Friday, Sept 5

Last day we proved the following (using the 'dots and dashes' idea) :

Theorem *The number of ways to place n indistinguishable balls in k distinguishable cells is* $\begin{pmatrix} n+k-1 \\ k-1 \end{pmatrix}$.

This result can be reformulated as

Theorem *The number of solutions in non-negative integers x_i to the equation*

$$x_1 + \cdots + x_k = n \tag{1}$$

is $\begin{pmatrix} n+k-1 \\ k-1 \end{pmatrix}$.

PROOF : Let x_i be the number of balls placed in the i:th cell above.

The solutions to (1) are called *ordered partitions* of n *into non-negative parts.*

There are three natural variations of the question answered by the above theorem, namely :

Question 1 In how many ways can n distinguishable balls be placed in k distinguishable cells ?

Question 2 In how many ways can n distinguishable balls be placed in k indistinguishable cells ?

Question 3 In how many ways can n indistinguishable balls be placed in k indistinguishable cells ?

Question 1 can be abswered easily :

Proposition *The number of ways to place n distinguishable balls into k distinguishable cells is k^n.*

PROOF : There is an obvious 1-1 correspondence between such placements and all functions from the set $\{1, 2, ..., n\}$ to the set $\{1, 2, ..., k\}$. Namely, given such a function f, we would put ball number i in the $f(i)$:th cell, for $i = 1, ..., n$.

So we only need to count these functions. Well, for each $i = 1, ..., n$, there are k choices for $f(i)$. So, by the multiplication principle (MP), the number of such functions is k^n, v.s.v.

Question 2 is already a lot harder.

DEFINITION : Let n, k be positive integers. The (n, k):th *Stirling number of the second kind*, denoted $S(n, k)$, is the number of ways to place n distinguishable balls into k indistinguishable cells so that no cell is left empty.

REMARK : Note that $S(n, k) = 0$ if $n < k$.

There is no really nice formula for the Stirling numbers, except in some special cases, for example (see also exercises for Vecka 1 and Hemuppgift 1) :

Proposition *If $n \geq 2$, then $S(n, 2) = 2^{n-1} - 1$.*

PROOF : Call the balls $1, ..., n$ and the two cells I and II. Let A be the subset of $\{1, ..., n\}$ denoting which balls are placed in cell I. Then A^c denotes which balls are placed in cell II, so the only choice is for A. That neither cell is to be left empty implies that we may choose for A any subset of $\{1, ..., n\}$ other than the whole set and the empty subset. There are thus $2^n - 2$ choices for A. Finally, to get $S(n, 2)$ we have to divide this number by $2! = 2$, since the two cells are indistinguishable. Hence $S(n, 2) = \frac{1}{2}(2^n - 2) = 2^{n-1} - 1$, v.s.v.

The following recurrence relation is helpful for more general computations of Stirling numbers :

Theorem *The Stirling numbers $S(n, k)$ satisfy the following recurrence relation :*

$$S(n, 1) = S(n, n) = 1, \tag{2}$$

$$S(n, k) = S(n-1, k-1) + k \cdot S(n-1, k). \tag{3}$$

PROOF : The relations (2) are obvious, so we turn to (3). Suppose we place balls $1, ..., n$ into k identical cells. Let's isolate the n:th ball and consider two possibilities :

Case 1 : The n:th ball is placed in a cell on its' own. Then the remaining $n-1$ balls are to be placed in $k-1$ identical cells, so that no cell is left empty. By definition, there are $S(n-1, k-1)$ ways to do this.

Case 2 : The n:th ball is not on its' own. How many options do we have in this case ? Well, first we have to place the remining $n-1$ balls in k cells so that no cell is left empty. There are $S(n-1, k)$ ways to do this. Then we have k choices for where to put the n:th ball. So, by the MP, we have in total $k \cdot S(n-1, k)$ possible choices in this case.

Altogether, then, we have $S(n-1, k-1) + k \cdot S(n-1, k)$ possible ways to distribute the balls, which proves (3).

We now turn to Question 3.

DEFINITION : Let n, k be positive integers. A placement of n indistinguishable balls into k indistinguishable cells so that no cell is empty is called a *(unordered) partition* of n *into* k *positive parts.* The number of such partitions is denoted $p(n, k)$.

The study of the functions $p(n, k)$ and the related functions

$$P(n,k) := \sum_{j \leq k} p(n,j),$$
$$p(n) := \sum_{k} p(n,k).$$

is a classical problem in combinatorial and analytic number theory (two branches of mathematics which you've maybe never even heard of !!). In particular, the problem of computing the function $p(n)$, which counts the total number of partitions of the positive integer n into positive parts, has attracted a great dela of attention. This problem is now considered essentially solved. The real breakthrough came in the 1920s with the following amazing result -

Theorem (Hardy and Ramanujan)

$$p(n) \sim \frac{e^{\pi\sqrt{\frac{2n}{3}}}}{4\sqrt{3}n}.$$

I don't want to discuss partitions much in this course. Chapter 26 in Biggs is devoted to them. See also the exercises.

Monday, Sept 8

Theorem *Suppose the sequence (u_n) of (complex) numbers satisfies the recurrence relation*

$$au_{n+2} + bu_{n+1} + cu_n = 0, \quad \forall\, n \geq 0,$$

for some constants a, b, c. Let α, β be the roots of the quadratic equation

$$ax^2 + bx + c = 0.$$

(i) If $\alpha \neq \beta$ then there exist constants K_1, K_2 such that

$$u_n = K_1\alpha^n + K_2\beta^n, \quad \forall\, n \geq 0.$$

(ii) If $\alpha = \beta$ then there exist constants K_1, K_2 such that

$$u_n = (K_1 + K_2 n)\alpha^n, \quad \forall\, n \geq 0.$$

I leave it as an exercise to the reader to generalise this theorem to linear recurrence relations with constant coefficients and of arbitrary degree.

A more powerful technique for attacking a wider class of recurrence relations is to use so-called *generating functions*.

DEFINITION : Let $(u_n)_0^\infty$ be any sequence of (complex) numbers. The *generating function* for the sequence (u_n) is the power series function

$$G(x) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} u_n x^n. \tag{4}$$

In applying generating functions to solve recurrence relations, one often uses the following well-known identity :

$$\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n, \qquad (\text{if } |t| < 1). \tag{5}$$

Let's begin by resolving a recurrence we already can solve using the theorem above, but this time using the generating function method.

EXAMPLE 1 : Solve the recurrence relation

$$u_0 = 3, \quad u_1 = 5,$$
$$u_n = 5u_{n-1} - 6u_{n-2}, \quad \forall\, n \geq 2.$$

Abbreviated Solution : Let $G(x)$ be the generating function of the sequence (u_n), as in (4). Observe that

$$xG(x) = \sum_{n=1}^{\infty} u_{n-1}x^n,$$
$$x^2G(x) = \sum_{n=2}^{\infty} u_{n-2}x^n.$$

Hence

$$(1-5x+6x^2)G(x) = \sum_{n=2}^{\infty}(u_n - 5u_{n-1} + 6u_{n-2})x^n + (u_0 + u_1x) - 5u_0x.$$

By the recurrence relation, the sum $\sum_2^{\infty}(\cdots)$ is identically zero. Hence

$$G(x) = \frac{(3+5x)-5(3x)}{1-5x+6x^2} = \frac{3-10x}{(1-2x)(1-3x)}.$$

We seek a partial fraction decomposition

$$\frac{3-10x}{(1-2x)(1-3x)} = \frac{A}{1-2x} + \frac{B}{1-3x},$$

and one readily computes that $A = 4$, $B = -1$. Finally, using the identity (5), we have the following explicit expression for $G(x)$ as a power series :

$$G(x) = \sum_{n=0}^{\infty}(4\cdot 2^n - 1\cdot 3^n)\,x^n.$$

Comparing with (4) it follows that

$$u_n = 4\cdot 2^n - 1\cdot 3^n = 2^{n+2} - 3^n, \quad \text{v.s.v.}$$

Now let's continue with an example not covered by the theorem above.

Example 2 (see Biggs 25.6.2) : Let q_n be the number of words of length n in the alphabet $\{a, b, c, d\}$ which contain an odd number of b:s. Find and solve a recurrence relation for q_n.

Abbreviated Solution : Let's divide the q_n allowed words of length n into two types :

(i) those that begin with a b. Then the remaining $n-1$ letters form a word which is not one of the q_{n-1} words of length $n-1$ containing an odd number of b:s. Since (by MP) there are in total 4^{n-1} words of length $n-1$ in our alphabet, if follows that there are $4^{n-1} - q_{n-1}$ words of type (i).

(ii) those that don't begin with a b. Then there are 3 choices for the first letter (a, c or d) and the remaining $n-1$ letters form one of the q_{n-1} words of length $n-1$ containing an odd number of b:s. Hence, by MP, there are $3q_{n-1}$ words of type (ii).

From the above analysis we deduce the recurrence relation

$$q_n = 4^{n-1} + 2q_{n-1}, \quad \forall\, n \geq 1.$$

By inspection, we also have the initial condition $q_0 = 0$. To solve the recurrence, we consider the generating function

$$G(x) := \sum_{n=0}^{\infty} q_n x^n. \tag{6}$$

We find that

$$\begin{aligned}
(1-2x)G(x) &= \sum_{n=1}^{\infty} (q_n - 2q_{n-1})x^n + q_0 x^0 \\
&= \sum_{n=1}^{\infty} 4^{n-1} x^n + 0 \\
&= \frac{1}{4} \sum_{n=1}^{\infty} (4x)^n \\
&= \frac{1}{4} \frac{4x}{1-4x} \\
&= \frac{x}{1-4x}.
\end{aligned}$$

Hence

$$G(x) = \frac{x}{(1-2x)(1-4x)}.$$

We seek a partial fraction decomposition

$$\frac{x}{(1-2x)(1-4x)} = \frac{A}{1-2x} + \frac{B}{1-4x},$$

and readily compute that $A = -\frac{1}{2}$, $B = \frac{1}{2}$. Using the identity (5) we thus obtain the explicit power series representation

$$G(x) = \sum_{n=0}^{\infty} \left(\frac{1}{2} \cdot 4^n - \frac{1}{2} \cdot 2^n \right) x^n.$$

Comparing with (6), we must have

$$q_n = \frac{1}{2} \left(4^n - 2^n \right).$$

Thursday, Sept 11

Our first task today is to generalise what we have previously called the *binomial theorem* and which may be stated as follows :

Let n be a positive integer and x any real number. Then

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k. \tag{7}$$

Here, $\binom{n}{k}$ is the number of ways to choose k objects from n (with order unimportant) and we have the formula

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!}. \tag{8}$$

Eq. (7) can be proved purely combinatorially, as we have seen in class. As a step toward generalising (7) we first note that, if $k > n$, then $\binom{n}{k} = 0$, since it is not possible in this case to choose k objects from n. This is consistent with the last formula in (8) (the first one makes no sense, since $(n-k)!$ is not defined when $n-k < 0$), since one of the factors in the numerator will be zero whenever $k > n$. Hence, the binomial theorem can be written in the form

$$(1+x)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k. \tag{9}$$

Then the thing to notice is that, because of the formula in (8), the HL of (9) is just the McLaurin expansion of the function $f(x) = (1+x)^n$. Recall that the McLaurin expansion of a function $f(x)$, which is infinitely differentiable in a neighbourhood of $x = 0$, is given by

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k.$$

There are some general conditions which guarantee that the McLaurin expansion of a C^∞-function $f(x)$ converges pointwise to $f(x)$ in a neighbourhood of $x = 0$. Since this is not a course in analysis, I don't want to go into any details here on this matter. But note that if n is any real (indeed complex) number, not just a positive integer, then the function $f(x) = (1+x)^n$ is a C^∞-function in the interval $|x| < 1$. So the function has a McLaurin expansion, and one can prove (we don't do so) that this expansion always converges to $f(x)$. This is the form in which we wish to generalise the binomial theorem. Let us state the result formally :

Generalised Binomial Theorem *Let z be any real (indeed complex) number and x a real number such that $|x| < 1$. Then*

$$(1+x)^z = \sum_{k=0}^{\infty} \binom{z}{k} x^k, \tag{10}$$

where

$$\binom{z}{k} \stackrel{def}{=} \frac{z(z-1)\cdots(z-k+1)}{k!}.$$

Later on in the lecture, we will see a cool application of this result !

DEFINITION : Let n be a non-negative integer. A *Dyck path of length* $2n$ is a path in the xy-plane from $(0,0)$ to $(2n,0)$ consisting of $2n$ steps, each of the form

$$(x,y) \mapsto (x+1, y \pm 1),$$

which in addition never goes below the x-axis.

DEFINITION : Let $n \geq 0$. The n^{th} *Catalan number*, denoted C_n, is defined to be the number of Dyck paths of length $2n$.

Theorem 1 *The Catalan numbers satisfy the following recurrence relation*

$$C_0 = 1, \tag{11}$$

$$C_n = \sum_{m=1}^{n} C_{m-1}C_{n-m}, \qquad \forall\, n \geq 1. \tag{12}$$

PROOF : (11) is obvious. For (12) we observe that $C_{m-1}C_{m-n}$ is the number of Dyck paths of length $2n$ which first intersect the x-axis at $(2m, 0)$.

Theorem 2

$$C_n = \frac{1}{n+1}\begin{pmatrix} 2n \\ n \end{pmatrix}.$$

Proof : We work with the generating function for the sequence (C_n), i.e.: the function

$$F(x) = \sum_{n=0}^{\infty} C_n x^n.$$

Using (11) and (12) we have that

$$\begin{aligned} x \cdot [F(x)]^2 = (xF(x)) \cdot F(x) = \left(\sum_{m=1}^{\infty} C_{m-1}x^m\right) \cdot \left(\sum_{t=0}^{\infty} C_t x^t\right) \\ = \sum_{n=1}^{\infty}\left(\sum_{m=1}^{n} C_{m-1}C_{n-m}\right) x^n \\ = \sum_{n=1}^{\infty} C_n x^n \\ = F(x) - C_0 \\ = F(x) - 1, \end{aligned}$$

i.e.:

$$x[F(x)]^2 = F(x) - 1. \tag{13}$$

We may consider (13) as a quadratic equation for $F(x)$, and hence there are two possible solutions, namely

$$F(x) = \frac{1 \pm \sqrt{1-4x}}{2x}.$$

Since $F(0) = C_0 = 1$, the correct solution must be to take the minus sign. We conclude that

$$F(x) = \frac{1 - \sqrt{1-4x}}{2x}.$$

To expand this in a power series, we use the generlised binomial theorem (10) for exponent $z = 1/2$.

$$
\begin{aligned}
F(x) &= \frac{1-\sqrt{1-4x}}{2x} \\
&= \frac{1}{2x}\left[1-(1-4x)^{1/2}\right] \\
&= -\frac{1}{2x}\sum_{n=1}^{\infty}\binom{1/2}{n}(-4x)^n \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n 4^{n+1}}{2}\binom{1/2}{n+1}x^n .
\end{aligned}
$$

So it remains to prove that, for every integer $n \geq 0$,

$$
\frac{(-1)^n 4^{n+1}}{2}\binom{1/2}{n+1} = \frac{1}{n+1}\binom{2n}{n}. \tag{14}
$$

We have

$$
\begin{aligned}
\binom{1/2}{n+1} &= \frac{1}{(n+1)!}\prod_{i=0}^{n}\left(\frac{1}{2}-i\right) \\
&= \frac{1}{(n+1)!}\cdot\frac{(-1)^n}{2^{n+1}}\cdot(1\cdot 3\cdot 5\cdots(2n-1)) \\
&= \frac{1}{(n+1)!}\cdot\frac{(-1)^n}{2^{n+1}}\cdot\frac{(2n)!}{2\cdot 4\cdot 6\cdots(2n)} \\
&= \frac{(-1)^n}{2^{n+1}}\cdot\frac{1}{(n+1)}\frac{(2n)!}{n!n!},
\end{aligned}
$$

from which (14) easily follows. This completes the proof of Theorem 2.

Monday, Sept 15

NOTATION/TERMINOLOGY : Let X be a set, A and B subsets of X. The *union* of A and B, denoted $A \cup B$, is the subset of X consisting of those elements which lie in either A or B, i.e.:

$$A \cup B \stackrel{\text{def}}{=} \{x \in X : x \in A \text{ or } x \in B\}.$$

The *intersection* of A and B, denoted $A \cap B$, consists of those elements of X which lie in both A and B, i.e.:

$$A \cap B \stackrel{\text{def}}{=} \{x \in X : x \in A \text{ and } x \in B\}.$$

The *set difference A minus B*, denoted $A \backslash B$, consists of those elements in A which are not in B, i.e.:

$$A \backslash B \stackrel{\text{def}}{=} \{x \in X : x \in A \text{ and } x \notin B\}.$$

EXAMPLE : $X = \mathbf{N}$, the set of natural numbers, $A = \{1, 2, 3, 4\}$, $B = \{2, 3, 5, 6\}$. Then

$$\begin{aligned} A \cup B &= \{1, 2, 3, 4, 5, 6\}, \\ A \cap B &= \{2, 3\}, \\ A \backslash B &= \{1, 4\}. \end{aligned}$$

NOTATION : If X is a finite set, then $|X|$ shall denote the number of elements in X. If X is an infinite set we write $|X| = \infty$.

Theorem (Inclusion-Exclusion or Sieve Principle) *Let X be a finite set and $A_1, ..., A_n$ be n subsets of X. Then*

$$\left| X \backslash \left(\bigcup_{i=1}^{n} A_i \right) \right| = |X| - \sum_{i=1}^{n} |A_i| \tag{15}$$
$$+ \sum_{i \neq j} |A_i \cap A_j| - \sum_{i \neq j \neq k} |A_i \cap A_j \cap A_k|$$
$$+ \cdots + (-1)^n |A_1 \cap \cdots \cap A_n|.$$

PROOF : Didn't bother with it. There is a proof in Biggs, Chapter 11.4 if you have the book and you're interested.

EXAMPLE : Let $n \geq 0$. A *derangement* of n objects is a permutation (rearrangement) of them such that no object is left in its' original position. More concretely, if we denote a permutation of the numbers $1, 2, ..., n$ by $a_1 a_2 \cdots a_n$, then a derangement is a permutation such that $a_i \neq i$ for $i = 1, ..., n$.

The number of derangements of n objects is denoted d_n. We seek information about the sequence (d_n). One may compute

$$d_1 = 0, \quad d_2 = 1, \quad d_3 = 2, \quad d_4 = 9,$$

etc. For example, the nine derangements of $1, 2, 3, 4$ are

$$2143 \;\; 2341 \;\; 2413 \;\; 3142 \;\; 3412 \;\; 3421 \;\; 4123 \;\; 4312 \;\; 4321.$$

Theorem

$$d_n = n! \times \left(\sum_{k=0}^{n} \frac{(-1)^k}{k!} \right). \tag{16}$$

In particular,

$$\frac{d_n}{n!} \to \frac{1}{e} \quad \textit{as } n \to \infty. \tag{17}$$

PROOF : Note that (17) follows from (16) upon inserting $x = -1$ into the well-known McLaurin expansion for the exponential function

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

To prove (16) we use the I-E principle. To simplify our notation, we'll henceforth denote the set $\{1, 2, ..., n\}$ by $[n]$. Let X denote the set of all permutations of $[n]$. We consider the n subsets

$$A_i = \{\pi \in X : \pi(i) = i\}, \quad i = 1, ..., n.$$

By definition,

$$d_n = \left| X \backslash \bigcup_{i=1}^{n} A_i \right|,$$

so we may use (15). A typical term on the rhs of (15) is

$$(-1)^k \sum_{i_1 \neq i_2 \neq \cdots \neq i_k} |A_{i_1} \cap \cdots \cap A_{i_k}|.$$

Each term in this sum just equals $(n-k)!$, since we are considering those permutations which leave some specified k of our n numbers fixed and permute the others arbitrarily. The number of terms in the sum is $\binom{n}{k}$. Hence the value of the sum, for a fixed k is

$$(-1)^k \binom{n}{k} \cdot (n-k)! = (-1)^k \frac{n!}{k!(n-k)!} \cdot (n-k)! = n! \times \frac{(-1)^k}{k!}.$$

Summing over k from zero to n, we obtain the HL of (16), v.s.v.

There is also a nice recurrence relation satisfied by the sequence (d_n), namely

Theorem

$$\begin{gathered} d_1 = 0, \quad d_2 = 1, \\ d_n = (n-1)(d_{n-1} + d_{n-2}), \quad \forall\, n \geq 3. \end{gathered} \tag{18}$$

Proof : Let $n \geq 3$. We divide the derangements of $[n]$ into two types :

(i) those derangements π such that, if $\pi(1) = i$ then $\pi(i) = 1$. Then π must include a derangement of the numbers $2, 3, ..., i-1, i+1, ..., n$. There are d_{n-2} possibilities for this derangement and $n-1$ possibilities for $i = \pi(1)$. Hence there are $(n-1)d_{n-2}$ derangements of type (i).

(ii) all other derangements. Let π be one such and let $\pi(1) = i$. If we now imagine identifying the numbers 1 and i, then we can think of π as including a derangement of the numbers $2, 3, ..., n$ (that i is 'moved' now means that it doesn't get sent back to 1). There are d_{n-1} possibilities for this derangement and $n-1$ possibilities for i, so there are $(n-1)d_{n-1}$ derangements of type (ii).

Adding, we get $d_n = (n-1)(d_{n-1} + d_{n-2})$, v.s.v.

See the Week 2 exercises for hints on how to derive (16) from (18) using exponential generating functions.