TMA 055 : Diskret matematik

Tentamen 100104

Lösningar

F.1 $60 = 2^2 \cdot 3 \cdot 5$ so $\phi(60) = \phi(2^2) \cdot \phi(3) \cdot \phi(5) = (2^2 - 2)(3-1)(5-1) = 2 \cdot 2 \cdot 4 = 16$. Hence, Euler's Theorem states that, if n is an integer relatively prime to 60, then

$$n^{16} \equiv 1 \pmod{60}.$$

Note that both 7 and 13 are relatively prime to 60. Hence (all congruences are modulo 60)

$$7^{50} = (7^{16})^3 \cdot 7^2 \equiv 1^3 \cdot 49 \equiv 49 \equiv -11,$$

and

$$13^{65} = (13^{16})^4 \cdot 13 \equiv 1^4 \cdot 13 \equiv 13.$$

Thus,

$$(7^{50} + 13^{65})^{12} \equiv (-11 + 13)^{12} = 2^{12} = (2^6)^2 = 64^2 \equiv 4^2 = 16.$$

So the answer is 16.

F.2 Let

$$G(x) = \sum_{n=0}^{\infty} u_n x^n$$

denote the generating function of the sequence (u_n). Let's rock !

$$\begin{aligned}
(1 - 5x + 4x^2)G(x) &= (u_0 + u_1 x) - 5(u_0 x) + \sum_{n=2}^{\infty} (u_n - 5u_{n-1} + 4u_{n-2})x^n \\
&= (2 + 3x) - 5(2x) + \sum_{n=2}^{\infty} 4^n x^n \\
&= 2 - 7x + \frac{16x^2}{(1-4x)} \\
&= \frac{44x^2 - 15x + 2}{(1-4x)}.
\end{aligned}$$

Since

$$1 - 5x + 4x^2 = (1 - x)(1 - 4x),$$

it follows that

$$G(x) = \frac{44x^2 - 15x + 2}{(1 - x)(1 - 4x)^2}.$$

We seek a partial fraction decomposition

$$\frac{44x^2 - 15x + 2}{(1 - x)(1 - 4x)^2} = \frac{A}{1 - x} + \frac{B}{(1 - 4x)} + \frac{C}{(1 - 4x)^2}. \tag{1}$$

Clearing denominators, we have

$$44x^2 - 15x + 2 = A(1 - 4x)^2 + B(1 - x)(1 - 4x) + C(1 - x).$$

Gathering coefficients, we get the following system of linear equations to solve

$$\begin{pmatrix} 1 & 1 & 1 \\ 8 & 5 & 1 \\ 16 & 4 & 0 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 2 \\ 15 \\ 44 \end{pmatrix}.$$

After the usual Gauβ elimination and back substitution (I omit the details), we get the solution

$$A = \frac{31}{9}, \quad B = -\frac{25}{9}, \quad C = \frac{4}{3}.$$

Substituting into (1) and using the relations

$$\frac{1}{1 - t} = \sum_{n=0}^{\infty} t^n,$$
$$\frac{1}{(1 - t)^2} = \sum_{n=0}^{\infty} (n + 1)t^n,$$

we conclude that

$$F(x) = \frac{31}{9} \sum_{n=0}^{\infty} x^n - \frac{25}{9} \sum_{n=0}^{\infty} 4^n x^n + \frac{4}{3} \sum_{n=0}^{\infty} (n + 1)4^n x^n.$$

Hence, it follows that

$$u_n = \frac{31}{9} - \frac{25}{9} \cdot 4^n + \frac{4}{3} \cdot (n+1) \cdot 4^n = \frac{31}{9} + \left(\frac{12n-13}{9}\right) 4^n.$$

F.3 *Step 0* : Since $4 \cdot 7 \equiv 1 \pmod 9$, the first congruence can be rewritten as

$$x \equiv 3 \cdot 7 \pmod 9 \;\Rightarrow\; x \equiv 3 \pmod 9.$$

Step 1 : We compute the inverse of $14 \cdot 25$ modulo 9. Since $14 \cdot 25 \equiv 5 \cdot 7 = 35 \equiv -1 \pmod 9$, we seek a solution to

$$-1 \cdot a_1 \equiv 1 \pmod 9.$$

So obviously a solution is $a_1 = -1$.

Step 2 : Compute the inverse of $9 \cdot 25$ modulo 14. Since $9 \cdot 25 \equiv -5 \cdot -3 = 15 \equiv 1 \pmod{14}$, we must solve

$$1 \cdot a_2 \equiv 1 \pmod{14}.$$

Obviously a solution is $a_2 = 1$.

Step 3 : Compute the inverse of $9 \cdot 14 = 126$ modulo 25. Since $126 \equiv 1 \pmod{25}$, we must solve

$$1 \cdot a_3 \equiv 1 \pmod{25}.$$

A solution is $a_3 = 1$.

Step 4 : A solution to the three congruences is given by

$$\begin{aligned} x &= 3 \cdot a_1 \cdot (14 \cdot 25) + 2 \cdot a_2 \cdot (9 \cdot 25) + 1 \cdot a_3 \cdot (9 \cdot 14) \\ &= 3 \cdot -1 \cdot 14 \cdot 25 + 2 \cdot 1 \cdot 9 \cdot 25 + 1 \cdot 1 \cdot 9 \cdot 14 \\ &= -474. \end{aligned}$$

Step 5 : The general solution is

$$x = -474 + (9 \cdot 14 \cdot 25) \cdot n = -474 + 3150n,$$

where n is an arbitrary integer. In particular, the smallest positive solution is $x = 2676$, got by taking $n = 1$.

F.4 Let h, d, s, c denote, respectively, the number of hearts, diamonds, spades and clubs chosen. Then we're looking for the number of possible solutions to the equation

$$h + d + s + c = 30 \tag{2}$$

with the restriction that each of h, d, s, c is a non-negative integer no greater than 13. We use the inclusion-exclusion principle. Let X denote the set of all solutions in non-negative integers to (2). Define four subsets H, D, S, C of X as follows :

$$\begin{aligned} H &= \{\text{those solutions for which } h > 13\}, \\ D &= \{\text{those solutions for which } d > 13\}, \\ S &= \{\text{those solutions for which } s > 13\}, \\ C &= \{\text{those solutions for which } c > 13\}. \end{aligned}$$

Then the number of solutions to (2) satisfying our requirements is precisely $|X \backslash (H \cup D \cup S \cup C)|$. By the I-E principle, we have

$$\begin{aligned} |X \backslash (H \cup D \cup S \cup C)| = |X| - |H| - |D| - |S| - |C| \\ +|H \cap D| + |H \cap S| + |H \cap C| + |D \cap S| + |D \cap C| + |S \cap C| \\ -|H \cap D \cap S| - |H \cap D \cap C| - |H \cap S \cap C| - |D \cap S \cap C| \\ +|H \cap D \cap S \cap C|. \end{aligned} \tag{3}$$

Note that, by symmetry,

$$|H| = |D| = |S| = |C| \tag{4}$$

and

$$|H \cap D| = |H \cap S| = |H \cap C| = |D \cap S| = |D \cap C| = |S \cap C|. \tag{5}$$

Also, since $30 < 3 \cdot 14$, each of the terms in (3) involving an intersection of three of more of the sets H, D, S, C is zero. So it will suffice to compute $|X|, |H|$ and $|H \cap D|$. We now recall from the lectures that the number of solutions in non-negative integers to the equation

$$x_1 + \cdots + x_n = k$$

is given by the binomial coefficient $\begin{pmatrix} n+k-1 \\ n-1 \end{pmatrix}$. This is also the number of ways to distribute n identical objects into k different cells. We immediately deduce that

$$|X| = \begin{pmatrix} 30+4-1 \\ 4-1 \end{pmatrix} = \begin{pmatrix} 33 \\ 3 \end{pmatrix}. \tag{6}$$

Regarding $|H|$, well we first have to choose 14 'hearts' and then have complete freedom how to distribute the remaining 16 'cards'. So

$$|H| = \begin{pmatrix} 16+4-1 \\ 4-1 \end{pmatrix} = \begin{pmatrix} 19 \\ 3 \end{pmatrix}. \tag{7}$$

Similarly, regarding $|H \cap D|$, we first choose 14 'hearts' and 14 'diamonds', then have complete freedom for how to distribute the remaining 2 'cards'. Thus,

$$|H \cap D| = \begin{pmatrix} 2+4-1 \\ 4-1 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}. \tag{8}$$

From (3), (4), (5), (6), (7) and (8), we obtain the final answer, namely

$$\begin{pmatrix} 33 \\ 3 \end{pmatrix} - 4 \cdot \begin{pmatrix} 19 \\ 3 \end{pmatrix} + 6 \cdot \begin{pmatrix} 5 \\ 3 \end{pmatrix}.$$

F.5 (i) The complete bipartite graph $K_{\frac{n}{2},\frac{n}{2}}$.

(ii) Apply Dijkstra's algorithm to build up the following tree

Step	Choice of edge	Labelling
1	sA	$A := 5$
2	sB	$B := 5$
3	BE	$E := 6$
4	EF	$F := 9$
5	sC	$C := 9$
6	EH	$H := 12$
7	FJ	$J := 13$
8	CD	$D := 15$
9	FI	$I := 15$
10	Ht/Jt	$t := 19$

Hence the shortest path from s to t has length 19. Depending on the choice you made in Step 10, there are two possibilities for the shortest path, namely

$$s \to B \to E \to H \to t,$$
$$s \to B \to E \to F \to J \to t.$$

F.6 (i) For each fixed i,

$$\sum_{k=i}^{n} \binom{n-i}{k-i} = \sum_{l=0}^{n-i} \binom{n-i}{l} = 2^{n-i}.$$

Then the outer sum over i becomes

$$\sum_{i=0}^{n} \binom{n}{i} \cdot 2^{n-i} = 3^n,$$

by the binomial theorem $(x = 1, y = 2)$. So the answer is 3^n.

(ii) Let $(x, y, z) \neq (0, 0, 0)$ be any solution. Let $d = \mathrm{GCD}(x, y, z)$. Then $(x/d, y/d, z/d)$ is also a solution. It follows that if there exists a non-trivial solution $(x, y, z) \neq (0, 0, 0)$, then such a solution exists satisfying, in addition, that $\mathrm{GCD}(x, y, z) = 1$. But no such solution can exist, for the following reason : 101 is a prime. Hence, by Fermat's Theorem, every integer a not divisible by 101 satisfies

$$a^{100} \equiv 1 \pmod{101}.$$

Thus, for any choice of the integers x, y, the VL must be congruent to $0, 2, 3$ or 5 modulo 101. And the HL must be congruent to either 0 or 4 modulo 101. Hence, if they are to be equal, then in particular they must both be congruent to zero modulo 101, and this is only possible if each of x, y and z is divisible by 101. But this contradicts the assumption that $\mathrm{GCD}(x, y, z) = 1$.