TMA 055 : Diskret Matematik

Inlämningsuppgift 4

Följande allmäna regler gäller för alla inlämningsuppgifterna (se också Kurs PM:et) :

1. Motivera (bevisa) alla svar.
2. Uppge vilka du samarbetet med.
3. Spara uppgifterna (med kommentarer) när du får tillbaka dem.

Att lämna in fredag den 4 oktober

OBS !! A. For Q.1 you will need to wait until Tuesday, as we haven't covered the material in class yet. Figures 1 and 2 will be drawn on the board during the lecture, and also during Monday's lektion.
B. I use several pieces of terminology which I have not given in the lectures. At the end of this document, you will find a list of definitions which should cover everything.

1. With respect to the graph in Figure 1,

(i) find a minimal weight spanning tree of the undirected graph got by removing all the arrows
(ii) find a shortest path from s to t
(iii) find a maximal flow from s to t and a minimal cut-set.

Obs! In all three parts it is important to show your calculations, so that it is clear you understand the relevant algorithms.

2 (i) Draw a planar graph G whose vertices are all the countries of Europe[1]

[1] don't include former Soviet republics, Cyprus, Turkey or Israel.

and in which two countries are joined by an edge if they share a common border. Then

(a) order the vertices in alphabetical order (in either English or Swedish), and apply the greedy algorithm with this ordering to color G.
What is $\chi(G)$?

(b) take the largest connected piece of your graph and compute $V =$ no. of vertices, $E =$ no. of edges and $R =$ no. of enclosed regions in G. Then compute $V - E + R$.

(ii) This time, take a graph based on the countries of South America instead. Repeat part **(b)** above. Notice anything ?

(iii) Repeat part **(b)** again, this time for the graph of a football (I mean a REAL football, that is, one that's sown together to form pentagons and hexagons). Notice anything ?

(iv) Explain everything you've noticed.

3. This exercise is about the Petersen graph P.

(i) Find simple cycles in P of lengths $5, 6, 8, 9$.
(ii) Show that P cannot be edge-colored with three or fewer colors.
(iii) Use the result of part **(ii)** to show that P has no Hamilton cycles.
(iv) Show that P is isomorphic to the graph in Figure 2.

(REMARKS (FOR THOSE WHO ARE INTERESTED) : (i) There is a theorem which says that 'nearly all' regular graphs of degree d can be edge-colored with d colors. P is one of the very few counterexamples. This is another one of the many strange properties of P

(ii) The *graph isomorphism problem*, that is, the problem of determining whether two arbitrarily chosen grpahs G, H are isomorphic, is another example of an algorithmic problem which seems to be very hard. As far as I know, the current status of the problem is that it is in NP (i.e.: no polynomial-time algorithm is known for solving it), but it is an open problem as to whether it is NP-complete. Hence it has the same status as the integer factorisation problem underlying the security of RSA.

Bonus problems

4 (i) Draw all pairwise non-isomorphic trees with 7 vertices.
(ii) Let G be a connected graph with n vertices. Show that if G has $n-1$ edges, then G is a tree.

5. Let G be any simple graph. Show that there is always SOME ordering of the vertices of G for which the greedy algorithm colors G with exactly $\chi(G)$ colors.

(REMARK : This isn't very useful as a practical way to compute $\chi(G)$, since there are $n!$ possible orderings of the vertices).

List of definitions

DEFINITION 1 : A cycle in a graph G is said to be *simple* if it does not cross any vertex more than once, other than the starting point (which is also the endpoint).

DEFINITION 2 : An *edge-coloring* of a graph G is a coloring of the edges of G such that edges with a vertex in common always get different colors.

DEFINITION 3 : Let $G = (V, E)$ and $H = (V^*, E^*)$ be graphs. G and H are said to be *isomorphic* if there exists a 1-1 mapping $\phi : V \to V^*$ such that

$$(v, w) \text{ is an edge in } E \Leftrightarrow (\phi(v), \phi(w)) \text{ is an edge in } E^*.$$