MAN 240 : Diskret matematik

Tentamen 240802

Lösningar

F.1 See Theorem 8.5 in the book. Here is a proof for the sake of completeness.

$\Leftarrow$ Suppose G is connected, with n nodes and $n-1$ edges. We prove by induction on n that G has no cycles. If $n=2$, then G is a single edge, and hence obviously has no cycles. Suppose G has k edges and $k+1$ nodes. Since

$$\sum_{v \in V(G)} \deg(v) = 2|E(G)| = 2k < 2(k+1),$$

it follows that G has a vertex v of degree 1. Let G^* be the graph obtained by removing from G the vertex v and its' single adjacent edge. Then G^* must still be connected. But G^* has k nodes and $k-1$ edges. By the induction hypothesis, G^* has no cycles. But then neither does G.

$\Rightarrow$ Suppose G is connected with n nodes and no cycles. We prove by induction on n that G has $n-1$ edges. Evidently, this is the case if $n=2$. So suppose G is connected, has $k+1$ nodes and no cycles. We must show G has k edges.

Let v be any vertex in G. Let G^* be the graph obtained by deleting from G the vertex v and all its' adjacent edges. Let $C_1, ..., C_t$ denote the connected components of G^*. Then

(i) since G has no cycles, neither has any C_i, hence by the induction hypothesis

$$|V(C_i)| = |E(C_i| + 1, \quad i = 1, ..., t. \tag{1}$$

(ii) since G is connected, there is at least one edge in G from v to each C_i. But since G has no cycles, there is therefore exactly one edge from v to each C_i, that is, $\deg(v) = t$. Hence, the total number of edges in G is

$$
\begin{aligned}
& t+\sum_{i=1}^{t}|E(C_i)| \\
&= t+\sum_{i=1}^{t}(|V(C_i|-1)\,, \quad \text{by (1)} \\
&= \sum_{i=1}^{t}|V(C_i| \\
&= |V(G)|-1,
\end{aligned}
$$

as required.

F.2 I label the nodes of the graph as follows : a to the left, z to the right as in the diagram ; reading downwards on the left b, c, d, e ; reading downwards on the right f, g, h, i.

(i) G has perfect matchings. An example of such a matching is that consisting of the 5 edges

$$\{a,b\}, \ \ \{c,d\}, \ \ \{e,i\}, \ \ \{f,z\}, \ \ \{g,h\}.$$

(ii) Use DFS, starting, say, from the vertex a, to build up the follwowing sequence of edges in a MST :

$$\{a,b\},\{a,e\},\{e,d\},\{d,h\},\{h,c\},\{c,f\},\{f,g\},\{g,z\},\{h,i\}.$$

The total weight of this tree is $1+3+2+3+1+1+2+2+3=18$.

(iii) The following is an example of a maximal flow

Edge	Flow	Edge	Flow
$\{a,b\}$	1	$\{d,h\}$	3
$\{a,c\}$	6	$\{e,h\}$	0
$\{a,d\}$	4	$\{e,i\}$	3
$\{a,e\}$	3	$\{g,f\}$	0
$\{c,b\}$	4	$\{h,g\}$	0
$\{d,c\}$	1	$\{i,h\}$	0
$\{e,d\}$	0	$\{f,z\}$	5
$\{b,f\}$	5	$\{g,z\}$	2
$\{c,f\}$	0	$\{h,z\}$	4
$\{c,g\}$	2	$\{i,z\}$	3
$\{c,h\}$	1		

This flow has strength 14. The corresponding minimal cut (S,T) is

$$S = \{a,b,c,d,f,g\}, \quad T = \{e,h,i,z\}.$$

Observe that

$$c(S,T) = c(a,e) + c(c,h) + c(d,h) + c(f,z) + c(g,z) = 3+1+3+5+2 = 14.$$

F.3 (i) $\Phi(G)$ is the minimal number of colors needed to color the edges of G, if edges with a vertex in common must receive different colors.

(ii) Theorem 10.2 in the book.

F.4 Let S have n elements, say $S = \{1,...,n\}$. Let E denote the collection of subsets of S having an even number of elements, and O the collection of subsets having an odd number of elements. We must describe a bijection $f : E \to O$. We do so explicitly as follows : Let $X \in E$, that is X is a subset of S having an even number of elements. There are two cases :

(i) if $n \in X$, set $f(X) = X \backslash \{n\}$.
(ii) if $n \notin X$, set $f(X) = X \cup \{n\}$.

It's easy to check that f is a bijection (if you're not convinced, write it out in full for $n = 4$, say).

F.5 Let

$$E(x) = \sum_{n=0}^{\infty} u_n \frac{x^n}{n!}$$

denote the exponential generating function of the sequence (u_n). Noting that

$$\sum_{n=0}^{\infty} n \frac{x^n}{n!} = xe^x,$$

we conclude that $E(x)$ satisfies the differential equation

$$2E'' - 7E' + 3E = xe^x. \tag{2}$$

We find the general solution to (2) in the usual way. First, the general solution of the homogeneous equation is

$$E_h(x) = C_1 e^{3x} + C_2 e^{\frac{1}{2}x}.$$

A particular solution to the inhomogeneous equation will have the form

$$E_p(x) = (Ax + B)e^x.$$

Substituting into the lhs of (2) we may solve for A, B as $A = -\frac{1}{2}$, $B = \frac{3}{4}$. Hence the general solution to (2) is

$$E(x) = C_1 e^{3x} + C_2 e^{\frac{1}{2}x} + \left(-\frac{1}{2}x + \frac{3}{4}\right) e^x.$$

To solve for C_1 and C_2 we use the initial conditions

$$u_0 = E(0) = 1, \qquad u_1 = E'(0) = 2.$$

After the required computation, we find that

$$C_1 = \frac{13}{20}, \qquad C_2 = -\frac{2}{5}.$$

We thus finally obtain the desired formula for u_n, namely

$$u_n = \frac{13}{20} \cdot 3^n - \frac{2}{5}\left(\frac{1}{2}\right)^n - \frac{n}{2} + \frac{3}{4}.$$

F.6 $f(n-2,k-1)$ is the number of sets satisfying the condition which contain the number n, and $f(n-1,k)$ is the number of sets satisfying the condition which do not contain n.

F.7 Let $K = \{x^2 \bmod 23 : x \in \mathbf{F}_{23}^*\}$.

The blocks of the design are the 23 translates of K modulo 23 - see Theorem 16.8. Explicitly,

$$1^2 \equiv 1,\ \ 2^2 \equiv 4,\ \ 3^2 \equiv 9,\ \ 4^2 \equiv 16,\ \ 5^2 \equiv 2,\ \ 6^2 \equiv 13,$$
$$7^2 \equiv 3,\ \ 8^2 \equiv 18,\ \ 9^2 \equiv 12,\ \ 10^2 \equiv 8,\ \ 11^2 \equiv 6,$$

so that

$$K = \{1,2,3,4,6,8,9,12,13,16,18\}.$$