SUPPLEMENTARY LECTURE NOTES : WEEK 45 Monday, November 8

Definition 1.1. A function $f : \mathbb{N} \to \mathbb{C}$ is said to be *multiplicative* if

$$f(mn) = f(m)f(n) \quad \text{whenever GCD}(m, n) = 1. \tag{1.1}$$

Example 1.2. The Euler ϕ -function is multiplicative. This follows immediately from the formula (see Proposition 8.1 in the main lecture notes)

$$\frac{\phi(n)}{n} = \prod_{p|n} \left(1 - \frac{1}{p}\right). \tag{1.2}$$

Example 1.3. For each $k \in \mathbb{N}_0$, let $\sigma_k : \mathbb{N} \to \mathbb{N}$ be the function defined by

$$\sigma_k(n) := \sum_{d|n} d^k, \tag{1.3}$$

where the sum is taken over all positive divisors d of n, including d = 1 and d = n. We set $d(n) := \sigma_0(n)$ and $\sigma(n) := \sigma_1(n)$. The function d(n) is called the *divisor function* and $\sigma(n)$ is called the *sum of divisors function*. It is easy to check (using FTA) that the functions σ_k are multiplicative for every $k \in \mathbb{N}_0$.

Example 1.4. Another important example of a multiplicative function is the so-called *Möbius function* μ defined by

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } p^2 | n \text{ for some prime } p, \\ (-1)^k, & \text{if } n = \prod_{i=1}^k p_i \text{, where the } p_i \text{ are distinct primes.} \end{cases}$$
(1.4)

There are important relationships between the functions ϕ , σ_k , μ and the Riemann ζ -function. For our present purposes, we need the following two relationships - more may be given later.

Lemma 1.5. For any $n \in \mathbb{N}$ one has

$$\frac{\phi(n)}{n} = \sum_{d|n} \frac{\mu(d)}{d}.$$
(1.5)

Proof. This is an immediate consequence of the definition of μ , FTA and the formula (1.2).

Lemma 1.6. If Re(s) > 1 then $\zeta(s) \neq 0$ and

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}.$$
 (1.6)

Proof. This is an immediate consequence of the definition of μ , FTA and the Euler product for ζ - see Theorem 5.3 in the main lecture notes.

We now use these facts to prove the following very nice theorem (see Remark 8.3 in the main lecture notes).

Theorem 1.7. For $n \in \mathbb{N}$ one has

$$\sum_{m=1}^{n} \phi(m) = \left(\frac{3}{\pi^2}\right) n^2 + O(n\log n).$$
(1.7)

Proof. By Lemma 1.5, one has

$$\sum_{m=1}^{n} \phi(m) = \sum_{m=1}^{n} m \cdot \left(\sum_{d|m} \frac{\mu(d)}{d} \right).$$
(1.8)

Set r := m/d and interchange the order of summation. One gets that

$$\sum_{m=1}^{n} \phi(m) = \sum_{d=1}^{n} \mu(d) \cdot \left(\sum_{r=1}^{\lfloor n/d \rfloor} r\right).$$
(1.9)

Now

$$\sum_{r=1}^{\lfloor n/d \rfloor} r = \frac{1}{2} \lfloor n/d \rfloor \left(\lfloor n/d \rfloor + 1 \right) = \frac{n^2}{2d^2} + O\left(\frac{n}{d}\right).$$
(1.10)

Hence

$$\sum_{m=1}^{n} \phi(m) = \frac{n^2}{2} \sum_{d=1}^{n} \frac{\mu(d)}{d^2} + O\left(n \sum_{d=1}^{n} \frac{\mu(d)}{d}\right).$$
(1.11)

We can bound the second term on the right as follows :

$$\left|\sum_{d=1}^{n} \frac{\mu(d)}{d}\right| \le \sum_{d=1}^{n} \left|\frac{\mu(d)}{d}\right| \le \sum_{d=1}^{n} \frac{1}{d} = \log n + O(1).$$
(1.12)

Hence the last term in (1.11) is $O(n \log n)$. Furthermore, by Lemma 1.6, as $n \to \infty$,

$$\sum_{d=1}^{n} \frac{\mu(d)}{d^2} \to \sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}.$$
 (1.13)

Regarding the speed of convergence, we have the estimate

$$\left|\sum_{d=n+1}^{\infty} \frac{\mu(d)}{d^2}\right| \le \sum_{d=n+1}^{\infty} \left|\frac{\mu(d)}{d^2}\right| \le \sum_{d=n+1}^{\infty} \frac{1}{d^2} = O(1/n).$$
(1.14)

Substituting (1.12), (1.13) and (1.14) into (1.11) yields (1.7).

Proposition 1.8. (i) Let f, g be two multiplicative functions. If f(n) = g(n) whenever n is a prime power, then f(n) = g(n) for every $n \in \mathbb{N}$. (ii) If f is a multiplicative function, then the function

$$g(n) := \sum_{d|n} f(d) \tag{1.15}$$

is also multiplicative.

The proof of these facts is simple and left to the reader. Here are two important applications :

Proposition 1.9. (*i*) For every $n \in \mathbb{N}$,

$$\sum_{d|n} \phi(d) = n. \tag{1.16}$$

(ii) Let $\nu : \mathbb{N} \to \mathbb{C}$ be given by

$$\nu(n) := \sum_{d|n} \mu(d).$$
 (1.17)

Then $\nu(1) = 1$ *and* $\nu(n) = 0$ *for all* n > 1*.*

Proof. For a proof of the first identity, see Exercise 4, Homework 2 from 2008. The proof of the second identity is left as an exercise. In both cases one uses part (i) of Proposition 1.8. \Box

The relationship between identities (1.5) and (1.16) is a special case of what is called the *Möbius Inversion Formula* :

Theorem 1.10. (Möbius Inversion Formula) Let $f : \mathbb{N} \to \mathbb{C}$ be any function and let the function $g : \mathbb{N} \to \mathbb{C}$ be defined by (1.15). Then

$$f(n) = \sum_{d|n} \mu(d) g\left(\frac{n}{d}\right).$$
(1.18)

Proof. By the definition of g, we can expand the right-hand side of (1.18) as a double sum

$$\sum_{d|n} \mu(d) \sum_{e|\frac{n}{d}} f(e).$$
(1.19)

Upon interchanging the order of summation we can rewrite this as

$$\sum_{e|n} f(e) \sum_{d|\frac{n}{e}} \mu(d).$$
(1.20)

The inner sum is just $\nu(n/e)$. Thus (1.18) follows from Proposition 1.9(ii).