MATEMATIK
Göteborgs Universitet
Peter Hegarty

Dag: 150827 Tid : 14.00 - 18.30 (**Obs! 4.5 hours**).
Hjälpmedel: Inga
Vakter: Gustav Kettil 0703-088304,
Peter Hegarty 0766-377873.

Tentamenskriving i Talteori (MMA 300)

≥ 50 points, including bonuses from the homeworks, required to pass. In Problems 2,4,6,8, any results that you use from the lecture notes may be just stated without proof.

1 (6p+9p) (a) State and prove a theorem classifying all primitive Pythagorean triples.
(b) Using the result of part (a), prove the case $n = 4$ of Fermat's Last Theorem.

2 (5p+7p) (a) Prove that the equation $a^2 - 3b^2 = 1$ has infinitely many solutions in positive integers.
(HINT: Factorise in $\mathbb{Z}[\sqrt{3}]$).
(b) Prove that the equation $x^2 + y^2 = z^3$ has infinitely many primitive solutions in positive integers, i.e.: solutions satisfying GCD$(x, y, z) = 1$.
(HINT: $\mathbb{Z}[\sqrt{-1}]$ is a unique factorisation domain).

3 (6p+9p) (a) State and prove Gauss' Lemma.
(b) State and prove the law of quadratic reciprocity (you may use Gauss' Lemma without proof).

4 (10p) Determine all primes p for which the congruence $x^2 \equiv 14 \pmod{p}$ has a solution.

5 (12p) Let p_n be the probability that two numbers chosen independently and uniformly at random from $\{1, 2, \ldots, n\}$ are relatively prime. Determine with proof the limit of p_n as $n \to \infty$.

6 (6p+6p) (a) Let $\sigma : \mathbb{N} \to \mathbb{N}$ be the sum of positive divisors function. Determine, with proof, each of the following limits:

$$\liminf_{n\to\infty} \frac{\sigma(n)}{n}, \quad \limsup_{n\to\infty} \frac{\sigma(n)}{n}, \quad \lim_{n\to\infty} \frac{\sigma(n)}{n^{1.001}}.$$

(b) Let $\mu : \mathbb{N} \to \mathbb{N}$ be the Möbius function. Prove that, if $\mathrm{Re}(s) > 1$, then

$$\frac{\zeta(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^s}.$$

7 (3p+11p) (a) State Chebyshev's inequality (no proof needed).
(b) Let $f(n)$ be the maximum size of a subset of $\{1, 2, \ldots, n\}$ containing distinct subset sums. Prove that

$$1 + \lfloor \log_2 n \rfloor \le f(n) \le \log_2 n + \frac{1}{2} \log_2 \log_2 n + O(1).$$

8 (10p) Let $r_3(n)$ denote the maximum size of a subset of $\{1, 2, \ldots, n\}$ containing no 3-term arithmetic progressions. Prove that

$$\liminf_{n\to\infty} \frac{r_3(n)}{n^{2/3}} > 0.$$

Obs! Tentan beräknas vara färdigrättad den 2 september. Då kan den hämtas i expeditionen (ankn. 3500) mellan kl. 11:00-13:00, alla vardagar utom onsdagar.