Permutation Patterns and Statistics

by

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Call two sequences of distinct integers $a_1a_2\ldots a_k$ and $b_1b_2\ldots b_k$ *order isomorphic* if they have the same pairwise comparisons, i.e., $a_i < a_j$ if and only if $b_i < b_j$ for all indices i, j. Let $\mathfrak{S}_n$ be the set of all permutations of $\{1, 2, \ldots, n\}$ viewed as sequences $\pi = a_1a_2\ldots a_n$. We say that $\sigma \in \mathfrak{S}_n$ contains $\pi \in \mathfrak{S}_k$ as a *pattern* if there is a subsequence σ' of σ which is order isomorphic to π. If σ does not contain π, we say it *avoids* π and use the notation

$$\mathrm{Av}_n(\pi) = \{\sigma \in \mathfrak{S}_n \ :\ \sigma \text{ avoids } \pi\}.$$

The theory of pattern avoidance contains many beautiful results. For example, all $\pi \in \mathfrak{S}_3$ avoid the same number of elements and $|\mathrm{Av}_n(\pi)| = C_n$, the nth Catalan number.

A *permutation statistic* is a function $\mathrm{st} : \mathfrak{S}_n \to \mathbb{N}$ where the range is the non-negative integers. Two famous statistics are the inversion number $\mathrm{inv}\,\pi$ (which counts the number of out-of-order pairs in π) and the major index $\mathrm{maj}\,\pi$ (which is the sum of the indices where π has a descent). Such statistics have been the object of intense study. For example, an early result of MacMahon showed that inv and maj are equidistributed on $\mathfrak{S}_n$.

In this talk, we will combine the notions of permutation patterns and permutation statistics by considering the generating functions

$$I_n(\pi) = \sum_{\sigma \in \mathrm{Av}_n(\pi)} q^{\mathrm{inv}\,\sigma}$$

and similarly for the major index. They turn out to have many wonderful properties, giving a strengthening of the concept of Wilf equivalence, q-analogues of the Catalan numbers and other well-known seqeuences, and connecting pattern avoidance with other parts of combinatorics such as the theory of integer partitions. No prior knowledge of permutations patterns or statistics will be assumed.

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