Extremal set systems

Peter Hegarty

May 15, 2000

N.B.: All references pertaining to the material of this talk are contained in an accompanying document called *Notes*.

1. Some things this talk could have been about but (mostly) isn't

- **Extremal graph theory :** Explores the relationships between various graph invariants. A large field with many well-known theorems and open problems.

  Examples : Ramsey theory, Turán's theorem, algorithmic problems etc. etc.

- **Extremal problems for hypergraphs :** Generalisations of the corresponding problems for graphs.

- **Designs :** A $t-(\nu,k,\lambda)$ design is a collection $\mathcal{C}$ of k-element subsets of a ν-set X

such that every t-element subset of X appears exactly λ times.

Existence and classification of designs is the basic theoretical problem. But for applications it may also be important to know 'how close one can get'.

- **Coding theory :** Obviously a subject of great practical importance.

- **Combinatorial number theory :** Here's a sample unsolved problem from this field :

  Question (Erdős \$ 500) : Does there exist an absolute constant $C > 0$ such that if $S \subseteq \{1, ..., n\}$ has all subset sums distinct, then $|S| \leq \log_2 n + C$?

2. Some notation and terminology

$[n] := \{1, 2, ..., n\}$

$\mathcal{F} \subseteq 2^{[n]}$ is called a *family* of sets.

$|\mathcal{F}|$:= the number of sets in the family.

For each $1 \leq i \leq n$, $\mathcal{F}_i := \{A \in \mathcal{F} : i \in A\}$.

$\Delta(\mathcal{F}) := \max_{1 \leq i \leq n} |\mathcal{F}_i|$.

$\binom{X}{k}$ is the family of all k-subsets of the set X.

$a(\mathcal{F})$ denotes the average size of a member of $\mathcal{F}$, i.e.:

$$a(\mathcal{F}) := \frac{1}{|\mathcal{F}|} \sum_{A \in \mathcal{F}} |A|.$$

Reverse lexicographic order :

Let A, B be finite subsets of $\mathbf{N}$. We say that $A <_L B$ if $A \subset B$ or $\max\{x \in A - B\} < \max\{x \in B - A\}$.

We denote by $\mathcal{L}(m)$ (resp. $\mathcal{L}(m,k)$) the family consisting of the m smallest members of $\mathbf{N}$ (resp. $\binom{\mathbf{N}}{k}$) in the reverse lexicographic ordering.

3. 'Lattice' problems

The oldest result in extremal set theory is

Sperner's theorem (1928) : If $\mathcal{F} \subseteq 2^{[n]}$ is an antichain, then $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$ with equality iff $\mathcal{F} = \binom{[n]}{\lfloor n/2 \rfloor}$ or $\mathcal{F} = \binom{[n]}{\lceil n/2 \rceil}$.

Definition : F is *union-closed* if $A, B \in \mathcal{F} \Rightarrow A \cup B \in \mathcal{F}$.

Frankl's (hopeless ?)
conjecture (1979) : If $\mathcal{F} \subseteq 2^{[n]}$ is UC, then $\Delta(\mathcal{F}) \geq \frac{1}{2}|\mathcal{F}|$.

A weaker conjecture : If $\mathcal{F}$ is UC then $a(\mathcal{F}) \geq \frac{1}{2}\log_2|\mathcal{F}|$.

A conjecture for lattices : Let L be a lattice with n elements. There exists a join-irreducible x of L such that the principal dual order ideal $V_x = \{y \in L : y \geq x\}$ has at most $n/2$ elements.

4. Intersecting families

Definition : $\mathcal{F}$ is *intersecting* if $A \cap B \neq \phi$ for all $A, B \in \mathcal{F}$.

Easy question : If $\mathcal{F} \subseteq 2^{[n]}$ is intersecting, what is $\max|\mathcal{F}|$?

Answer : $|\mathcal{F}| \leq 2^{n-1}$.

Not so easy question : If $\mathcal{F} \subseteq \binom{[n]}{k}$ is intersecting, with $k \leq n/2$, what is $\max|\mathcal{F}|$?

Answer (Erdós-Ko-Rado theorem) : $|\mathcal{F}| \leq \binom{n-1}{k-1}$.

Generalisations of these results start from the following definition :

Definition : The family $\mathcal{F}$ is said to be *s-wise t-intersecting* if $|A_1 \cap \; ... \; \cap A_s| \geq t$ for all $A_1, ..., A_s \in \mathcal{F}$.

5. Ideals

Definition : $\mathcal{F}$ is called a *(lower) ideal* if, whenever $A \in \mathcal{F}$ and $B \subseteq A$, then $B \in \mathcal{F}$.

Kleitman's theorem : If $\mathcal{F}, \mathcal{G} \subseteq 2^{[n]}$ are ideals, then

$$|\mathcal{F} \cap \mathcal{G}| \geq \frac{|\mathcal{F}||\mathcal{G}|}{2^n}.$$

Notation :

$\nabla(\mathcal{F}, \mathcal{G}) := \{(A, B) : A \in \mathcal{F}, B \in \mathcal{G}, A \cap B = \emptyset\}.$

Theorem : If $\mathcal{F}$ and $\mathcal{G}$ are ideals, then

$$|\nabla(\mathcal{F}, \mathcal{G})| \geq (|\mathcal{F}||\mathcal{G}|)^{\frac{1}{2}\log_2 3}.$$

This result follows from another theorem

Theorem (Seymour and Hajela 1985) : Let $\mathcal{F}, \mathcal{G} \subseteq \{0,1\}^n$. Then

$$|\mathcal{F} + \mathcal{G}| \geq (|\mathcal{F}||\mathcal{G}|)^{\frac{1}{2}\log_2 3},$$

where

$$\mathcal{F} + \mathcal{G} = \{f + g : f \in \mathcal{F}, g \in \mathcal{G}\}.$$

The most outstanding open problem on ideals is

Chvátal's conjecture : If $\mathcal{I}$ is an ideal and $\mathcal{F} \subseteq \mathcal{I}$ is intersecting, then $|\mathcal{F}| \leq \Delta(\mathcal{I})$.

There are some half-decent partial results on this conjecture (so it's perhaps not 'hopeless' !). For example

Theorem (Berge 1975) : Any ideal $\mathcal{I}$ is the disjoint union of pairs of disjoint sets, together with $\emptyset$ if $|\mathcal{I}|$ is odd. In particular, if $\mathcal{F} \subseteq \mathcal{I}$ is intersecting, then $|\mathcal{F}| \leq \lfloor \frac{1}{2}|\mathcal{I}| \rfloor$.

Theorem (Miklós 1984) : Chvátal's conjecture holds if $\mathcal{I}$ contains an intersecting subfamily of size $\lfloor \frac{1}{2}|\mathcal{I}| \rfloor$.

6. Isoperimetric problems

Definition 1 : Let $\mathcal{F} \subseteq \binom{X}{k}$. For any $l < k$, the *l-shadow* of $\mathcal{F}$, denoted $\sigma_l(\mathcal{F})$, is defined by

$$\sigma_l(\mathcal{F}) = \left\{ B \in \binom{X}{l} : \exists\, A \in \mathcal{F} \text{ with } B \subset A \right\}.$$

Kruskal-Katona-Schützenberger theorem (1963, 1966, 1959) :

$$|\sigma_l(\mathcal{F})| \geq |\sigma_l(\mathcal{L}(|\mathcal{F}|, k))|.$$

Definition 2 : The *boundary* $\partial(\mathcal{F})$ of a family $\mathcal{F} \subseteq 2^{[n]}$ is defined by

$$\partial(\mathcal{F}) = \{B \subseteq [n] : B \notin \mathcal{F},\ \exists\, A \in \mathcal{F} \text{ with } |A \Delta B| = 1\}.$$

Definition 2(a) : Let $A \subseteq [n]$, $r > 0$. The *ball* centered at X and of radius r, denoted $\mathcal{B}(X, r)$, is defined by

$$\mathcal{B}(X, r) = \{A \subseteq [n] : |A \Delta X| \leq r\}.$$

A family $\mathcal{F} \subseteq 2^{[n]}$ is called a *generalised ball* if there exists some X, r such that $\mathcal{B}(X, r) \subseteq \mathcal{F} \subset \mathcal{B}(X, r+1)$.

Discrete isoperimetric theorem (Harper 1966) : Given $|\mathcal{F}|$, $|\partial(\mathcal{F})|$ is minimised by some generalised ball.

Definition 3 : Let $\mathcal{F} \subseteq \binom{[n]}{k}$. The *$k$-boundary* of $\mathcal{F}$, denoted $\kappa(\mathcal{F})$, is defined by

$$\kappa(\mathcal{F}) = \left\{ B \in \binom{[n]}{k} : B \notin \mathcal{F},\ \exists\ A \in \mathcal{F} \text{with } |A \Delta B| = 2 \right\}.$$

Open Problem (no conjectures !) : Given $k, |\mathcal{F}|$, how should one minimise $|\kappa(\mathcal{F})|$?

We want to state one last result

Definition 4 : For $\mathcal{F} \subseteq 2^{[n]}$ we define

$$\mathcal{P}(\mathcal{F}) := \{(A, B) : A \in \mathcal{F}, B \in \partial(\mathcal{F}), |A \Delta B| = 1\}.$$

Theorem : Given $|\mathcal{F}|$, $|\mathcal{P}(\mathcal{F})| \geq |\mathcal{P}(\mathcal{L}(|\mathcal{F}|))|$.

6(a). A general technique - 'shifting'

- **General shift**

  Let $\mathcal{F} \subseteq 2^{[n]}$. Pick U, V disjoint subsets of $[n]$. The (U, V)-*shift* S_{UV} is defined by

  $S_{UV}(\mathcal{F}) = \{S_{UV}(A) : A \in \mathcal{F}\}$ where

  $$S_{UV}(A) = \begin{cases} A - U + V =: \bar{A}, & \text{if } U \subseteq A, \\ & V \cap A = \emptyset \\ & \text{and } \bar{A} \notin \mathcal{F}, \\ A, & \text{otherwise.} \end{cases}$$

- **Down shifting**

  $D_i(\mathcal{F}) = \{D_i(A) : A \in \mathcal{F}\}$ where

  $$D_i(A) = \begin{cases} A - \{i\}, & \text{if } i \in A \text{ and } A - \{i\} \notin \mathcal{F}, \\ A, & \text{otherwise.} \end{cases}$$

- **Sideways shifting**

  $S_{ij}(\mathcal{F}) = \{S_{ij}(A) : A \in \mathcal{F}\}$ where

$$S_{ij}(A) = \begin{cases} A - \{j\} + \{i\} =: \bar{A}, & \text{if } j \in A, i \notin A \\ & \text{and } \bar{A} \notin \mathcal{F}, \\ A, & \text{otherwise.} \end{cases}$$