Extra lecture notes : Day 15

The extra stuff today was a short introduction to *posets*.

DEFINITION : Let S be any set. A relation $\leq$ on S is said to be a *partial ordering* if

(i) $\leq$ is *reflexive*, i.e.: $a \leq a \; \forall \; a \in S$,
(ii) $\leq$ is *anti-symmetric*, i.e.: if $a \leq b$ and $b \leq a$ then $a = b$,
(iii) $\leq$ is *transitive*, i.e.: if $a \leq b$ and $b \leq c$ then $a \leq c$.

DEFINITION : A set S together with a partial ordering $\leq$ on it is called a *poset*. Notation : $(S, \leq)$.

DEFINITION : Two elements a, b of a poset $(S, \leq)$ are said to be *comparable* if either $a \leq b$ or $b \leq a$.

A subset T of S is said to be *totally ordered* by $\leq$ if every pair of elements from T are comparable.

A totally ordered subset of a poset is called a *chain*.

A subset $T \leq S$ such that no pair of elements from T are comparable is called an *antichain*.

EXAMPLE 0 : The ordinary $\leq$ relation on any subset of the real numbers is a total ordering. On the other hand, there is no 'natural' total ordering of the complex numbers.

EXAMPLE 1 : Let X be any set and take $S = 2^X$, the collection of subsets of X. Define the relation $\leq$ on S by

$$A \leq B \text{ if } A \subseteq B.$$

Then $(S, \leq)$ is a poset. It is called the *Boolean algebra* on the set X.

EXAMPLE 2 : Let S be the set of positive integers. Define the relation $\leq$ by

$$n_1 \leq n_2 \text{ if } n_1 \mid n_2.$$

If we let X be a set which contains infinitely many copies of each prime number, then the Fundamental Theorem of Arithmetic implies that $(S, \leq)$ can be identified with that part of the Boolean algebra on X consisting of the finite subsets of X.

As preparation for the next result, note that if $X = \{1, ..., n\}$, then for any k with $0 \leq k \leq n$, the subsets of X of size k form an antichain in 2^X of size $\left(\begin{array}{c} n \\ k \end{array} \right)$.

DEFINITION : Let $f : \mathbf{Z} \to \mathbf{R}$ be a sequence of real numbers. The sequence is said to be *unimodal* if there exists an integer k_0 such that
(i) $f(k_0)$ is a maximum for f,
(ii) $f(k)$ is an increasing function of k for $k \leq k_0$,
(iii) $f(k)$ is a decreasing function of k for $k \geq k_0$.

Proposition *Let n be a fixed positive integer, and consider the sequence $f(k)$ given by*

$$f(k) := \begin{cases} \left(\begin{array}{c} n \\ k \end{array} \right), & \textit{if } 0 \leq k \leq n, \\ 0, & \textit{otherwise.} \end{cases}$$

Then $f(k)$ is unimodal with a maximum at $k = \lfloor n/2 \rfloor$.

PROOF : Using the formula for the binomial coefficients we have that, for $k < n$,

$$\left(\begin{array}{c} n \\ k+1 \end{array} \right) = \frac{n-k}{k+1} \left(\begin{array}{c} n \\ k \end{array} \right).$$

Hence $f(k+1) \geq f(k)$ if and only if $n - k \geq k + 1 \Leftrightarrow k \leq (n-1)/2 \Leftrightarrow k < \lfloor n/2 \rfloor$, v.s.v.

Theorem (Sperner 1929) *Let n be a positive integer and $X = \{1, ..., n\}$. The largest size of an antichain in 2^X is* $\left(\begin{array}{c} n \\ \lfloor n/2 \rfloor \end{array} \right)$.

PROOF : Not given. See [1], Chapter 6.

Theorem (Dilworth) *Let $(S, \leq)$ be any finite poset (i.e.: the set S is finite). Then the maximum size of an antichain in S equals the minimum number of pairwise disjoint chains needed to cover all the elements of S.*

PROOF : Not given. See [1], Chapter 6.

Corollary (Erdös) *Let n be a positive integer. Then in any reordering of the integers $1, 2, ..., n^2 + 1$, there exists either an increasing or a decreasing subsequence of length $n + 1$.*

PROOF : Let $S = \{1, 2, ..., n^2 + 1\}$ and let π be any permutation of S. Define a relation $\leq_\pi$ on S by

$$i \leq_\pi j \text{ if } i \leq j \text{ and } \pi(i) \leq \pi(j).$$

One easily checks that

(i) $\leq_\pi$ is a partial ordering on S,

(ii) a chain in $(S, \leq_\pi)$ corresponds to an increasing subsequence in the reordering π,

(iii) an antichain in $(S, \leq_\pi)$ corresponds to a decreasing subsequence in the reordering π.

By (iii), if there exists no decreasing subsequence of length $n + 1$, then there exists no antichain of size $n + 1$ in $(S, \leq_\pi)$. By Dilworth's Theorem, this implies that S can be covered by at most n pairwise disjoint chains. But S has $n^2 + 1$ elements, so at least one of these chains must have size greater than n. By (ii), this would imply that the reordering π contained an increasing subsequence of length $n + 1$.

REFERENCE

[1] J.H. van Lint and R.M. Wilson, A Course in Combinatorics, Cambridge University Press, 1992.