Iterated integrals and integrability

In this note, we give a (hopefully interesting) example of a bounded, non-Riemann integrable function $f : \Delta = [0,1] \times [0,1] \to \mathbf{R}$ both of whose iterated Riemann integrals exist and are equal to zero. That is,

$$\int_0^1 \left(\int_0^1 f(x,y)dx \right) dy = \int_0^1 \left(\int_0^1 f(x,y)dy \right) dx = 0, \tag{1}$$

but $\int\int_\Delta f \; dxdy$ does not exist. For the example, we construct a dense subset S of Δ which meets every vertical and horizontal line through Δ in exactly one point. We then take $f = \chi_S$, the *characteristic function* of S[1], i.e.:

$$\chi_S(p) = \begin{cases} 1, & if \; p \in S. \\ 0, & if \; p \notin S. \end{cases} .(2)$$

Since S is dense in Δ, for any covering of Δ by a finite number of rectangles, each rectangle contains points belonging to both S and S^c. Hence, any step function $\Phi \leq f$ must satisfty $\Phi \leq 0$, and any step function $\Psi \geq f$ must satisfy $\Psi \geq 1$. Hence, $\int\int_\Delta (\Psi - \Phi) \geq 1$ for all such Φ, Ψ which proves that f is not Riemann integrable.

On the other hand, since S meets each horizontal line in exactly one point, for each fixed y the section $f(x,y)$ is identically zero except in one point where it equals 1. Such a function is Riemann integrable with integral 0, that is $\int_0^1 f(x,y)dx = 0$ for each fixed y. Hence

$$\int_0^1 \left(\int_0^1 f(x,y)dx \right) dy = \int_0^1 0 \; dy = 0. \tag{3}$$

The same applies to the other iterated integral.

So how to construct the set S ? Well, S shall include the points $(0,0)$ and $(1,1)$ plus all points of the form $(x, \zeta(x))$ where $\zeta : (0,1) \to (0,1)$ is a bijective function with the property that the image of any open interval $I \subseteq (0,1)$ is a dense subset of $(0,1)$. If we identify $(0,1)$ with $\mathbf{R}$ via the map $t \longmapsto \tan(\pi t - \frac{\pi}{2})$, then it suffices now to find a bijection on $\mathbf{R}$ with the above property. First, a definition :

DEFINITION : A function $\zeta : \mathbf{R} \to \mathbf{R}$ is said to be $\mathbf{Q}$-*linear* if

[1]Note that the function f is discontinuous at every point of Δ.

(I) $\zeta(x+y)=\zeta(x)+\zeta(y)$ for all $x, y \in \mathbf{R}$.

(II) $\zeta(qx)=q\zeta(x)$ for all $q \in \mathbf{Q}$, $x \in \mathbf{R}$.

Now we shall prove

Proposition. *There exists a* $\mathbf{Q}$*-linear bijection* $\zeta : \mathbf{R} \to \mathbf{R}$ *with the property that, for any interval* $I \subseteq \mathbf{R}$, $\zeta(I)$ *is dense in* $\mathbf{R}$.

The proof of this result shall be accomplished in two steps :

STEP 1 : Any $\mathbf{Q}$-linear map from $\mathbf{R}$ to $\mathbf{R}$ which is discontinuous at $x=0$ has the property that every interval is mapped to a dense subset of $\mathbf{R}$[2].

PROOF : Let $\zeta : \mathbf{R} \to \mathbf{R}$ be $\mathbf{Q}$-linear and suppose that ζ is discontinuous at $x=0$. Hence, there exists $\epsilon > 0$ and a sequence $x_n \to 0$ such that $|\zeta(x_n)| > \epsilon$ for all n.

Let I be an open interval. We must show that $\zeta(I)$ is dense in $\mathbf{R}$. Since ζ is additive, $\zeta(x+I)=\zeta(x)+\zeta(I)$ for all $x \in \mathbf{R}$, and since any translate of a dense set is still dense, it suffices to consider the case when $0 \in I$. To show that $\zeta(I)$ is dense in $\mathbf{R}$, we must show that, for any $s \in \mathbf{R}$ and any $\delta > 0$, there exists $x \in I$ such that $|\zeta(x)-s| < \delta$.

So let x and δ be given. To simplify the argument, we assume that s and all $\zeta(x_n)$ in the above sequence are positive (it's easy to see that it suffices to consider this case). Let $N = \lceil \frac{x}{\epsilon} \rceil$. Since $x_n \to 0$, there exists $n >> 0$ such that $Nx_n \in I$. Since ζ is $\mathbf{Q}$-linear, $\zeta(Nx_n) = N\zeta(x_n) > N\epsilon > s$. Since $\mathbf{Q}$ is dense in $\mathbf{R}$ there exists $q \in (0,1)$ such that $|q\zeta(Nx_n)-s| < \delta$. Now set $x := qNx_n$. Then $x \in I$ and, using $\mathbf{Q}$-linearity again, $|f(x)-s| < \delta$. q.e.d.

STEP 2 : There exists a $\mathbf{Q}$-linear bijection on $\mathbf{R}$ which is discontinuous at $x=0$.

PROOF : $\mathbf{R}$ is an infinite-dimensional vector space over $\mathbf{Q}$. From the theory of infinite-dimensional vector spaces (see, for example [1]), $\mathbf{R}$ has a basis $\{e_\lambda\}_{\lambda \in \Lambda}$ as a vector space over $\mathbf{Q}$. The cardinality of Λ is c (continuum). Every 1-1 map $\Lambda \leftrightarrow \Lambda$ may be identified with a $\mathbf{Q}$-linear bijection on $\mathbf{R}$.

[2]Note for the record that any $\mathbf{Q}$-linear map from $\mathbf{R}$ to $\mathbf{R}$ which is discontinuous at $x=0$ must be discontinuous at every point of $\mathbf{R}$.

The number of such maps is c^c.

On the other hand, it is an easy exercise to show that every $\mathbf{Q}$-linear bijection on $\mathbf{R}$ which is continuous at $x = 0$ is a straight line through the origin. Each such line is identified by a single real number, namely its' slope, so the cardinality of the set of all these lines is c.

Since $c < c^c$ by Cantor's theorem (see [2], ch. 1), it follows that there must exist (a lot of !) $\mathbf{Q}$-linear bijections from $\mathbf{R}$ to $\mathbf{R}$ which are discontinuous at $x = 0$. This completes the proof of our proposition.

References

[1] Hungerford T.W., Algebra (GTM series no.73), Springer-Verlag (1974).

[2] Simmons G.F., Introduction to topology and modern analysis, McGraw-Hill (1963).