ 0, & \text{if } j \ne i \end{cases} \tag{3.3}$$

and

$$\sum_{i=0}^{k} q_{j,i} = \begin{cases} m, & \text{if } j = 0, \\ 0, & \text{if } j > 0. \end{cases} \tag{3.4}$$

Set

$$Q_{j,i} := \begin{cases} \{q_{j,i}\}, & \text{if } j = 0, \\ q_{j,i}, & \text{if } j > 0. \end{cases} \tag{3.5}$$

We may write each of the numbers $Q_{j,i}$ as a fraction in lowest terms, say

$$Q_{j,i} = \frac{u_{j,i}}{v_{j,i}}, \quad u_{j,i} \in \mathbb{Z},\ v_{j,i} \in \mathbb{N},\ \text{GCD}(u_{j,i}, v_{j,i}) = 1. \tag{3.6}$$

We shall prove Theorem 1.8 by induction on $m$. The case $m = 1$ follows from Theorem 1.5. If $m > 1$ then, in order to apply the induction hypothesis, it suffices, by Theorem 1.7, to find any pair $i_1, i_2 \in \{1, ..., k\}$ such that $\theta_{i_1} + \theta_{i_2} \in \mathbb{Z}$. Hence this is all we need to do to finish the proof. Using Lemma 2.1, we shall deduce it as a consequence of the requirement that the function $\epsilon(N)$, given by (2.11), be constant for all $N \gg 0$. In a way which we will make rigorous in what follows, that lemma will allow us to ignore the influence of all but one of $\theta_1, ..., \theta_d$ - for simplicity, we select $\theta_1$ (see eqs. (3.11) and (3.28)) - and then reduce the proof of the theorem to a purely combinatorial problem (Proposition 3.3 below).

To begin with, define positive integers $L_0, L$ by

$$L_0 := \text{LCM}\{v_{0,i} : q_{1,i} \ne 0\}, \quad L := \text{LCM}\{|u_{1,i}| : q_{1,i} \ne 0\}. \tag{3.7}$$

For each $i$ such that $q_{1,i} \neq 0$, define the numbers $U_i, V_i$ by

$$Q_{0,i} =: \frac{U_i}{L_0}, \quad q_{1,i} =: \frac{L}{V_i}. \tag{3.8}$$

Finally, we set

$$\begin{cases} a_i := L_0 V_i, \;\; b_i := -U_i V_i, & \text{if } q_{1,i} > 0, \\ c_j := -L_0 V_j, \;\; d_j := -U_j V_j, & \text{if } q_{1,j} < 0. \end{cases} \tag{3.9}$$

We shall use Lemma 2.1 to establish the following claim :

**Claim 3.2.** *If the function $\epsilon(N)$ is constant for all $N \gg 0$, then for every $t \in \mathbb{Z}$, we have an equality of multisets*

$$\bigcup_{q_{1,i}>0} S(a_i, tb_i) = \bigcup_{q_{1,j}<0} S(c_j, td_j). \tag{3.10}$$

Suppose the claim were false. Then clearly it must fail for some non-negative $t$. Choose such a $t$ and let $\eta_t$ be an element of the multiset difference. Without loss of generality, $\eta_t$ occurs more often on the left-hand side of (3.10), say $r$ times on the left-hand side and $s$ times on the right-hand side, with $r > s$. Now let $\delta$ be a sufficiently small, positive real number - how small it should be will become clear below. By Lemma 2.1, we can find arbitrarily large integers $n$ satisfying

$$n \equiv 1 \pmod{L_0 L}, \quad \delta < \{n\theta_1\} < \delta + e^{-1/\delta}, \quad \{n\theta_i\} < \delta^3, \; i = 2, ..., d. \tag{3.11}$$

Let $n_0$ be any positive integer satisfying (3.11). Let $N_+$ (resp. $N_-$) be the least positive integer which is divisible by $n_0$, congruent to $t$ modulo $L_0 L$ and greater than $\frac{1}{\delta L_0 L} n_0 \eta_t$ (resp. $\frac{1-\delta}{\delta L_0 L} n_0 \eta_t$). Then the point is that, provided $\delta$ is sufficiently small, for every $i = 1, ..., k$ we have

$$\lfloor N_+\{\theta_i\} + \{N_+ q_{0,i}\} \rfloor - \lfloor N_-\{\theta_i\} + \{N_- q_{0,i}\} \rfloor = \begin{cases} 1, & \text{if } q_{1,i} > 0 \text{ and } \eta_t \in S(a_i, tb_i), \\ -1, & \text{if } q_{1,i} < 0 \text{ and } \eta_t \in S(c_i, td_i), \\ 0, & \text{otherwise.} \end{cases} \tag{3.12}$$

This in turn is easily seen to imply that

$$\epsilon(N_+) - \epsilon(N_-) = s - r \neq 0. \tag{3.13}$$

Since the numbers $N_+$ and $N_-$ can be made arbitrarily large, this would mean that the function $\epsilon(N)$ could not be constant for $N \gg 0$, a contradiction which establishes Claim 3.2.

We state the next assertion as a separate proposition, as the reader may find it interesting in its own right. It is also the crucial combinatorial ingredient in this section :

**Proposition 3.3.** *Let $a_1, ..., a_\mu, c_1, ..., c_\nu$ be positive integers and $b_1, ..., b_\mu, d_1, ..., d_\nu$ be any integers. If, for every $t \in \mathbb{Z}$, we have an equality of multisets*

$$\bigcup_{i=1}^{\mu} S(a_i, tb_i) = \bigcup_{j=1}^{\nu} S(c_j, td_j), \tag{3.14}$$

*then $\mu = \nu$, and we can reorder so that, for each $i = 1, ..., \mu$, $a_i = c_i$ and $b_i \equiv d_i \pmod{a_i}$.*

The proof of the proposition will employ the following facts :

**Lemma 3.4.** *Let $p$ be a prime, $l$ a non-negative integer, $l_1, ..., l_\chi$ integers each strictly greater than $l$ and $b, d_1, ..., d_\chi$ any integers. Suppose that, as sets,*

$$S(p^l, b) \subseteq \bigcup_{j=1}^{\chi} S(p^{l_j}, d_j). \tag{3.15}$$

*Then,*

*(i) $S(p^l, b)$ equals the disjoint union of some subset of the terms on the right-hand side of (3.15),*

*(ii) Let $L$ be the maximum of the $l_j$. Then for some $\xi_1 \in \{0, 1, ..., p^L - 1\}$ there exists, for each $\xi_2 \in \{0, 1, ..., p-1\}$, some $j$ such that*

$$l_j = L \quad \textit{and} \quad d_j \equiv \xi_1 + \xi_2 p^{L-1} \pmod{p^L}. \tag{3.16}$$

*Proof. of Lemma 3.4.* These are standard observations which can be proven in various ways. For example, one can consider the $p$-ary rooted tree $\mathcal{T}$, whose nodes are all the progressions $S(p^i, u)$, where $0 \le i \le L$ and $0 \le u < p^i$, and in which, for $i < L$, the node $S(p^i, u)$ has the $p$ daughters $S(p^{i+1}, u + vp^i)$, $v = 0, 1, ...p-1$. Eq. (3.15) expresses the hypothesis that the rooted subtree $T_0$ under a certain node $x$ is, apart from the node $x$ itself, entirely contained inside the union of a collection $T_1, ..., T_\chi$ of rooted subtrees at strictly lower levels. Part (i) then asserts that some subset of the $T_1, ..., T_\chi$ are pairwise disjoint and their union equals $T_0 \backslash \{x\}$. This is simple to prove, for example by induction on the depth of $T_0$. Part (ii) is then also an immediate consequence of the rooted tree structure. □

*Proof. of Proposition 3.3.* We shall perform an induction on several different parameters. First of all, let $n$ be the total number of distinct primes which divide at least one of the moduli $a_i$ or $c_j$. If $n = 0$ then each individual AP is just $\mathbb{Z}$ and the proposition simply asserts the obvious fact that they must then be equal in number, i.e.: that $\mu = \nu$. So now suppose $n > 0$ and that the proposition is true for all smaller values of $n$. Let $p := p_1 < \cdots < p_n$ be the distinct primes which divide at least one modulus. Let $p^k$ denote the highest power of $p$ dividing any modulus and partition the moduli into subsets $M_0, M_0', ..., M_k, M_k'$, where

$$M_l := \{i : p^l \,||\, a_i\}, \; M_l' := \{j : p^l \,||\, c_j\}, \quad l = 0, ..., k. \tag{3.17}$$

By the Chinese Remainder Theorem, for each $i = 1, ..., \mu$ (resp. each $j = 1, ..., \nu$) we can write

$$S(a_i, tb_i) = S(p^{l_i}, tb_i) \cap S(A_i, tb_i) \quad (\text{resp. } S(c_j, td_j) = S(p^{l_j'}, td_j) \cap S(C_j, td_j)), \tag{3.18}$$

where $p^{l_i} \,||\, a_i$ and $A_i = a_i / p^{l_i}$ (resp. $p^{l_j'} \,||\, c_j$ and $C_j = c_j / p^{l_j'}$). Let $\xi \in \{0, 1, ..., p^k - 1\}$ and let $t$ be any integer s.t. $t \equiv 1 \pmod{p^k}$. Considering the intersection of both sides

of (3.14) with $S(p^k, \xi)$ we find that, as multisets,

$$\bigcup_{i: b_i \equiv \xi \pmod{p^k}} S(A_i, tb_i) = \bigcup_{j: d_j \equiv \xi \pmod{p^k}} S(C_j, td_j). \tag{3.19}$$

Now note that a necessary and sufficient condition for (3.14) to hold for every $t \in \mathbb{Z}$ is that it do so for any $t$ divisible only by those primes dividing some $a_i$ or $c_j$. Applying this observation to (3.19) instead, we deduce that the latter equality holds for every $t \in \mathbb{Z}$. Since there are exactly $n-1$ primes dividing some $A_i$ or $C_j$, we can apply the induction hypothesis to conclude that, for each $i$ such that $b_i \equiv \xi \pmod{p^k}$, there exists a $j$ such that $S(A_i, b_i) = S(C_j, d_j)$. For such a pair $(i, j)$ it follows that

$$S(a_i, b_i) \supseteq S(c_j, d_j) \Leftrightarrow l_i \leq l'_j. \tag{3.20}$$

Now we introduce the second induction parameter, which is the total number of APs involved in (3.14), i.e.: on the quantity $\mu + \nu$. It is clear that Proposition 3.3 holds if $\mu = \nu = 1$, so suppose $\mu + \nu > 2$ and that the proposition holds for any smaller value of $\mu + \nu$. If there were any pair $(i, j)$ whatsoever such that $S(a_i, b_i) = S(c_j, d_j)$, then we could immediately cancel this pair from (3.14) and apply the induction on $\mu + \nu$ to deduce the proposition. Hence, we may assume no such pair exists.

Let $l_{\min}$ (resp. $l'_{\min}$) denote the smallest value of $l$ (resp. $l'$) such that the set $M_l$ (resp. $M'_l$) is non-empty. We claim that $l_{\min} = l'_{\min}$. To see this, set $t := p^k$ in (3.14) and consider the contribution of both sides to numbers which are divisible by $p^l$ but not $p^{l+1}$, where $l = \min\{l_{\min}, l'_{\min}\}$. These contributions cannot be equal if $l_{\min} \neq l'_{\min}$, since then only one side would give a non-empty contribution. In fact, we can deduce much more. Let $l := l_{\min}$. It is clear that, for every $t^* \in \mathbb{Z}$, we have equality of multisets

$$\bigcup_{p^l || a_i} S(A_i, t^* b_i) = \bigcup_{p^l || c_j} S(C_j, t^* d_j). \tag{3.21}$$

By induction on the first parameter $n$, the total number of prime divisors of the $a_i$ and $c_j$, we can deduce that the progressions $S(A_i, b_i)$ for which $p^l \,||\, a_i$ and the progressions $S(C_j, d_j)$ for which $p^l \,||\, c_j$ are equal in pairs. This fact will be exploited later on.

For the next step in the argument, consider any $i$ for which $l_i = l_{\min}$. For each $\xi$ such that $S(p^{l_i}, b_i) \supseteq S(p^k, \xi)$ we can find, as shown earlier, some $j$ such that $S(A_i, b_i) = S(C_j, d_j)$ and $S(a_i, b_i) \supseteq S(c_j, d_j)$. Clearly, the multiset union of all these $S(c_j, d_j)$ must contain $S(a_i, b_i)$ and thus (3.20) and Lemma 3.4(i) imply that some subset of the $S(c_j, d_j)$ are pairwise disjoint and their union equals $S(a_i, b_i)$. To summarise, for any $i$ such that $l_i = l_{\min}$, we can find a set of $j$'s such that

$$S(C_j, d_j) = S(A_i, b_i) \text{ for each } j \text{ and } S(a_i, b_i) = \bigsqcup_j S(c_j, d_j). \tag{3.22}$$

These conditions imply that

$$S(p^{l_i}, b_i) = \bigsqcup_j S(p^{l'_j}, d_j). \tag{3.23}$$

If, in (3.22), we had $S(a_i, b_i) = S(c_j, d_j)$ for some $j$, then we could apply the induction on $\mu + \nu$. Hence we may assume that $l'_j > l_i$ for each $j$ in (3.23), and therefore Lemma 3.4(ii) applies to the $d_j$ in this union.

Now take $t = p$ in (3.14) and assume for the moment that there is some pair $(i_1, j_1)$ such that $S(a_{i_1}, pb_{i_1}) = S(c_{j_1}, pb_{j_1})$ and $p^{l_{\min}} \mid\mid a_{i_1}$. Then from (3.14) it would follow that, for every $t \in \mathbb{Z}$, we have the equality of multisets

$$\bigcup_{i \neq i_1} S(a_i, t(pb_i)) = \bigcup_{j \neq j_1} S(c_j, t(pb_j)). \tag{3.24}$$

Applying the induction hypothesis on $\mu + \nu$, we could then conclude that the arithmetic progressions $S(a_i, pb_i), i \neq i_1$ and $S(c_j, pd_j), j \neq j_1$ are equal in pairs. But then, by applying Lemma 3.4(ii) to any union of the type (3.22)-(3.23), we would find that there must, after all, be a pair $(i, j)$ such that $S(a_i, b_i) = S(c_j, d_j)$, so that the induction on $\mu + \nu$ yields the proposition.

Thus, finally, we may assume there is no pair $(i_1, j_1)$ satisfying the above requirements. But, from (3.21) we know that the progressions $S(a_i, p^l b_i)$ for which $p^l \mid\mid a_i$ and the progreesions $S(c_j, p^l d_j)$ for which $p^l \mid\mid c_j$, are equal in pairs. So we introduce a third and final induction parameter, namely the smallest integer $m$ such that there exists at least one pair $(i_1, j_1)$ such that $S(a_{i_1}, p^m b_{i_1}) = S(c_{j_1}, p^m d_{j_1})$ and $p^l \mid\mid a_{i_1}$. We know that $m$ is finite. But, if $m > 1$, then applying the previous argument for $m = 1$ to the multiset relation

$$\bigcup_{i=1}^{\mu} S(a_i, t(p^{m-1} b_i)) = \bigcup_{j=1}^{\nu} S(c_j, t(p^{m-1} d_j)), \quad \text{for all } t \in \mathbb{Z}, \tag{3.25}$$

we could conclude that the progressions $S(a_i, p^{m-1} b_i)$ and $S(c_j, p^{m-1} d_j)$ are equal in pairs, thus contradicting the definition of $m$.

This final contradiction completes the proof of Proposition 3.3. □

We can now complete the proof of Theorem 1.8. Let $i_1, ..., i_r$ be the indices for which $q_{1,i} \neq 0$. Our goal is to find a pair $u, v$ such that $\theta_{i_u} + \theta_{i_v} \in \mathbb{Z}$. Claim 3.2 and Proposition 3.3 already imply that we can pair off the $\theta_{i_j}$ such that the sum of each pair is in $\mathbb{Z}$, modulo their dependence on $\theta_2, ..., \theta_d$. Precisely, let $V_1$ be the $\mathbb{Q}$-vector subspace of $V$ spanned by $\theta_2, ..., \theta_d$. Then Claim 3.2 and Proposition 3.3 imply that $r$ is even, say $r = 2s$, and the indices $i_1, ..., i_r$ can be reordered so that, for $t = 1, ..., s$,

$$q_{1,i_{2t-1}} > 0, \quad q_{1,i_{2t}} = -q_{1,i_{2t-1}}, \quad q_{0,i_{2t-1}} + q_{0,i_{2t}} \in \mathbb{Z} \tag{3.26}$$

and hence

$$\theta_{i_{2t-1}} + \theta_{i_{2t}} = z_t + v_{1,t}, \quad \text{for some } z_t \in \mathbb{Z} \text{ and } v_{1,t} \in V_1. \tag{3.27}$$

Hence we would be done if we could find any $t$ for which $v_{1,t} = 0$. We can locate such a $t$ by a more refined application of Lemma 2.1. Let $\delta$ be a very small positive real number - how small is necessary will again become clear in due course. By Lemma 2.1, we can find arbitrarily large integers $n$ satisfying

$$n \equiv 0 \pmod{L_0} \text{ and } \delta^{2i-1} < \{\theta_i\} < \delta^{2i-1} + e^{-1/\delta}, \text{ for } i = 1, ..., d. \tag{3.28}$$

Let $M_1$ be the maximum of the numbers $q_{1,i_{2t-1}}$, $t = 1, ..., s$, and let $\mathcal{T}_1 := \{t : q_{1,i_{2t-1}} = M_1\}$. Now let

$$M_{2,+} := \max\{q_{2,i_{2t-1}} : t \in \mathcal{T}_1\}, \quad M_{2,-} := \min\{q_{2,i_{2t}} : t \in \mathcal{T}_1\}. \tag{3.29}$$

We claim that $M_{2,-} = -M_{2,+}$. Suppose this is not the case, and without loss of generality that $M_{2,+} > -M_{2,-}$. Let $\mathcal{T}_2 := \{t \in \mathcal{T}_1 : q_{2,i_{2t-1}} = M_{2,+}\}$. We shall prove a

contradiction to the assumption that the function $\epsilon(N)$ is constant for $N \gg 0$. Fix a very small $\delta > 0$, let $n_2$ be any integer satisfying (3.28) and take

$$N_{2,+} := 2n_2 \cdot \lceil \frac{1}{2(\delta M_1 + \delta^3 M_{2,+})} \rceil, \quad N_{2,-} := \frac{N_{2,+}}{2}. \tag{3.30}$$

Then the point is that, provided $\delta$ is small enough,

$$(\lfloor N_{2,+}\{\theta_i\} \rfloor, \lfloor N_{2,-}\{\theta_i\} \rfloor) = \begin{cases} (1,0), & \text{if } i = i_t \text{ for some } t \in \mathcal{T}_2, \\ \text{either } (0,0) \text{ or } (-1,-1), & \text{otherwise.} \end{cases} \tag{3.31}$$

Hence,

$$\epsilon(N_{2,-}) - \epsilon(N_{2,+}) = |\mathcal{T}_2| \neq 0, \tag{3.32}$$

giving the desired contradiction, since the numbers $N_{2,\pm}$ can be made arbitrarily large.

So we have shown that $M_{2,+} = M_{2,-}$. Let $M_2 := M_{2,+}$. With $\mathcal{T}_2$ as defined above we have, for each $t \in \mathcal{T}_2$, that

$$q_{\xi,i_{2t-1}} = M_\xi = -q_{\xi,i_{2t}}, \text{ for } \xi = 1, 2, \tag{3.33}$$

which in turn implies that, if $V_2$ is the $\mathbb{Q}$-vector subspace of $V$ spanned by $\theta_3, ..., \theta_d$, then, for each $t \in \mathcal{T}_2$,

$$\theta_{i_{2t-1}} + \theta_{i_{2t}} = z_t + v_{2,t}, \quad \text{for some } z_t \in \mathbb{Z} \text{ and } v_{2,t} \in V_2. \tag{3.34}$$

The idea now is to iterate the same kind of argument to produce a sequence of non-empty sets of indices

$$\mathcal{T}_1 \supseteq \mathcal{T}_2 \supseteq \cdots \supseteq \mathcal{T}_d \tag{3.35}$$

such that, for any $j = 1, ..., d$ and any $t \in \mathcal{T}_j$,

$$\theta_{i_{2t-1}} + \theta_{i_{2t}} = z_t + v_{j,t}, \quad \text{for some } z_t \in \mathbb{Z} \text{ and } v_{j,t} \in V_j. \tag{3.36}$$

Since $V_d = \{0\}$ we will be done at the $d$:th and final step of this process.

We have already described in detail the first two steps of the process, but for the sake of completeness, let us describe just one further step. Let

$$M_{3,+} := \max\{q_{3,i_{2t-1}} : t \in \mathcal{T}_2\}, \quad M_{3,-} := \min\{q_{3,i_{2t}} : t \in \mathcal{T}_2\}. \tag{3.37}$$

We claim that $M_{3,-} = -M_{3,+}$. Suppose this is not the case, and without loss of generality that $M_{3,+} > -M_{3,-}$. Let $\mathcal{T}_3 := \{t \in \mathcal{T}_2 : q_{3,i_{2t-1}} = M_{3,+}\}$. We shall prove a contradiction to the assumption that the function $\epsilon(N)$ is constant for $N \gg 0$. Fix a very small $\delta > 0$, let $n_3$ be any integer satisfying (3.28) and take

$$N_{3,+} := 2n_3 \cdot \lceil \frac{1}{2(\delta M_1 + \delta^3 M_2 + \delta^5 M_{3,+})} \rceil, \quad N_{3,-} := \frac{N_{3,+}}{2}. \tag{3.38}$$

Then the point is that, provided $\delta$ is small enough,

$$(\lfloor N_{3,+}\{\theta_i\} \rfloor, \lfloor N_{3,-}\{\theta_i\} \rfloor) = \begin{cases} (1,0), & \text{if } i = i_t \text{ for some } t \in \mathcal{T}_3, \\ \text{either } (0,0) \text{ or } (-1,-1), & \text{otherwise.} \end{cases} \tag{3.39}$$

Hence,

$$\epsilon(N_{3,-}) - \epsilon(N_{3,+}) = |\mathcal{T}_3| \neq 0, \tag{3.40}$$

giving the desired contradiction, since the numbers $N_{3,\pm}$ can be made arbitrarily large.

So we have shown that $M_{3,+} = M_{3,-}$. Letting $M_3 := M_{3,+}$ and with $\mathcal{T}_3$ as above, we have shown that

$$t \in \mathcal{T}_3 \Rightarrow q_{\xi,i_{2t-1}} = M_\xi = -q_{\xi,i_{2t}}, \quad \text{for } \xi = 1, 2, 3, \tag{3.41}$$

from which (3.36) immediately follows for $j = 3$.

Hence, as we have already noted, by iterating the argument as far as $j = d$ we will find that, for any $t \in \mathcal{T}_d$, $\theta_{i_{2t-1}} + \theta_{i_{2t}} \in \mathbb{Z}$. Since the set $\mathcal{T}_d$ will certainly be non-empty, the proof of Theorem 1.8 is complete.

We close this section by indicating how to prove Theorem 1.9. In the notation of the statement of that theorem, if all the $\alpha_i$ are irrational, then the result follows immediately from Theorem 1.8. So it suffices to show that we cannot have an irreducible $m$-EEC in which there are both rational and irrational moduli present. To accomplish this, it suffices to show that the irrational moduli must themselves constitute an $m'$-EEC for some $m'$. Let the representation function $r(N)$ be as in (2.8). As before, the requirement is that $r(N) = m$ for all $N \gg 0$. Let us separate representations of $N$ coming from irrational and rational moduli separately and write

$$r(N) = r_{\text{irr}}(N) + r_{\text{rat}}(N). \tag{3.42}$$

Now the point is that, no matter what the rational moduli are, there must be some $a \in \mathbb{N}$ such that the function $r_{\text{rat}}(N)$ is constant on any congruence class modulo $a$. Hence, the same must be true of $r_{\text{irr}}(N)$, for all $N \gg 0$. But now one may check that this is enough to be able to push through the entire proof of Theorem 1.8 and deduce that the irrational moduli can be paired off so that each pair sums to an integer. Theorem 1.9 follows at once.

## 4. THE INHOMOGENEOUS CASE

In the previous section, we employed Weyl equidistribution (Lemma 2.1) to reduce the characterisation of homogeneous $m$-EEC's with irrational moduli to a purely combinatorial problem about multiset unions of arithmetic progressions (Proposition 3.3). The first part of this approach carries over to the inhomogeneous setting, but the second part seems to be more difficult and we do not resolve it to our satisfaction in this paper. Nevertheless, we can at least explain why Question 1.11 has a negative answer and why families of inhomogeneous $m$-EEC's may have additional structure.

We begin with some terminology :

**Definition 4.1.** A *system of parameters* $\mathcal{S} = (\mu, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{\phi})$ shall consist of a positive integer $\mu$ and three $\mu$-tuples

$$\boldsymbol{a} = (a_1, ..., a_\mu), \quad \boldsymbol{b} = (b_1, ..., b_\mu), \quad \boldsymbol{\phi} = (\phi_1, ..., \phi_\mu), \tag{4.1}$$

where the $a_i$ are positive integers and the $b_i, \phi_i$ any integers. We consider all the tuples as unordered, i.e.: we do not distinguish between systems based on the same three tuples but with the entries reordered. The number $\mu$ is called the *size* of the system. We say that the aystem is *homogeneous* if $\boldsymbol{\phi} = \boldsymbol{0}$, otherwise *inhomogeneous*.

**Definition 4.2.** Let $\mathcal{S} = (\mu, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{\phi})$ and $\mathcal{S}' = (\nu, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{\psi})$ be two systems of parameters. We say that these two systems are *complementary* if, for every $t \in \mathbb{Z}$, we have an equality of multisets

$$\bigcup_{i=1}^{\mu} S(a_i, \phi_i + tb_i) = \bigcup_{j=1}^{\nu} S(c_j, \psi_j + td_j). \tag{4.2}$$

The study of $m$-EEC's of Beatty sequences can be reduced to that of complementary systems of parameters. In the case of homogeneous sequences and systems, this reduction was established in Claim 3.2. The same arguments carry over to the inhomogeneous setting. Indeed, let notation be as in eqs. (3.1)-(3.8) and assume that all $\gamma_i \in \mathbb{Q}$ - the general case can also be reduced to this one. Write $\gamma_i = \frac{g_i}{h_i}$, a fraction in lowest terms, and set

$$H := \mathrm{LCM}\{h_i : i = 1, ..., k\}, \quad \gamma_i =: \frac{G_i}{H}. \tag{4.3}$$

Then the analogoue of (3.9) in the inhomogeneous setting is

$$\begin{cases} a_i := L_0 V_i H, \;\; b_i = -U_i V_i H, \;\; \phi_i := -L_0 V_i G_i, & \text{if } q_{1,i} > 0, \\ c_j := -L_0 V_j H, \;\; d_j := -U_j V_j H, \;\; \psi_j := -L_0 V_j G_j, & \text{if } q_{1,j} < 0. \end{cases} \tag{4.4}$$

Using the same methods as in Section 3, one may show that if the function $\epsilon(N)$ of (2.11) is constant for $N \gg 0$, then for all $t \in \mathbb{Z}$ we must have equality of multisets

$$\bigcup_{q_{1,i}>0} S(a_i, \phi_i + tb_i) = \bigcup_{q_{1,j}<0} S(c_j, \psi_j + td_j). \tag{4.5}$$

In fact, it is not hard to see from our earlier analysis that when $d = 1$, i.e.: $\dim(V) = 2$, then equality in (4.5) for all $t \in \mathbb{Z}$ is also sufficient for constancy of $\epsilon(N)$.

At this point, there remains a gap in our understanding, since we do not know what is the 'right' generalisation of Proposition 3.3 to inhmogeneous systems of parameters. However, we shall explain why Question 1.11 has a negative answer. We need some more termoinology.

**Definition 4.3.** Let $\mathcal{S} = (\mu, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{\phi})$ and $\mathcal{S}' = (\nu, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{\psi})$ be two systems of parameters. We say that $\mathcal{S}'$ is a *subsystem* of $\mathcal{S}$ if $\nu \leq \mu$ and there is a $\nu$-element subset $\{i_1, ..., i_\nu\}$ of $\{1, ..., \mu\}$ such that

$$\boldsymbol{c} = (a_{i_1}, ..., a_{i_\nu}), \;\; \boldsymbol{d} = (b_{i_1}, ..., b_{i_\nu}), \;\; \boldsymbol{\psi} = (\phi_{i_1}, ..., \phi_{i_\nu}). \tag{4.6}$$

A *decomposition* of $\mathcal{S}$ is a collection $\mathcal{S}^1, ..., \mathcal{S}^k$ of subsystems of $\mathcal{S}$ based on index sets whose disjoint union is $\{1, ..., \mu\}$. We write

$$\mathcal{S} = \bigsqcup_{i=1}^{k} \mathcal{S}^i. \tag{4.7}$$

The decomposition is said to be *trivial* if $k = 1$, otherwise *non-trivial*. It is *complete* if each $\mathcal{S}^i$ has size one.

**Definition 4.4.** A system of parameters $\mathcal{S} = (\mu, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{\phi})$ is said to be *exact* if, for each $t \in \mathbb{Z}$, the multiset $\cup_{i=1}^{\mu} S(a_i, \phi_i + tb_i)$ is an exact cover of the underlying set, in other words, if every integer occurring in the multiset occurs the same number of times.

A decomposition (4.7) of $\mathcal{S}$ is called *exact* if each $\mathcal{S}^i$ is exact. Note that any complete decomposition is exact, but the converse need not be true.

**Drfinition 4.5.** A pair $(\mathcal{S}, \mathcal{S}')$ of complementary systems is said to be *reducible/exact/ completely reducible* if there are non-trivial/exact/complete decompositions

$$\mathcal{S} = \bigsqcup_{i=1}^{k} \mathcal{S}^i, \quad \mathcal{S}' = \bigsqcup_{i=1}^{k} (\mathcal{S}')^i \tag{4.8}$$

for which the pairs $(\mathcal{S}^i, (\mathcal{S}')^i)$, $i = 1, ..., k$, are each complementary/exact/equal.

Proposition 3.3 states that any complementary pair of homogeneous systems of parameters is completely reducible. In general, however, a complementary pair need be neither reducible nor exact - see Example 4.8 below. Together with the following fact, this explains why Question 1.11 has a negative answer :

**Proposition 4.6.** *If an (irreducible) $m$-EEC with irrational moduli has the form (1.8) then, with notation as in Sections 2-4, the systems of parameters $\mathcal{S} = (\mu, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{\phi})$ and $\mathcal{S}' = (\nu, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{\psi})$ defined by (4.4) form an exact (irreducible) complementary pair. The latter condition is also sufficient when $dim(V) = d + 1 = 2$. In fact, the notations in (1.8) and (4.4) are consistent, up to a normalising factor and shifts $(\boldsymbol{\phi} \mapsto \boldsymbol{\phi} + t\boldsymbol{b}, \boldsymbol{\psi} \mapsto \boldsymbol{\psi} + t\boldsymbol{d})$.*

The verification of these assertions is a tedious recapitulation of earlier work. We shall therefore content ourselves with giving two further examples. The first illustrates the correspondences in Proposition 4.6, the second demonstrates the existence of inexact complementary pairs and hence of $m$-EEC's not of the form (1.8).

**Example 4.7.** Let $\alpha \in (1, \infty) \backslash \mathbb{Q}$. Then $\{S(\alpha, 0), S(\frac{\alpha}{\alpha-1}, 0)\}$ is an EEC by Beatty's theorem. Two exact covers of $\mathbb{Z}$ by APs are given by

$$\{S(3,0), S(3,1), S(3,2)\} \text{ and } \{S(2,0), S(4,1), S(4,3)\}. \tag{4.9}$$

From this data we can build, as in (1.8), the following irreducible, inhomogeneous EEC :

$$\{S(3\alpha,0), S(3\alpha,\alpha), S(3\alpha,2\alpha)\} \cup \left\{ S\left(\frac{2\alpha}{\alpha-1}, \frac{\alpha}{\alpha-1}\right), S\left(\frac{4\alpha}{\alpha-1}, \frac{\alpha}{\alpha-1}\right), S\left(\frac{4\alpha}{\alpha-1}, \frac{3\alpha}{\alpha-1}\right) \right\}. \tag{4.10}$$

In the notation of (2.5), we have $k = 6$ and the following table of values

| $i$ | $\theta_i$ | $\gamma_i$ |
|---|---|---|
| 1 | $\frac{1}{3\alpha}$ | $0$ |
| 2 | $\frac{1}{3\alpha}$ | $-1/3$ |
| 3 | $\frac{1}{3\alpha}$ | $-2/3$ |
| 4 | $\frac{\alpha-1}{2\alpha}$ | $-1/2$ |
| 5 | $\frac{\alpha-1}{4\alpha}$ | $-1/4$ |
| 6 | $\frac{\alpha-1}{4\alpha}$ | $-3/4$ |

Then (3.2) will become

$$\theta_1 = \theta_2 = \theta_3, \;\; \theta_4 = -\frac{3}{2}\theta_1 + \frac{1}{2}, \;\; \theta_5 = \theta_6 = \frac{1}{2}\theta_1. \tag{4.11}$$

In (3.7) and (4.3) we'll obtain the values

$$L_0 = 4, \quad L = 3, \quad H = 12, \tag{4.12}$$

and for the remaining variables in (3.8), (4.3) and (4.4) the table of values

| $i$ | $U_i$ | $V_i$ | $G_i$ | $a_i$ | $b_i$ | $\phi_i$ | $c_i$ | $d_i$ | $\psi_i$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 3 | 0 | 144 | 0 | 0 | | | |
| 2 | 0 | 3 | 8 | 144 | 0 | $-96$ | | | |
| 3 | 0 | 3 | 4 | 144 | 0 | $-48$ | | | |
| 4 | 2 | $-2$ | 0 | | | | 96 | 48 | 0 |
| 5 | 1 | $-4$ | 9 | | | | 192 | 48 | 144 |
| 6 | 1 | $-4$ | 3 | | | | 192 | 48 | 48 |

Dividing everything by the normalising factor of $48$, we see that (4.5) becomes the assertion that, for every $t \in \mathbb{Z}$,

$$S(3,0) \cup S(3,-2) \cup S(3,-1) = S(2,t) \cup S(4,3+t) \cup S(4,1+t). \tag{4.13}$$

Notice that this equality is irreducible and that, when $t = 0$, it coincides with that between the pair of exact covers we started with in (4.9).

**Example 4.8.** Let $\mathcal{S} = (\mu, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{\phi})$ and $\mathcal{S}' = (\nu, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{\psi})$ be systems for which

$$b_i \equiv 0 \pmod{a_i}, \;\; i = 1, ..., \mu, \quad c_j \equiv 0 \pmod{d_j}, \;\; j = 1, ..., \nu. \tag{4.14}$$

Then both sides of (4.5) are independent of $t$, so it suffices for complementarity to have the multiset equality

$$\bigcup_{i=1}^{\mu} S(a_i, \phi_i) = \bigcup_{j=1}^{\nu} S(c_j, \psi_j). \tag{4.15}$$

Consider the solution of (4.15) given by

$$S(1,0) \cup S(6,0) = S(2,0) \cup S(3,0) \cup S(6,1) \cup S(6,5). \tag{4.16}$$

One readily checks that this equality is irreducible and inexact. Hence any corresponding complementary pair of systems satisfying (4.14) will be both irreducible and inexact. This is the simplest example we found of an inexact complementary pair, in that the value of $\mu + \nu = 6$ is minimal (note that one must have $\min\{\mu, \nu\} > 1$), and likewise with the moduli $a_i, c_j$.

We can use this data to construct an irreducible $2$-EEC of Beatty sequences with irrational moduli, which does not have the form (1.8). In the notation of (3.2), we choose $d = 1$, $k = 6$. Condition (4.14) will be satisfied if $q_{0,i} \in \mathbb{Z}$ for all $i$. Then it is easy to check that, with the following assignments, (4.4) reduces (4.15) to (4.16) :

| $i$ | $\theta_i$ | $\gamma_i$ |
|---|---|---|
| 1 | $\theta_1$ | 0 |
| 2 | $z_2 + 6\theta_1$ | 0 |
| 3 | $z_3 - 2\theta_1$ | 0 |
| 4 | $z_4 - 3\theta_1$ | 0 |
| 5 | $z_5 - \theta_1$ | $1/6$ |
| 6 | $z_6 - \theta_1$ | $5/6$ |

Here $\theta_1$ is any positive irrational and the $z_i$ are integers. By (3.4), we have $m = z_2 + \cdots + z_6$. Since each $\theta_i > 0$, the minimum possible value of $m$ is thus $m = 4$, obtained by choosing $z_2 = 0$, $z_3 = z_4 = z_5 = z_6 = 0$ and $\theta_1 < 1/3$. This will yield an irreducible $4$-EEC of Beatty sequences with irrational moduli, which does not have the form (1.8). However, as promised above, we can do better and construct an irreducible $2$-EEC instead. The point is that, formally, in the proof of Claim 3.2, there is no requirement that the $\theta_i$ in (3.2) be positive, and also nothing changes if we shift any $\theta_i$ by an integer. So, if we set $\theta := -\theta_1$, we can define a new family of Beatty sequences by

| $i$ | $\theta_i'$ | $\gamma_i$ |
|---|---|---|
| 1 | $1 + \theta$ | 0 |
| 2 | $1 + 6\theta$ | 0 |
| 3 | $-2\theta$ | 0 |
| 4 | $-3\theta$ | 0 |
| 5 | $-\theta$ | $1/6$ |
| 6 | $-\theta$ | $5/6$ |

This yields an irreducible $2$-EEC provided $-1/6 < \theta < 0$. By the way, consider the function $\epsilon(N)$ of (2.11). Let $x := \{N\theta\}$. Then

$$\epsilon(N) = f(x) = \{x\} + \{6x\} + \{-2x\} + \{-3x\} + \{-x + 1/6\} + \{-x + 5/6\}. \quad (4.17)$$

Since $\{N\theta\}$ is equidistributed in $[0, 1)$, constancy of $\epsilon(N)$ for $N \gg 0$ is equivalent to constancy of $f(x)$ for $x \in [0, 1)$. One readily checks that $f(x) = 2$ for all $x \in [0, 1)$.

In general, given an irreducible and inexact solution to (4.15), one can construct, as in Example 4.8, a corresponding irreducible $m$-EEC not of the form (1.8), where $m = \min\{\mu, \nu\}$. It is easy to see how (4.16) can be generalised to give examples of irreducible and inexact solutions of (4.15), for any value of $\min\{\mu, \nu\} > 1$. Hence we deduce

**Theorem 4.9.** *For every $m > 1$, there exist irreducible $m$-EEC's of Beatty sequences with irrational moduli, not having the form (1.8).*

## 5. A FRACTIONAL BEATTY THEOREM

The notion of exact $m$-cover in Definition 1.2 clearly does not make sense if $m$ is not an integer. However, one might imagine various ways of extending the notion to non-integer $m$. Here, we only take a first tentative step, which nevertheless may prove instructive. We shall prove a 'fractional version' of Beatty's theorem.

Let $p, q$ be relatively prime positive integers. Let $\alpha_1, \alpha_2$ be positive irrationals satisfying

$$\frac{1}{\alpha_1} + \frac{1}{\alpha_2} = \frac{p}{q}. \tag{5.1}$$

As in Section 2, denote $\theta_i := 1/\alpha_i,\ i = 1, 2$. Let $p_0, p_1 \in \{0, 1, ..., q-1\}$ be the integers defined by

$$p \equiv p_0 \ (\text{mod } q), \quad \frac{p_1}{q} < \{\theta_1\} < \frac{p_1 + 1}{q}. \tag{5.2}$$

Let $r(N)$ be the representation function of (2.8) and set

$$R(N) := \sum_{M=1}^{N} r(M). \tag{5.3}$$

We will prove the following result :

**Theorem 5.1.** *For every $N \in \mathbb{N}$ one has*

$$R(qN-1) = \begin{cases} pN - \lceil p/q \rceil, & \text{if } p_1 < p_0, \\ pN - \lfloor p/q \rfloor, & \text{if } p_1 \geq p_0. \end{cases} \tag{5.4}$$

*Moreover,*

*(A) If $q = 1$, then $r(N) = p$ for every $N \in \mathbb{N}$.*
*(B) If $q = 2$, then $r(N) \in \{\lfloor p/2 \rfloor, \lceil p/2 \rceil\}$ for every $N \in \mathbb{N}$.*
*(C.i) If $q > 2$ and $p_1 < p_0$, then*

$$r(N) \in \{\lfloor p/q \rfloor, \lceil p/q \rceil, \lceil p/q \rceil + 1\}, \quad \text{for every } N \in \mathbb{N}. \tag{5.5}$$

*If, for each $i = 0, 1, 2$, we let*

$$S_i := \{N \in \mathbb{N} : r(N) = \lfloor p/q \rfloor + i\}, \tag{5.6}$$

*then each $S_i$ has asymptotic density, say $d(S_i) = d_i$, where*

$$d_0 = \left(1 - \frac{p_0}{q}\right) + d_2, \tag{5.7}$$

$$d_1 = \frac{p_0}{q} - 2d_2, \tag{5.8}$$

$$d_2 = \frac{1}{q}\left[\frac{p_0^2 + p_1^2 - (p_0 - p_1)}{2q} - p_1\{\theta_1\}\right]. \tag{5.9}$$

*(C.ii) If $q > 2$ and $p_1 \geq p_0$, then*

$$r(N) \in \{\lfloor p/q \rfloor - 1, \lfloor p/q \rfloor, \lceil p/q \rceil\}, \quad \text{for every } N \in \mathbb{N}. \tag{5.10}$$

*If, for each $i = 0, 1, 2$, we let*

$$T_i := \{N \in \mathbb{N} : r(N) = \lceil p/q \rceil - i\}, \tag{5.11}$$

*then each $T_i$ has asymptotic density, say $d(T_i) = \delta_i$, where*

$$\delta_0 = \frac{p_0}{q} + \delta_2, \tag{5.12}$$

$$\delta_1 = \left(1 - \frac{p_0}{q}\right) - 2\delta_2, \tag{5.13}$$

$$\delta_2 = \frac{1}{q}\left[\frac{4p_0p_1 + (p_1 - p_0) - (p_0^2 + p_1^2)}{2q} - (2p_1 - p_0) + (q - p_1)\{\theta_1\}\right]. \tag{5.14}$$

**Remark 5.2.** The interesting thing in this result is that, when $q > 2$, the function $r(N)$ cannot take on just the values $\lfloor p/q \rfloor$ and $\lceil p/q \rceil$. Nevertheless, $r(N)$ never takes on more than three distinct values, and each value is assumed on a fairly regular set. Thus the family $\{S(\alpha_1, 0), S(\alpha_2, 0)\}$ is always, in some sense, 'close to an exact $p/q$-cover'.

*Proof.* Eqs. (2.11) and (2.12) here become

$$\epsilon(N) = \{N\theta_1\} + \{N\theta_2\}, \tag{5.15}$$

$$r(N) = \frac{p}{q} + (\epsilon(N) - \epsilon(N + 1)). \tag{5.16}$$

Define $c_N \in \{0, 1, ..., q - 1\}$ by $c_N \equiv Np \pmod q$. Since (5.1) implies that

$$N\theta_1 + N\theta_2 \equiv \frac{c_N}{q} \pmod 1, \tag{5.17}$$

it follows that

$$\epsilon(N) = \begin{cases} c_N/q, & \text{if } \{N\theta_1\} < c_N/q, \\ 1 + c_N/q, & \text{if } \{N\theta_1\} > c_N/q. \end{cases} \tag{5.18}$$

In particular,

$$\epsilon(1) = \begin{cases} p_0/q, & \text{if } p_1 < p_0, \\ 1 + p_0/q, & \text{if } p_1 \geq p_0, \end{cases} \tag{5.19}$$

whereas

$$\text{if } q|N, \text{ then } c_N = 0 \text{ and } \epsilon(N) = 1. \tag{5.20}$$

From (5.16) it follows that, for any $N_1 > N_2$,

$$R(N_1) - R(N_2) = \left(\frac{p}{q}\right)(N_1 - N_2) - (\epsilon(N_1 + 1) - \epsilon(N_2 + 1)). \tag{5.21}$$

In particular, if $N \equiv -1 \pmod q$, then (5.20) implies that

$$R(N + q) - R(N) = pN. \tag{5.22}$$

Furthermore,

$$R(q - 1) = \frac{p(q - 1)}{q} + (\epsilon(1) - 1) = p - \lfloor p/q \rfloor - p_0/q + (\epsilon(1) - 1). \tag{5.23}$$

From (5.19), (5.22) and (5.23), one easily deduces (5.4). Now we turn to the proofs of statements (A), (B) and (C). The first of these is just Theorem 1.7, and it is immediately implied by (5.16) and (5.20). Using (5.18) we also quickly deduce (B). For (C) we

need to work a little more. We shall prove the statements of (C.i) rigorously - similar arguments give (C.ii). It is already clear from (5.16) that $r(N)$ must be one of the four numbers $\lfloor p/q \rfloor + i$, $i \in \{-1, 0, 1, 2\}$. If $r(N) = \lfloor p/q \rfloor - 1$ then it means that $\epsilon(N+1) - \epsilon(N) = 1 + \frac{p_0}{q}$. By (5.18), this happens if and only if

$$\{N\theta_1\} < \frac{c_N}{q}, \quad \{(N+1)\theta_1\} > \frac{c_{N+1}}{q}, \quad c_{N+1} = c_N + p_0 < q. \tag{5.24}$$

In particular, these conditions are unsatisfiable if $\{\theta_1\} < \frac{p_0}{q}$, in other words if $p_1 < p_0$. This proves (5.5). The set $S_2$ consists of all those $N \in \mathbb{N}$ for which $\epsilon(N+1) - \epsilon(N) = \frac{p_0}{q} - 2$. By (5.18), we have explicitly,

$$S_2 = \left\{ N \in \mathbb{N} : \{N\theta_1\} > \frac{c_N}{q}, \ \ \{(N+1)\theta_1\} < \frac{c_{N+1}}{q}, \ \ c_{N+1} = c_N + p_0 - q \geq 0 \right\}. \tag{5.25}$$

That this set has an asymptotic density follows from Lemma 2.1, which we can also use to compute $d_2$ explicitly. Note that (5.7) and (5.8) would follow from (5.9) and the fact that

$$d_0 + d_1 + d_2 = 1. \tag{5.26}$$

Hence, it just remains to compute $d_2$. From (5.25) we deduce that $N \in S_2$ if and only if

$$q - p_0 \leq c_N \leq q - 1 \tag{5.27}$$

and

$$\max \left\{ \frac{c_N}{q}, 1 - \{\theta_1\} \right\} < \{N\theta_1\} < \frac{c_N + p_0}{q} - \{\theta_1\}. \tag{5.28}$$

By Lemma 2.1, we thus have

$$d_2 = \frac{1}{q} \left[ \sum_{j=q-p_0}^{q-p_1-1} \left( \frac{j + p_0 - q}{q} \right) + \sum_{j=q-p_1}^{q-1} \left( \frac{p_0}{q} - \{\theta_1\} \right) \right]. \tag{5.29}$$

It is now just a tedious exercise to verify (5.9). □

## 6. Open questions

In this paper we showed how the classification of $m$-EEC's of Beatty sequences with irrational moduli can be reduced to that of complementary pairs of systems of parameters, the latter problem being purely arithmetical. We proved that every homogeneous complementary pair is completely reducible, but that there exist inhomogeneous complementary pairs which are neither reducible nor exact. There one might let things rest, but we feel that something is still missing, that it should be possible to prove some more insightful structural result for arbitrary complementary pairs. This is admittedly a vague hypothesis. Equally vague, but still enticing, is the question of how to push further the notion of $m$-cover, when $m$ is not an integer. Theorem 5.1 may provide some hints, but let us stop before we cross over the threshold into the realm of idle speculation !

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