Att lämna in senast onsdag den 20 september, kl 1700

REGLER : Q.1 is worth 30 points, each other question 20 points. The Bonus Problems are worth 15 points each. Thus a total of 180 points are available. A score of 140 points will correspond to $11\frac{1}{3}$ points for the course (see points system as described on homepage).

1. The Swedish Research Council has 15 million crowns of grant money available to distribute among 15 applicants, of which 5 are from each of Chalmers, KTH and Lund. Each award will be a multiple of 100,000 crowns. In how many ways can the money be distributed if

(i) there are no further restrictions ?

(ii) equal amounts of money must go to each of the three universities ?

(iii) exactly 7 projects will be funded, and each award must be of a minimum of 500,000 crowns ?

2. Investigate the behaviour of the function $C(n, k)$ as a function of k for a fixed n, and prove a general result which allows one to answer rigorously, for example, the following question :

"If a fair coin is tossed six million times, what is the most likely number of heads to result ?"

Bonus : Same problem, except this time you want to be able to answer the same type of question for a fair dice instead of a fair coin, and with "heads" replaced by "a one".

3. Explain combinatorially the following identities :

(i) If n, m, k are three positive integers, then

$$\binom{n+m}{k} = \sum_{i=0}^{k} \binom{n}{i} \binom{m}{k-i}.$$

(ii) If $N \geq n \geq r$ are positive integers, then

$$\binom{N}{n} = \sum_{k=1}^{N} \binom{k-1}{r-1} \binom{N-k}{n-r}.$$

4 (see 5.5.12 in Grimaldi) Let A be a subset of $\{1, 2, ..., 29\}$ consisting of 9 elements. Show that, among the 5-element subsets of A, there must exist two whose elements have the same sum.

(HINT : Compare the number of 5-element subsets with the number of available sums and use the Pigeonhole Principle).

5. Find with proof a recurrence relation for the number s_n of n-bit ternary words which do not contain the pattern 11. Hence, by solving this recurrence, or otherwise, find an explicit formula for s_n.

Bonus : Find with proof a recurrence relation for the number t_n of n-bit binary words which do not contain the pattern 010. You do not need to solve the recurrence !

6. How many 15-decimal-digit positive integers contain each of the digits 2,3,5 at least once.

(REMARK : Remember that the leading digit can't be a 0.)

7. If A is a set of integers, we denote by $A + A$ the set of all integers which can be expressed in the form $a_1 + a_2$, where a_1, a_2 are (not necessarily distinct) elements of A. For example, if $A = \{2, 3, 4, 7\}$, then all possible sums are :

$2 + 2 = 4, 2 + 3 = 5, 2 + 4 = 6, 2 + 7 = 9, 3 + 3 = 6, 3 + 4 = 7, 3 + 7 = 10, 4 + 4 = 8, 4 + 7 = 11, 7 + 7 = 14.$

Hence in this case $A + A = \{4, 5, 6, 7, 8, 9, 10, 11, 14\}$.

(i) If the set A contains n elements, prove that the set $A + A$ contains at least $2n - 1$ elements.
(ii) Give with proof a complete characterisation of those finite sets A of integers for which $|A + A| = 2 \cdot |A| - 1$.

(NOTE : Partial credit will be awarded if you guess the right answer but cannot prove it.)