**TMA 055 : Diskret matematik**

**Tentamen 201004**

**Lösningar**

**F.1** Det är lätt att se att SGD$(13, 21) = 1$ genom att snabbt faktorisera bägge talen : 13 är ett primtal och $21 = 3 \cdot 7$. Därför vet vi att det finns heltal $x_0, y_0$ så att

$$13x_0 + 21y_0 = 1. \tag{1}$$

Vi hittar först en lösning till (1) genom att köra Euklides algoritm fram och tillbaka. Framåt får vi

$$\begin{aligned} 21 &= 1 \cdot 13 + 8, \\ 13 &= 1 \cdot 8 + 5, \\ 8 &= 1 \cdot 5 + 3, \\ 5 &= 1 \cdot 3 + 2, \\ 3 &= 1 \cdot 2 + 1, \\ 2 &= 2 \cdot 1 + 0. \end{aligned}$$

Bakåt får vi då

$$\begin{aligned} 1 &= 3 - 2 \\ &= 3 - (5 - 3) \\ &= 2 \cdot 3 - 5 \\ &= 2 \cdot (8 - 5) - 5 \\ &= 2 \cdot 8 - 3 \cdot 5 \\ &= 2 \cdot 8 - 3 \cdot (13 - 8) \\ &= 5 \cdot 8 - 3 \cdot 13 \\ &= 5 \cdot (21 - 13) - 3 \cdot 13 \\ &= 5 \cdot 21 - 8 \cdot 13. \end{aligned}$$

Därmed har vi hittat lösningen $x_0 = -8$, $y_0 = 5$. Genom att multiplicera dessa med 2000 så får vi en lösning $(x_1, y_1)$ till

$$13x + 21y = 2000, \tag{2}$$

nämligen $x_1 = -16000$, $y_1 = 10000$. Den allmäna lösningen till (2) ges då av

$$x = -16000 + 21n, \tag{3}$$
$$y = 10000 - 13n \tag{4}$$

där $n$ är ett godtyckligt heltal. Vi är nu intresserade av lösningar för vilka både $x > 0$ och $y > 0$.

Å ena sidan

$$x > 0 \Leftrightarrow -16000 + 21n > 0 \Leftrightarrow 21n > 16000 \Leftrightarrow n \geq 762. \tag{5}$$

Å andra sidan

$$y > 0 \Leftrightarrow 10000 - 13n > 0 \Leftrightarrow 13n < 10000 \Leftrightarrow n \leq 769. \tag{6}$$

Från (5) och (6) får vi åtta möjligheter för $n$, nämligen $n = 762, 763, ..., 769$. Till sist sätter vi in dessa åtta värden i (3) och (4) så får vi åtta lösningar :

$$x = 2,\ y = 94 \quad x = 23,\ y = 81 \quad x = 44,\ y = 68, \quad x = 65,\ y = 55$$
$$x = 86,\ y = 42 \quad x = 107,\ y = 29 \quad x = 128,\ y = 16 \quad x = 149,\ y = 3.$$

**F.2** The homogeneous equation is

$$2u_{n+2} - 11u_{n+1} + 15u_n = 0.$$

The characteristic equation for this is

$$2x^2 - 11x + 15 = 0,$$

which factorises as

$$(2x - 5)(x - 3) = 0,$$

and hence has the two roots $x = 5/2$, $x = 3$. Hence the general solution to the homogeneous equation is

$$u_n^h = C_1 \cdot \left(\frac{5}{2}\right)^n + C_2 \cdot 3^n.$$

Since $3^n$ is already a solution to the homogeneous equation, our guess for a particular solution should have the form

$$u_n^p = A \cdot n \cdot 3^n + Bn + C.$$

Substituting into the recurrence relation, the requirement on $A$ is that

$$A \cdot \Big[2(n+2)3^{n+2} - 11(n+1)3^{n+1} + 15n3^n\Big] = 3^n, \tag{7}$$

whereas the requirement on $B$ and $C$ is that

$$2[B(n+2)+C] - 11[B(n+1)+C] + 15[Bn+C] = n. \tag{8}$$

From (7) we deduce that $A = 1/3$. From (8) we deduce that $B = 1/6$, $C = 7/36$. Hence the general solution to our recurrence relation is

$$u_n = C_1 \cdot \left(\frac{5}{2}\right)^n + C_2 \cdot 3^n + n \cdot 3^{n-1} + \frac{n}{6} + \frac{7}{36}.$$

It remains to insert the initial conditions :

$$n = 0 \Rightarrow u_0 = 1 = C_1 + C_2 + \frac{7}{36},$$
$$n = 1 \Rightarrow u_1 = 2 = \frac{5}{2}C_1 + 3C_2 + 1 + \frac{1}{6} + \frac{7}{36}.$$

Solving, we obtain $C_1 = 32/9$, $C_2 = -99/36$. Hence the final answer is

$$u_n = \frac{32}{9} \cdot \left(\frac{5}{2}\right)^n - \frac{99}{36} \cdot 3^n + n \cdot 3^{n-1} + \frac{n}{6} + \frac{7}{36}.$$

**F.3** *Step 0* : Since $2 \cdot 5 \equiv 1 \pmod 9$, the first congruence can be rewritten as

$$x \equiv 5 \pmod 9.$$

Since $3 \cdot (-4) \equiv 1 \pmod{13}$, the second congruence can be rewritten as

$$x \equiv -8 \equiv 5 \pmod{13}.$$

Since $4 \cdot 5 \equiv 1 \pmod{19}$, the third congruence can be rewritten as

$$x \equiv 15 \equiv -4 \pmod{19}.$$

*Step 1* : We compute the inverse of $13 \cdot 19$ modulo 9. Since $13 \cdot 19 \equiv 4 \cdot 1 \equiv 4 \pmod 9$, we seek a solution to

$$4a_1 \equiv 1 \pmod 9.$$

It's easy to spot that a solution is $a_1 = 7$.

*Step 2* : Compute the inverse of $9 \cdot 19$ modulo 13. Since $9 \cdot 19 \equiv 9 \cdot 6 = 54 \equiv 2 \pmod{13}$, we must solve

$$2a_2 \equiv 1 \pmod{13}.$$

A solution is $a_2 = 7$.

*Step 3* : Compute the inverse of $9 \cdot 13 = 117$ modulo 19. Since $117 \equiv 3 \pmod{19}$, we must solve

$$3a_3 \equiv 1 \pmod{19}.$$

A solution is $a_3 = 13$.

*Step 4* : A solution to the three congruences is given by

$$\begin{aligned} x = 5 \cdot a_1 \cdot (13 \cdot 19) + 5 \cdot a_2 \cdot (9 \cdot 19) - 4 \cdot a_3 \cdot (9 \cdot 13) \\ = 5 \cdot 7 \cdot 13 \cdot 19 + 5 \cdot 7 \cdot 9 \cdot 19 - 4 \cdot 13 \cdot 9 \cdot 13 \\ = 8546. \end{aligned}$$

*Step 5* : The general solution is

$$x = 8546 + (9 \cdot 13 \cdot 19) \cdot n = 8546 + 2223n,$$

where $n$ is an arbitrary integer. In particular, the smallest positive solution is $x = 1877$, got by taking $n = -3$.

**F.4** To avoid confusing notation I relabel the vertices as $a, b, c, d, e, f, g, h$, where $a \equiv 1$, $b \equiv 2$ etc.

**(i)** Clearly, $\chi(G_0) \geq 3$ since $G_0$ contains many triangles. On the other hand, if we apply the greedy algorithm to the nodes ordered alphabetically, then we get a 3-coloring, namely (the colors are $1, 2, 3$)

| | | | |
|---|---|---|---|
| a | 1 | e | 1 |
| b | 2 | f | 3 |
| c | 2 | g | 1 |
| d | 3 | h | 2 |

Hence $\chi(G_0) = 3$.

**(ii)** Starting with the null flow $f \equiv 0$, one can find the following sequence of $f$-augmenting paths from $a$ to $h$ (these are not the only choices, of course) :

$$\begin{aligned} a - b - f - h, &\quad \epsilon = 5, \\ a - d - g - h, &\quad \epsilon = 3, \\ a - c - e - h, &\quad \epsilon = 4, \\ a - b - e - h, &\quad \epsilon = 2, \\ a - d - g - f - b - e - h, &\quad \epsilon = 2. \end{aligned}$$

This yields the following maximal flow with $|f| = 16$

| Edge | Flow | Edge | Flow | Edge | Flow |
|---|---|---|---|---|---|
| $(a,b)$ | 7 | $(c,e)$ | 4 | $(e,h)$ | 8 |
| $(a,c)$ | 4 | $(d,c)$ | 0 | $(f,h)$ | 5 |
| $(a,d)$ | 5 | $(d,g)$ | 5 | $(g,f)$ | 2 |
| $(b,e)$ | 4 | $(e,f)$ | 0 | $(g,h)$ | 3 |
| $(b,f)$ | 3 | | | | |

The corresponding cut of minimal capacity is

$$S = \{a,b,c,d,f,g\}, \qquad T = \{e,h\}.$$

We verify that

$c(S,T) = c(b,e) + c(c,e) + c(f,h) + c(g,h) = 4 + 4 + 5 + 3 = 16 = |f|$, v.s.v.

**F.5 (i)**

$$\binom{16}{3} \cdot \binom{36}{2} + \binom{16}{4} \cdot \binom{36}{1} + \binom{16}{5}.$$

**(ii)** Let $X$ denote the set of all functions from $\{1, ..., 100\}$ to $\{1,2,3\}$, and define three subsets of $X$ as follows :

$$\begin{aligned} A &:= \{f \in X : \mathrm{Im}(f) \subseteq \{1,2\}\}, \\ B &:= \{f \in X : \mathrm{Im}(f) \subseteq \{1,3\}\}, \\ C &:= \{f \in X : \mathrm{Im}(f) \subseteq \{2,3\}\}. \end{aligned}$$

Then the quantity we must compute is

$$|X| - |A \cup B \cup C|. \tag{9}$$

By the sieve principle, we know that

$$\begin{aligned} |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| \\ -|A \cap C| - |B \cap C| + |A \cap B \cap C|. \end{aligned} \tag{10}$$

Everything can be easily computed using MP. We have

$$\begin{aligned} |X| = 3^{100}, \\ |A| = |B| = |C| = 2^{100}, \\ |A \cap B| = |A \cap C| = |B \cap C| = 1^{100} = 1, \\ |A \cap B \cap C| = 0. \end{aligned}$$

Substituting everything into (9) and (10), we find that the answer to the question is

$$3^{100} - 3 \cdot 2^{100} + 3.$$

**(iii)** By Fermat's theorem, for any integer $a$ we have that either $a^6 \equiv 1 \pmod 7$ or $7|a$. Now it is not possible to express 3 as a sum of two numbers from the set $\{0, 1\}$. Hence if $(x, y, z)$ is any solution to our equation, we may deduce that each of $x, y, z$ must be divisible by 7. In other words, there exists no solution for which $\mathrm{GCD}(x, y, z) = 1$. But if $(x, y, z)$ is any non-zero solution and $d = \mathrm{GCD}(x, y, z)$, then $(x', y', z') := (x/d, y/d, z/d)$ is a non-zero solution for which $\mathrm{GCD}(x', y', z') = 1$.

Hence we may conclude that no non-zero solutions exist.