MATEMATIK Dag : 101218 Tid : 8.30 - 13.00 (**Obs! 4.5 hours**).
Göteborgs Universitet Hjälpmedel : Inga
Peter Hegarty Vakt : Peter Hegarty 076-8998364,
Ragnar Freij 070-3088304.

Tentamenskriving i Talteori (MMA 300)

≥ 50 points, including bonuses from the homeworks, required to pass.

1 (15p) Prove that every non-negative integer is a sum of four integer squares, and that 'four' cannot be replaced by any smaller number.
(REMARK : If you use Minkowski's Theorem, you may just quote it).

2 (10p) Prove that the equation $x^2 - 2y^2 = 1$ has infinitely many integer solutions.
(HINT : 'Multiply' solutions in a suitable ring).

3 (15p) (i) State Gauss' lemma on Legendre symbols.
(ii) Using the Lemma, or otherwise, state and prove the Law of Quadratic Reciprocity.

4 (10p) Determine, with proof, whether the congruence

$$9x^2 - 3x + 11 \equiv 0 \pmod{1237}$$

has a solution. (REMARK : 1237 is a prime.)

5 (15p) (i) For a subset $A \subseteq \mathbb{N}_0$ and a positive integer h, define
(a) the unordered h-fold representation function $r_{A,h} : \mathbb{N}_0 \to \mathbb{N}_0$.
(b) what is meant by saying that A is an *asymptotic basis of (exact) order* h.
(ii) Prove that the 2-fold representation function of an infinite subset $A \subseteq \mathbb{N}_0$ cannot be ultimately constant.

6 (10p) Let $A \subseteq \mathbb{N}_0$ satisfy

$$1/2 < \underline{d}(A) < 1.$$

Prove that A is an asymptotic basis of order 2.

7 (15p) (i) State Szemerédi's Regularity Lemma. Define clearly what is meant by the term 'ϵ-*regular*' in the statement of the theorem.
(ii) State the Triangle Counting Lemma.
(iii) Assuming the results above, prove that if $A \subseteq \mathbb{N}_0$ and $\overline{d}(A) > 0$, then A contains a 3-term arithmetic progression.

8 (10p) Prove that the equation

$$x + y = 5z$$

is irregular.

Obs! Tentan beräknas vara färdigrättad den 5 januari. Då kan den hämtas i mottagningsrummet mellan kl. 12:30-13:00. Tentamensresultat lämnas också ut per telefon 772 35 09 *efter* kl. 14:00.