MATEMATIK	Dag : 070117 Tid : 8.30 - 13.30.
Göteborgs Universitet	Hjälpmedel : Inga
Peter Hegarty	Vakt : Peter Hegarty 0733-428321.

Tentamenskriving i Talteori (MAN 640)

$\geq$ 12.5 poäng, inkl. inlämningsuppgifterna, ger godkännt.

1 (3p). Prove that at most 64/73 of all integers (in a sense which will be obvious when you solve the problem) can be expressed in the form

$$7a^{72} - 3b^{72} + 11c^{72} - 13d^{72} + 21e^{72} - 16f^{72},$$

for any integers a, b, c, d, e, f.

(HINT : Fermat + Pigeonholes).

2 (3p). Find, with proof, all primes which can be written as the sum of two integer squares.

3 (3p). State and prove Gauss reciprocity law.

(OBS! Gauss Lemma or any results about Gauss sums may be quoted without proof.)

4 (2p+2p) (i) Determine (with proof) $H(-83)$ and all reduced positive-definite binary quadratic forms of this discriminant.
(ii) Give a variable substitution which converts the form

$$127x^2 + 67xy + 9y^2$$

to a reduced form (OBS! the form has discriminant -83).

5 (3p). Compute the quadratic irrational whose continued fraction expansion is $[3; 1, 2, 1, 2, 1, 2, ...]$.

6 (0.5p+2.5p) (i) Write down a formula for the Riemann zeta-function as an infinite product over the primes, indicating the range of its validity.
(ii) Using this, or otherwise, determine the average value of $\phi(n)/n$, i.e.:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \frac{\phi(n)}{n}.$$

7 (3p). Determine with proof an explicit positive integer n_0 such that, for all $n \geq n_0$, given any colouring of the integers $1, 2, ..., n$ in at most three colours, there must exist a monochromatic arithmetic progression of length 3.

8 (3p). Let n be a positive integer. Show that, if A is a subset of $\{1, 2, ..., n\}$ containing no solutions to the equation $4x = y + z$, then $|A| \leq \lceil 3n/4 \rceil$.

(NOTE : $3n/4$ can be improved to $63n/110$, up to an absolute constant. But this requires a lot more work to prove !)

Obs! Tentan beräknas vara färdigrättad den 24 januari. Då kan den hämtas i mottagningsrummet mellan kl. 12:30-13:00. Tentamensresultat lämnas också ut per telefon 772 35 09 *efter* kl. 14:00.