MAN 240 (2003) : Inlämningsuppgift 2

In the following, to translate from old Biggs to new Biggs, replace chapter 12 by chapter 19, chapter 18 by chapter 25 and chapter 19 by chapter 26.

A. Solve the following exercises in old Biggs :

12.7.4, 12.7.7, 12.7.12, 18.3.3, 18.6.2, 18.7.17.

Note : 18.7.17 is the same as 12.7.11 except that in the latter you are 'given' the answer and asked to verify it, whereas in the former you 'derive' the answer methodically using generating functions.

B. Let $(u_n)_{n=0}^{\infty}$ b a sequence of numbers. The *exponential generating function* of the sequence is the power series

$$E(x) = \sum_{n=0}^{\infty} u_n \frac{x^n}{n!}.$$

(i) Suppose the sequence (u_n) satisfies the recurrence relation

$$au_{n+2} + bu_{n+1} + cu_n = f_n,$$

where (f_n) is some 'known' sequence. Show that in that case the e.g.f. satisfies the differential equation

$$aE''(x) + bE'(x) + cE(x) = f(x),$$

where

$$f(x) = \sum_{n=0}^{\infty} f_n \frac{x^n}{n!}.$$

(Note : The point of this exercise is to show those of you familiar with differential equations the explicit connection between linear recurrence relations with constant coefficients and differential equations of the same type).

(ii) Let d_n denote the number of derangements of $[n]$ as usual. In class we proved that

$$d_n = (n-1)(d_{n-1} + d_{n-2}), \qquad n \geq 3.$$

Using this or otherwise, prove that

$$d_n = n d_{n-1} + (-1)^n, \qquad n \geq 2. \tag{1}$$

(Note : With regard to the 'otherwise', I don't know of any more direct combinatorial proof of (1). It would be interesting if anyone could find one !).

Now use (1) to prove that the exponential generating function of the sequence d_n (we define $d_0 = 1$) is

$$E(x) = \frac{e^{-x}}{1-x}.$$

From this, recover the usual explicit formula for d_n (which we also derived in class).

C. Prove combinatorially/algebraically (i.e.: without introducing any differential calculus) the binomial theorem for exponent 1/2. That is, prove that if $|x| < 1$ then

$$(1+x)^{1/2} = \sum_{n=0}^{\infty} \binom{1/2}{n} x^n, \tag{2}$$

where, by definition,

$$\binom{1/2}{n} = \frac{1}{n!} \prod_{i=0}^{n-1} \left(\frac{1}{2} - i\right).$$

(Hint : Square both sides of (2)).

D (i). Let n be a positive integer. A permutation $\pi : [n] \to [n]$ is said to be 132-*avoiding* if there does not exist any triple (i, j, k) such that

$$i < j < k \leq n,$$
$$\pi(i) < \pi(j) > \pi(k) > \pi(i).$$

Let C_n denote the number of 132-avoiding permutations of $[n]$. Show that C_n is the n^{th} Catalan number.

(ii) Let $n \geq 2$ and let C_{n-1} denote the number of UNORDERED $(n-1)$-tuples $(x_1, ..., x_{n-1})$ of integers which satisfy

$$0 \leq x_i \leq n-1, \qquad i = 1, ..., n-1,$$
$$n \text{ divides } \sum_{i=1}^{n-1} x_i.$$

Show that C_n is the n^{th} Catalan number.

(Note : In part **(i)** I think the easiest method is to show that C_n satisfies the same recurrence relation as the Catalan numbers, whereas in part **(ii)** I think it is easier to show directly the same explicit formula for C_n as for the Catalan numbers. Once again, it would be very interesting to see alternative proofs !).

E. Before doing this exercise, you should read Chapter 19 in Biggs, in particular 19.1-19.4.

(i) Solve the example at the end of 19.4 without using generating functions.
(ii) Prove theorem 19.1 using generating functions.

(N.B.: I consider part **(i)** 'hard', though I know a solution. I can't do part **(ii)** !).