TWO-GROUPS IN WHICH AN AUTOMORPHISM INVERTS PRECISELY HALF THE ELEMENTS

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Abstract

We classify all finite 2-groups G for which the maximum proportion of elements inverted by an automorphism of G is a half. These groups constitute 10 isoclinism families.

1 Introduction

Let l be a rational number in $(0,1]$. We call the finite group G an l-group if the maximum proportion of elements inverted by an automorphism of G is $l = l(G)$. A good deal of work has been done in investigating the structure of l-groups for various values of l (see [3] [4] [5] [6]). In this paper we complete the classification of $\frac{1}{2}$-groups, begun by Fitzpatrick in [1], who considered the case where G is not a 2-group, obtaining the following classification:

THEOREM 1.1 [1] *Let G be a finite group which is not a 2-group, and which has no abelian direct factor. Then G is a $\frac{1}{2}$-group if and only if it is one of the following types:*
Type I. $\mid G \mid = 2^a 3^b$, $G/Z(G) \cong A_4$.
Type II. $\mid G \mid = 2^a 3$, $G/Z(G) \cong C_2 \times C_2 \times S_3$, *$G$ is generated modulo $Z(G)$ by $\{w, v, u, t\}$ where $[w,u] = [v,u] = [w,t] = [v,t] = t^3 = 1$, $[t,u] = t$, $[v,w] \neq 1$ and $[v,w], u^2, v^2, w^2$ all lie in $Z(G)$.*

The classification in the 2-group case involves many more types of groups, which in fact constitute 10 isoclinism classes. We are not aware of any simple conceptual 'reason' as to why our groups should constitute isoclinism classes

- this just seems to come out from the proof. Before stating the theorem, a couple of remarks on notation : C_2^n will denote the elementary abelian 2-group of rank n. The identifiers used below for groups of order 16, 32 and 64 are taken from the CAYLEY library of finite groups[1]. The numbering of the isoclinism families of groups of order 128 is taken from [2].

THEOREM 1.2 *Let G be a finite 2-group. Then G is a $\frac{1}{2}$-group if and only if it belongs to one of the following isoclinism families:*

Family I. $G/Z(G) \cong G' \cong C_2^3$. *A stem group for the family is the group* 64/73.

Family II. $G/Z(G) \cong C_2^4$ *and* $G' \cong C_2^3$, *where* $G/Z = < Zu, Zv, Zw, Zx >$ *and* $G' = < z_1, z_2, z_3 >$ *and the following relations hold:*
$[u,w] = [u,x] = z_1$; $[v,w] = z_2$; $[v,x] = z_3$; $[u,v] = [w,x] = 1$
A stem group for the family has order 128 *and belongs to isoclinism family no.* 29 *among groups of that order.*

Family III. $G/Z(G) \cong C_2^4$ *and* $G' \cong C_2^2$, *where* $G/Z = < Zu, Zv, Zw, Zx >$ *and* $G' = < z_1, z_2 >$ *and the following relations hold:*
$[u,w] = [u,x] = z_1$; $[v,w] = z_2$; $[v,x] = z_1 z_2$; $[u,v] = [w,x] = 1$
A stem group for the family is the group 64/214.

Family IV. $G/Z(G) \cong D_4 \times C_2$ *and* $G' \cong C_2 \times C_4$, *where* $G/Z = < Zu, Zv, Zw \mid u^2, v^4, w^2 \in Z >$ *and* $G' = < z, a \mid z^2 = a^4 = 1,\ z \in Z >$ *and the following relations hold:*
$[a,v] = [u,w] = 1$; $[a,u] = [a,w] = a^2$; $[v,w] = a$; $[v,u] = az$; $[u,vw] = z$
A stem group for the family is the group 64/128.

Family V. $G/Z(G) \cong C_2^5$ *and* $G' \cong C_2^2$, *where* $G/Z = < Zu, Zv, Zw, Zx, Zy >$ *and* $G' = < z_1, z_2 >$ *and the following relations hold:*
$[u,v] = [u,w] = [v,w] = [u,x] = [v,y] = [w,y] = [x,y] = 1$; $[v,x] = z_1$; $[w,x] = z_2$; $[u,y] = z_1 z_2$
A stem group for the family has order 128 *and belongs to isoclinism family no.* 34 *among groups of that order.*

[1] This can be accessed using GAP. For all information about GAP, one may consult its' home page at http://www.math.rwth-aachen.de/LDFM/GAP/

Family VI. *$G/Z(G) \cong D_4 \times C_2 \times C_2$ and $G' \cong C_4$, where $G/Z = < Zu, Zv, Zw, Zx \mid u^2, v^4, w^2, x^2 \in Z >$ and $G' = < a >$ and the following relations hold:*
$[u,v] = [u,w] = [w,x] = [u,a] = [v,a] = 1$; $[u,x] = [w,a] = [x,a] = a^2$; $[v,w] = [v,x] = a$
A stem group for the family is the group 64/257.

Family VII. *$G/Z(G) \cong D_4 \times C_2$ and $G' \cong C_2 \times C_4$, where $G/Z = < Zu, Zv, Zw \mid u^2, v^4, w^2 \in Z >$ and $G' = < z, a \mid z^2 = a^4 = 1,\ z \in Z >$ and the following relations hold:*
$[a,u] = [a,w] = [u,v] = 1$; $[a,v] = a^2$; $[u,w] = z$; $[v,w] = a$
A stem group for the family is the group 64/140.

Family VIII. *$G/Z(G)$ is isomorphic to the group* 16/3 *and $G' \cong C_2 \times C_4$. $G/Z = < Zu, Zv$ and $G' = < z, a \mid z^2 = a^4 = 1,\ z \in Z >$ and the following relations hold:*
$[u,v] = a$; $[a,u] = z$; $[a,v] = a^2 z$
A stem group for the family is the group 64/8.

Family IX. *$G/Z(G) \cong D_4 \times C_2$ and $G' \cong C_4$, where $G/Z = < Zu, Zv, Zw \mid u^2, v^4, w^2 \in Z >$ and $G' = < a >$ and the following relations hold:*
$[a,v] = [a,w] = [u,w] = 1$; $[a,u] = [v,w] = a^2$; $[u,v] = a$
A stem group for the family is the group 32/43.

Family X. *$G/Z(G) \cong$ 16/3 and $G' \cong C_2^2$, where $G/Z = < Zu, Zv >$ and $G' = < z, a \mid z \in Z >$ and the following relations hold:*
$[a,u] = [a,v] = z$; $[u,v] = a$
A stem group for the family is the group 32/6.

In each case, the mapping $\alpha : G \to G$ which sends every element of $Z(G)$, and each of the representatives of the cosets in the given generating system of G/Z, to its inverse, can be extended to an automorphism of G which inverts precisely half its elements.

We note that all of the groups of types I-X are metabelian and of nilpotency class at most three.

2 Notation and Terminology

Most of the notation is standard. Here we list those terms which may not be so, many of which are taken from [1].

For $x \in G$, $C_G(x) = \{g \in G \mid [x,g] = 1\}$.
For $H \subseteq G$, $C_G(H) = \{g \in G \mid [g,H] = 1\}$.
Similarly, $N_G(x)$ and $N_G(H)$ denote the normalizers of x and H respectively.
I_x will denote the inner automorphism sending g to $x^{-1}gx$ $(g \in G)$.
For $\alpha \in \mathrm{Aut}(G)$, $I_G(\alpha) = \{g \in G \mid g\alpha = g^{-1}\}$.

3 Proof of Theorem

One may check that all of the groups of types I-X possess an abelian subgroup of index 4. We make this the starting point of the proof :

LEMMA 3.1 *If G is a $\frac{1}{2}$-group, then G possesses an abelian subgroup of index 4, but no abelian subgroup of smaller index.*

Proof. We shall be referring extensively to the classification of $> \frac{1}{2}$-groups, given as Theorem 4.13 in [3]. This class of groups contains all groups containing an abelian subgroup of index 1 or 2, from which the second statement of the lemma immediately follows.

Let G be a $\frac{1}{2}$-group and α a $\frac{1}{2}$-automorphism of G. Then the set $S = \{I_g \alpha I_{g^{-1}} \mid g \in I_G(\alpha)\}$ consists entirely of $\frac{1}{2}$-automorphisms. Let A be a subgroup of G of largest possible order which is inverted elementwise by β, as β ranges over S. Obviously, A is abelian. Let $(G : A) = 2^k$ and write a coset decomposition of G relative to A as follows:

$$G = A \cup Ax_2 \cup ... \cup Ax_{2^k} \tag{1}$$

By Lemma 2.2 of [1], two cases arise :

Case I. Each x_i can be chosen to lie in $I_G(\alpha)$. Relative to a suitable ordering of the cosets, $(A : C_A(x_2)) = (A : C_A(x_3)) = 4$ and $(A : C_A(x_i)) = 2$ for $i > 3$. We call Ax_2 and Ax_3 the '$\frac{1}{4}$-cosets' and all the other cosets '$\frac{1}{2}$-cosets'.
Case II. Relative to a suitable ordering of the cosets, all x_i $(i \neq 2)$ can be chosen to lie in $I_G(\alpha)$, and $Ax_2 \cap I_G(\alpha) = \phi$. Then $(A : C_A(x_i)) = 2$ for all

$i > 2$. We shall term Ax_2 the 'blank' coset in this case.

Suppose, in either case, that $(A : C_A(x_i)) = 2$ and let $g = ax_i \in Ax_i$. Then $< C_A(g), g > \subseteq I_G(I_{a^{-1}}\alpha)$ so, by maximality of $\mid A \mid$, it follows in particular that $g^2 \in A$ and that $g \in N_G(A)$.

Now suppose *Case I* applies. $(G : N_G(A)) \leq \frac{2^k}{2^{k-2}}$ so we can assume that $A \lhd G$. Let Ax_2 and Ax_3 be the $\frac{1}{4}$-cosets. Then Ax_2x_3 is a $\frac{1}{2}$-coset. Suppose there exists $x_4 \notin < A, x_2, x_3 >$. Clearly, $< A, x_2x_3, x_4 >$ is of type II and $< A, x_2x_4, x_4 >$ of type III as listed in [3], whence there exist distinct elements z_1, z_2 of $Z(G)$ such that

$$[A, x_2x_3] = [A, x_4] = < z_1 > \qquad [A, x_2x_4] = < z_2 > \tag{2}$$

But $< A, x_2x_3, x_2x_4, x_3x_4 >$ is also of type II in [3] and thus $[A, x_2x_4] = [A, x_2x_3]$ which contradicts (2), thus establishing that $(G : A) = 4$.

Next suppose *Case II* applies. It is clear that $A \lhd G$ and that G/A is elementary 2-abelian. It is convenient to refine the notation in (1) slightly and write $G/A = \ < Ax_1, ..., Ax_k >$ with Ax_1x_2 as the blank coset. Also set $B = < A, x_1x_2 >$. Suppose $C_A(x_1) \neq C_A(x_2)$. Replace A by $A^* = < C_A(x_1), x_1 >$ whence, since Ax_1x_2 is blank, the coset decomposition of B relative to A^*, and hence of G relative to A^*, must involve two $\frac{1}{4}$-cosets, so that $(G : A^*) = (G : A) = 4$.

We may thus assume that $C_A(x_1) = C_A(x_2)$. It is our aim to show that if $(G : A) > 4$, then G is a $> \frac{1}{2}$-group of type II in [3].

So suppose that $(G : A) > 4$. Let

$$G_1 = < A, x_1, x_3, ..., x_k > \qquad G_2 = < A, x_2, x_3, ..., x_k > \tag{3}$$

Clearly, each G_i is of type II in [3], so there exist z_1, z_2 of order 2 in $Z(G)$ such that $G_i' = < z_i >$. But, writing $A = < C_A(x_3), a_3 >$, it follows from [3] that $[a_3, x_3] = z_1 = z_2 = z$, say. It is now easy to deduce that $G' = < z >$ and that G/Z is elementary abelian. It remains to show that a generating system may be chosen for G/Z whose coset representatives satisfy the same commutator relations as the groups of type II in [3]. Let $A = C_A(x_1) \cup C_A(x_1)a$ and $A = C_A(x_i) \cup C_A(x_i)a_i$ for $3 \leq i \leq k$. The centre of $< A, x_1, x_2 >$ is $C_A(x_1) \cup C_A(x_1)ax_1x_2$. Since $< A, x_1, x_3 >$ is of type II in [3], $C_A(x_1) \neq C_A(x_3)$ and it follows that we can choose a_3 such that $[a_3, x_1] = 1$. If $[x_1, x_3] = z$ then $[a_3x_1, x_3] = 1$ and we may choose the coset representative x_1, and similarly x_2, to commute with x_3. It is now clear that $C_B(x_3) = C_A(x_3) \cup C_A(x_3)ax_1x_2$ and that the centre Z^* of the group $< A, x_1, x_2, x_3 > = < B, x_1, x_3 >$ is $(C_A(x_1) \cup C_A(x_1)ax_1x_2) \cap (C_A(x_3) \cup C_A(x_3)ax_1x_2)$, with B/Z^* elementary abelian of order 4. It is easy to see how this line of reasoning may be pursued to obtain $G/Z(G) = < Za, Za_3, ..., Za_k, Zx_1, ..., Zx_k >$ elementary abelian of order 2^{2k-2} with the appropriate commutator relations being satisfied by the coset representatives.

We have now completed the proof of Lemma 3.1.
We now divide the proof of the theorem into two parts, as suggested by the two cases which arose in the proof of Lemma 3.1. So let G be a $\frac{1}{2}$-group, and a subgroup A of G be defined as in the proof of the lemma. Let α be a $\frac{1}{2}$-automorphism which inverts A elementwise.

Assume first that *Case I* of the lemma applies when G is decomposed relative to A and, in addition, that $A \lhd G$. Then for all $x \in I_G(\alpha)$ and $a \in A$ we have that $x^{-1}ax \in A$, so an application of α gives $x^2 \in C_G(A) = A$. Thus $G/A \cong C_2 \times C_2$. Write $G = < A, x_1, x_2 >$ with Ax_1x_2 the $\frac{1}{2}$-coset and $x_1, x_2 \in I_G(\alpha)$.

Assume firstly that $C_A(x_1) = C_A(x_2) = Z(G)$. Then A/Z, being of order 4, has 2 possible structures, each of which leads to a family of $\frac{1}{2}$-groups.

If $A/Z = < Za, Zb >$ is elementary abelian, with $[a, x_1x_2] = 1$, then $[a, x_i^2] = [a^2, x_i] = 1$ implies that $[a, x_i] = z$ of order 2 in Z, for $i = 1, 2$. Similarly, $[b, x_i^2] = [b^2, x_i] = 1$ gives $[b, x_i] = z_i$ of order 2 in Z, with $z_1 \neq z_2$. If $[x_1, x_2] \neq 1$ then $bx_1x_2 \in I_G(\alpha)$. An easy calculation gives $[x_1, x_2] = z_1z_2$ and hence $[bx_1, bx_2] = 1$. Thus we may choose the coset representatives of Ax_1 and Ax_2 to commute. Clearly $z \neq z_i$, since $[ab, x_i] \neq 1$, for $i = 1, 2$. However z may equal z_1z_2, in which case it is now clear that G belongs to **Family III** in the theorem. Similarly, if $z \neq z_1z_2$, then G is easily seen to belong to **Family II**.

If $A/Z = < Za >$ is cyclic, the relations $[a, x_i^2] = 1 \neq [a^2, x_i]$ imply that $[a, x_i] = a^2z_i$ for some $z_i \in Z$, for $i = 1, 2$. Easy calculations show that $[a, x_1x_2] \neq 1$ implies $[a, x_1] \neq [a, x_2]$, whereas $[a^2, x_1x_2] = 1$ implies $[a, x_1]^2 = [a, x_2]^2$. Next, since $(x_1x_2)^2 \in A$, an application of α gives $[(x_1x_2)^2, x_1] = 1$ and hence $(x_1x_2)^2 \in Z$. Thus, since both x_1^2 and x_2^2 belong to Z, we also have $[x_1, x_2] \in Z$. Since x_1 and x_2 are both in $I_G(\alpha)$ it is easily deduced that they commute. It is now easily verified that G belongs to **Family IV** in our classification theorem.

We now assume that $C_A(x_1) \neq C_A(x_2)$. It is easy to see that A/Z is abelian, and non-cyclic, of order 8.

First we take $A/Z \cong C_2 \times C_2 \times C_2$. Write $A/Z = < Za, Zb, Zc >$. Put

$C_A(x_1) = < Z, a >$, $C_A(x_1x_2) = < Z, b, c >$ and $C_A(x_2) = < Z, abc >$. The relations $[a, x_2^2] = [a^2, x_2] = 1$ imply that $[a, x_2] \in C_A(x_2)$ and hence that $[a, x_2] \in C_A(x_1)$. Thus $[a, x_2] = z_3$ of order 2 in Z. By a similar argument, $[b, x_i] = z_i$ of order 2 in Z $(i = 1, 2)$. Since $[b, x_1x_2] = 1$ we have $z_1 = z_2 = z$. Similarly $[c, x_i] = z^*$ of order 2 in Z and clearly $z \neq z^*$. However $[abc, x_2] = 1$ so $z_3zz^* = 1$. Now if $[x_1, x_2] \neq 1$ then $ax_1x_2 \in I_G(\alpha)$, so $[x_1, x_2] = z_3$ which in turn implies that $[ax_1, x_2] = 1$. Hence the coset representatives of Ax_1 and Ax_2 may be chosen to commute. Since $(x_1x_2)^2 \in A$, an application of α gives $(x_1x_2)^2 \in C_A(x_1) \cap C_A(x_2) = Z$. Since $[x_1, x_2] \in Z$, we also have $x_1^2 \in Z$ and $x_2^2 \in Z$. Thus G belongs to **Family V** in the theorem.

Secondly, we take $A/Z \cong C_2 \times C_4$ and write $A/Z = < Za, Zb \mid a^2 \in Z,\ b^4 \in Z >$. Let $C_A(x_1) = < Z, a >$, $C_A(x_2) = < Z, ab^2 >$ and $C_A(x_1x_2) = < Z, b >$. Now $[a^2, x_2] = [a, x_2^2] = 1$ tells us that $[a, x_2] = z_2$ of order 2 in Z. Since $[ab^2, x_2] = 1$ we have $[b^2, x_2] = z_2$and since $[b^2, x_1x_2] = 1$ we have $[b^2, x_1] = z_2$. Next, easy calculations give $[b, x_i] = b^2z_i^* \ \exists\ z_i^* \in Z,\ i = 1, 2$. The relations $[b, x_1x_2] = 1$ and $[b^2, x_1] = z_2$ give $z_1^* = z_2^* = z$, and hence $[b, x_1] = [b, x_2]$ of order 4. Finally, if $[x_1, x_2] \neq 1$ we have $ax_1x_2 \in I_G(\alpha)$ and hence $[ax_1, x2] = 1$, showing that once again the coset representatives of Ax_1 and Ax_2 may be chosen to commute. Summarising, we have $G' = < b^2z >$ cyclic of order 4, and $G/Z = < Za, Zb, Zx_1, Zx_2 \mid a^2, b^4, x_1^2, x_2^2 \in Z >$ of order 32 subject to the following commutator relations :

$[a, x_1] = [ab^2, x_2] = [b, x_1x_2] = [x_1, x_2] = 1$;
$[a, x_2] = [b^2, x_2] = [b^2, x_1] = z_2$;
$[b, x_1] = [b, x_2] = b^2z$, and $b^4z^2 = z_2$.

Thus G belongs to **Family VI** in the theorem.

We now remain with *Case I* but assume that A is not normal in G. There is a homomorphism of G into S_4 whose kernel is $K = \text{Core } A$. It is easy to see that $G/K \cong D_4$ and that G/Z is non-abelian of order 16, with $(K : Z) = 2$. Write $G/K = < Kx_1, Kx_2 \mid x_1^2 \in K,\ x_2^4 \in K >$ and let $B = < K, x_2^2 >$. Then B is elementwise inverted by α, $B \lhd G$ with $G/B \cong C_2 \times C_2$, and Bx_1x_2 is a blank coset since Bx_1 and Bx_2 are clearly $\frac{1}{2}$-cosets. Thus *Cases I* and *II* overlap here. We aim to show that G belongs to one of the families **VII-X** in the theorem. Clearly $C_B(x_1) \neq C_B(x_2)$. Since $(G : Z) = 16$, the analysis divides into 2 cases according as to whether $(B : C_B(x_1x_2)) = 2$ or 4. It further subdivides according as to whether $x_1^2 \in Z$ or not. Hence there are 4 cases in all to consider and, as we shall see, each will give rise to a single family of $\frac{1}{2}$-groups.

Firstly, we assume that $(B : C_B(x_1x_2)) = 4$ and $x_1^2 \in Z$. Obviously $Z = C_B(x_1x_2)$ and B/Z is elementary abelian of order 4. Write $B/Z = < Za, Zc >$ where $[c, x_2] = [a, x_1] = 1$. $[a, x_2]$ and $[c, x_1]$ are thus distinct elements z_1, z_2 respectively of order 2 in Z. We have $x_1^2 = z_3$ and $x_2^2 = cz_4$ for some $z_3, z_4 \in Z$. Since $(x_1x_2)^2 \in Z$ it follows that $[x_1, x_2] = cz_{12}$ for some $z_{12} \in Z$. The relation $[x_1^2, x_2] = 1$ now gives $(cz_{12})^2 = z_1$, and thus $G' = < z_2, cz_{12} >$ is isomorphic to $C_2 \times C_4$. It is also clear that $G/Z \cong D_4 \times C_2$ and now easily verified that G belongs to **Family VII**.

Secondly we assume that $(B : C_B(x_1x_2)) = 4$ and that $x_1^2 \notin Z$. As before B/Z is elementary abelian and we can write $A/Z = < Za, Zc >$ so that $[a, x_1] = [c, x_2] = 1$ and $[c, x_1], [a, x_2]$ are distinct elements z_1, z_2 respectively of order 2 in Z. We have $x_1^2 = az_3$ and $x_2^2 = cz_4$ for some $z_3, z_4 \in Z$. It follows that $[x_1, x_2] = acz_{12}$ for some $z_{12} \in Z$. By expressing the commutator $[x_1, x_2^2]$ in two different ways we find that $(acz_{12})^2 = z_1z_2$. Thus $G' = < z_1, acz_{12} >$ is once again isomorphic to $C_2 \times C_4$. It is now easily concluded that G belongs to **Family VIII**.

Thirdly, we assume that $(B : C_B(x_1x_2)) = 2$ and $x_1^2 \in Z$. $(x_1x_2)^2 \in B$ so an application of α shows eventually that $(x_1x_2)^2 \in Z$. Write $B/Z = < Za, Zc >$ with $[a, x_1] = [c, x_2] = 1$. Then $[x_1, x_2] = cz_{12}$ for some $z_{12} \in Z$. The relation $[x_1^2, x_2] = 1$ implies that $(cz_{12})^2 = z = [c, x_1] = [a, x_2]$. Thus $G' = < cz_{12} >$ is cyclic of order 4 and $G/Z = < Za, Zx_1, Zx_2 >$ is isomorphic to $D_4 \times C_2$. G obviously belongs to **Family IX**.

Fourthly, we assume that $(B : C_B(x_1x_2)) = 2$ and $x_1^2 \notin Z$. Again write $B/Z = < Za, Zc >$ with $[a, x_1] = [c, x_2] = 1$. Then $[x_1, x_2] = acz_{12}$ for some $z_{12} \in Z$. The relation $[x_1, x_2^2] = z = [x_1, c] = [x_2, a]$ gives $(acz_{12})^2 = 1$. Thus $G' = < z, acz_{12} >$ is elementary abelian of order 4. $G/Z = < Zx_1, Zx_2 >$ with $x_1^4, x_2^4 \in Z$ and $[x_1, x_2] \in Zx_1^2x_2^2$. G clearly belongs to **Family X**.

At this stage, we have obtained all of the groups appearing in our classification theorem except those from **Family I**. Referring to Lemma 3.1, we have also analysed only the situation which pertains when *Case I* of the lemma applies. It remains, therefore, to analyse the situation when *Case II* of the lemma applies.

It is clear, first of all, that $G/A \cong C_2 \times C_2$. We write $G = < A, x_1, x_2 >$ where Ax_1x_2 is the blank coset and $x_1, x_2 \in I_G(\alpha)$. It is not too difficult to see that the groups of types **II-X** already investigated will arise again if $C_A(x_1) \neq C_A(x_2)$. This is because we can replace A by $A^* = < C_A(x_1), x_1 >$, so that the decomposition of G relative to A^* involves two

$\frac{1}{4}$-cosets.

So we can assume that $C_A(x_1) = C_A(x_2) = Z(G)$. Clearly, $G/Z \cong C_2 \times C_2 \times C_2$. Thus $G' \cong C_2 \times C_2$ or $C_2 \times C_2 \times C_2$. But since the coset Ax_1x_2 is blank, it is quite easy to see that $G' \cong C_2 \times C_2$ would imply that G had an abelian subgroup of index 2, in which case G would be a $> \frac{1}{2}$-group. Thus $G' \cong C_2 \times C_2 \times C_2$ and G belongs to **Family I**.

We have now completed the proof of our classification theorem in full.

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