TMA 055 : Diskret matematik

Tentamen 150105

Lösningar

F.1 $66 = 2 \cdot 3 \cdot 11$ so $\phi(66) = \phi(2) \cdot \phi(3) \cdot \phi(11) = (2-1)(3-1)(11-1) = 1 \cdot 2 \cdot 10 = 20$. Hence, Euler's Theorem states that, if n is an integer relatively prime to 66, then

$$n^{20} \equiv 1 \pmod{66}.$$

Note that both 7 and 13 are relatively prime to 66. Hence (all congruences are modulo 66)

$$7^{82} = (7^{20})^4 \cdot 7^2 \equiv 1^4 \cdot 49 \equiv 49,$$

and

$$13^{141} = (13^{20})^7 \cdot 13 \equiv 1^7 \cdot 13 \equiv 13.$$

Thus,

$$(7^{82} + 13^{141})^4 \equiv (49+13)^4 = 62^4 \equiv (-4)^4 = 256 \equiv -8 \equiv 58.$$

So the answer is 58.

F.2 The homogeneous equation is

$$3u_{n+2} - 10u_{n+1} - 8u_n = 0.$$

The characteristic equation for this is

$$3x^2 - 10x - 8 = 0,$$

which factorises as

$$(3x+2)(x-4) = 0,$$

and hence has the two roots $x = 4$, $x = -2/3$. Hence the general solution to the homogeneous equation is

$$u_n^h = C_1 \cdot 4^n + C_2 \cdot \left(-\frac{2}{3}\right)^n.$$

Since 4^n is already a solution to the homogeneous equation, our guess for a particular solution should have the form

$$u_n^p = A \cdot n \cdot 4^n + Bn + C.$$

Substituting into the recurrence relation, the requirement on A is that

$$A \cdot \left[3(n+2)4^{n+2} - 10(n+1)4^{n+1} - 8n4^n\right] = 4^n, \tag{1}$$

whereas the requirement on B and C is that

$$3[B(n+2)+C] - 10[B(n+1)+C] - 8[Bn+C] = n. \tag{2}$$

From (1) we deduce that $A = 1/56$. From (2) we deduce that $B = -1/15$, $C = 4/225$. Hence the general solution to our recurrence relation is

$$u_n = C_1 \cdot 4^n + C_2 \cdot \left(-\frac{2}{3}\right)^n - \frac{n}{56} \cdot 4^n - \frac{n}{15} + \frac{4}{225}.$$

It remains to insert the initial conditions :

$$n = 0 \Rightarrow u_0 = 1 = C_1 + C_2 + \frac{4}{225},$$
$$n = 1 \Rightarrow u_1 = 1 = 4C_1 - \frac{2}{3}C_2 + \frac{1}{14} - \frac{1}{15} + \frac{4}{225}.$$

Solving, we obtain $C_1 = 617/1764$, $C_2 = 3099/4900$. Hence the final answer is

$$u_n = \frac{617}{1764} \cdot 4^n + \frac{3099}{4900} \cdot \left(-\frac{2}{3}\right)^n + \frac{n}{14} \cdot 4^{n-1} - \frac{n}{15} + \frac{4}{225}.$$

F.3 Det är lätt att se att SGD$(17, 31) = 1$ eftersom båda två är primtal. Därför vet vi att det finns heltal x_0, y_0 så att

$$17x_0 + 31y_0 = 1. \tag{3}$$

Vi hittar först en lösning till (3) genom att köra Euklides algoritm fram och tillbaka. Framåt får vi

$$\begin{aligned} 31 &= 1 \cdot 17 + 14, \\ 17 &= 1 \cdot 14 + 3, \\ 14 &= 4 \cdot 3 + 2, \\ 3 &= 1 \cdot 2 + 1, \\ 2 &= 2 \cdot 1 + 0. \end{aligned}$$

Bakåt får vi då

$$\begin{aligned}
1 &= 3 - 2 \\
&= 3 - (14 - 4 \cdot 3) \\
&= 5 \cdot 3 - 14 \\
&= 5 \cdot (17 - 14) - 14 \\
&= 5 \cdot 17 - 6 \cdot 14 \\
&= 5 \cdot 17 - 6 \cdot (31 - 17) \\
&= 11 \cdot 17 - 6 \cdot 31.
\end{aligned}$$

Därmed har vi hittat lösningen $x_0 = 11$, $y_0 = -6$. Genom att multiplicera dessa med 2000 så får vi en lösning (x_1, y_1) till

$$17x + 31y = 2000, \tag{4}$$

nämligen $x_1 = 22000$, $y_1 = -12000$. Den allmäna lösningen till (4) ges då av

$$x = 22000 - 31n, \tag{5}$$

$$y = -12000 + 17n \tag{6}$$

där n är ett godtyckligt heltal. Vi är nu intresserade av lösningar för vilka både $x > 0$ och $y > 0$.

Å ena sidan

$$x > 0 \Leftrightarrow 22000 - 31n > 0 \Leftrightarrow 31n < 22000 \Leftrightarrow n \leq 709. \tag{7}$$

Å andra sidan

$$y > 0 \Leftrightarrow -12000 + 17n > 0 \Leftrightarrow 17n > 12000 \Leftrightarrow n \geq 706. \tag{8}$$

Från (7) och (8) får vi fyra möjligheter för n, nämligen $n = 706, 707, 708, 709$. Till sist sätter vi in dessa åtta värden i (5) och (6) så får vi åtta lösningar :

$$x = 114,\ y = 2 \quad x = 83,\ y = 19 \quad x = 52,\ y = 36, \quad x = 21,\ y = 73.$$

F.4 (i) The graphs are not isomorphic. For example, in G_2, the vertices t, w, z form a triangle of vertices, each of degree 4. There is no triangle in G_1 in which each of the vertices has degree 4.

(ii) Starting with the null flow $f \equiv 0$, one can find the following sequence of f-augmenting paths from a to z (these are not the only choices, of course) :

$$\begin{aligned} a-b-d-z, &\quad \epsilon = 8, \\ a-g-i-z, &\quad \epsilon = 6, \\ a-h-i-z, &\quad \epsilon = 6, \\ a-g-d-z, &\quad \epsilon = 6, \\ a-g-b-d-z, &\quad \epsilon = 3, \\ a-h-g-b-d-z, &\quad \epsilon = 3. \end{aligned}$$

This yields the following maximal flow with $|f| = 32$

Edge	Flow	Edge	Flow	Edge	Flow
(a,b)	8	(g,b)	6	(h,i)	6
(a,g)	15	(h,g)	3	(i,d)	0
(a,h)	9	(g,d)	6	(d,z)	20
(b,d)	14	(g,i)	6	(i,z)	12

The corresponding cut of minimal capacity is

$$S = \{a,b,d,g,h\}, \qquad T = \{i,z\}.$$

We verify that

$$c(S,T) = c(g,i) + c(h,i) + c(d,z) = 6 + 6 + 20 = 32 = |f|, \text{ v.s.v.}$$

F.5 (i) If the first digit is odd, then we have

- 5 choices for the first digit
- 5 choices for the position and the choice of the other odd digit
- 5 choices for each of the other four digits (which is even).

Thus, by MP, we have in total 5^7 possible numbers in this case.

If, on the other hand, the first digit is even, then we have

- 4 choices for the first digit
- $C(5,2) = 10$ choices for the positions of the odd digits
- 5 choices for each digit other than the first one.

Thus we have in total $5^5 \cdot 4 \cdot 10 = 5^6 \cdot 8$ possible numbers in this case.

The total number of possibilities is thus

$$5^7 + 5^6 \cdot 8 = 5^6 \cdot 13 = 203125.$$

(ii) First note that the VL is congruent to $x^2 + y^2$ modulo 4, since $-3 \equiv 1 \pmod 4$.

Now if n is an even number, then $n^2 \equiv 0 \pmod 4$, whereas if n is odd, then $n^2 \equiv 1 \pmod 4$. Hence the sum of two squares is always congruent to $0 + 0 = 0$ or $0 + 1 = 1$ or $1 + 1 = 2$ modulo 4. But $11 \equiv 3 \pmod 4$, which proves that the equation can have no solution, since it can't even have a solution modulo 4.

(iii) Denote by q_n the number of words of length n. Clearly $q_1 = 4$. Now let $n > 1$. If the first letter in the word is an a, then d cannot appear among the remaining $n - 1$ symbols, but otherwise we can choose them freely. Hence there are 3^{n-1} possible words of this type. On the other hand, if the first symbol is either b, c or d, then the same conditions as at the outset apply for the remaining symbols. By MP, there are thus $3q_{n-1}$ words of this type. We conclude that the desired recurrence relation is

$$q_1 = 4, \qquad q_n = 3q_{n-1} + 3^{n-1} \ \ \forall\, n > 1.$$