TMA 055 : Diskret matematik

Tentamen 200107

Lösningar

F.1 The homogeneous equation is

$$u_n - 6u_{n-1} + 5u_{n-2} = 0.$$

The characteristic equation for this is

$$x^2 - 6x + 5 = 0,$$

which factorises as

$$(x-1)(x-5) = 0,$$

and hence has the roots $x = 1$ and $x = 5$. Hence the general solution to the homogeneous equation is

$$u_n^h = C_1 + C_2 \cdot 5^n.$$

Since 3 is already a solution to the homogeneous equation, our guess for a particular solution should have the form

$$u_n^p = A \cdot 2^n + Bn.$$

Substituting into the recurrence relation, the requirement on A is that

$$A \cdot [2^n - 6 \cdot 2^{n-1} + 5 \cdot 2^{n-2}] = 2^n,$$

whereas the requirement on B is that

$$B[n - 6(n-1) + 5(n-2)] = 3.$$

Solving, we obtain $A = -4/3$ and $B = -3/4$. Hence the general solution to our recurrence relation is

$$u_n = C_1 - \frac{3n}{4} + C_2 \cdot 5^n - \frac{4}{3} \cdot 2^n.$$

It remains to insert the initial conditions :

$$n = 0 \Rightarrow u_0 = 1 = C_1 + C_2 - \frac{4}{3},$$
$$n = 1 \Rightarrow u_1 = 1 = C_1 - \frac{3}{4} + 5C_2 - \frac{8}{3}.$$

Solving, we obtain $C_1 = 87/48$, $C_2 = 25/48$. Hence the final answer is

$$u_n = \frac{87}{48} - \frac{3n}{4} + \frac{25}{48} \cdot 5^n - \frac{4}{3} \cdot 2^n.$$

F.2 $104 = 2^3 \cdot 13$ so $\phi(104) = 104(1 - \frac{1}{2})(1 - \frac{1}{13}) = 48$. Hence, Euler's Theorem states that, if n is an integer relatively prime to 104, then

$$n^{48} \equiv 1 \pmod{104}.$$

Note that both 3 and 11 are relatively prime to 104. Hence (all congruences are modulo 104)

$$3^{97} = (3^{48})^2 \cdot 3^1 \equiv 1^2 \cdot 3 \equiv 3,$$

and

$$11^{146} = (11^{48})^3 \cdot 11^2 \equiv 1^3 \cdot 121 \equiv 121 \equiv 17.$$

Thus,

$$(3^{97} + 11^{146} + 1)^{46} \equiv (3 + 17 + 1)^{46} = 21^{46}.$$

But 21 is also relatively prime to 104, so we can apply Euler's theorem again and deduce that

$$21^{46} = 21^{48} \cdot 21^{-2} \equiv 1 \cdot 21^{-2} \equiv 21^{-2}.$$

Now $21 \cdot 5 = 105 \equiv 1 \pmod{104}$, so $21^{-1} \equiv 5 \pmod{104}$. Thus we deduce that $21^{-2} \equiv 5^2 \equiv 25 \pmod{104}$.

So the answer is 25.

F.3 (i) There are no Euler cycles as most of the nodes have odd degree. There are Hamilton cycles, for example

$$s \to G \to H \to t \to E \to F \to D \to C \to B \to A \to s.$$

(ii) Starting with the null flow $f \equiv 0$, one can find the following sequence of f-augmenting paths from s to t (these are not the only choices, of course) :

$$\begin{aligned} s - G - F - H - t, &\quad \epsilon = 6, \\ s - D - C - E - t, &\quad \epsilon = 6, \\ s - A - B - t, &\quad \epsilon = 4, \\ s - A - C - E - t, &\quad \epsilon = 2, \\ s - A - D - F - H - t, &\quad \epsilon = 2. \end{aligned}$$

This yields the following maximal flow with $|f| = 20$

Edge	Flow	Edge	Flow	Edge	Flow	Edge	Flow
(s, G)	6	(A, B)	4	(G, H)	0	(C, E)	8
(s, D)	6	(D, C)	6	(F, H)	8	(B, C)	0
(s, A)	8	(D, F)	2	(H, E)	0	(H, t)	8
(A, D)	2	(G, F)	6	(F, E)	0	(E, t)	8
(A, C)	2	(F, G)	0	(E, C)	0	(B, t)	4

The corresponding cut of minimal capacity is

$$S = \{s, A, C, D, E, G, H\}, \qquad T = \{B, t\}.$$

We verify that

$$c(S, T) = c(H, t) + c(E, t) + c(A, B) = 8 + 8 + 4 = 20 = |f|, \text{ v.s.v.}$$

F.4 (i) The easiest way to solve this is to use a symmetry argument. Clearly, there will be the same number of hands containing more red than black cards as vice versa. As 9 is an odd number, this means that the answer is exactly one half of the total number of possible hands, i.e.: $\frac{1}{2}C(52, 9)$.

(ii) We shall use the inclusion-exclusion principle to sieve out the disallowed hands. Let X denote the set of all possible 9-card hands. Thus

$$|X| = C(52, 9). \tag{1}$$

Let A, B resp. C denote the subsets of X consisting of those hands containing no 2:s, 3:s resp. 5:s. Thus the number we are interested in is $|X \backslash (A \cup B \cup C)|$. By the I-E principle (eq.(15) in the lecture notes), this number is given by

$$|X| - |A| - |B| - |C| + |A \cap B| + |A \cap C| + |B \cap C| - |A \cap B \cap C|.$$

By obvious symmetry, this simplifies to

$$|X| - 3|A| + 3|A \cap B| - |A \cap B \cap C|. \tag{2}$$

Consider $|A|$. Since there are no 2:s in the hand, we have only 48 cards to choose from. Thus $|A| = C(48, 9)$. Similarly, $|A \cap B| = C(44, 9)$ and $|A \cap B \cap C| = C(40, 9)$. Substituting everything into (2), the final answer is

$$C(52, 9) - 3 \cdot C(48, 9) + 3 \cdot C(44, 9) - C(40, 9).$$

(iii) As cards in the same colour are identical, all that matters is the number of cards in each colour. Thus this is just a reformulation of the problem of placing 9 indistinguishable balls in 4 distinguishable boxes. So the answer is $C(9 + 4 - 1, 4 - 1) = C(12, 3) = 220$.

F.5 (i) Definition 4, page 19 in the lecture notes.
(ii) Eq.(21) on page 19 in the lecture notes. You are referred to your class notes for a proof.

F.6 (i) Let G have n nodes and n edges. Since G is connected, it has a spanning tree T. Then T has $n - 1$ edges. Now $\chi(T) = 2$ for any tree (it has no cycles at all, in particular no odd cycles, hence is bipartite : Proposition 22, page 43 in the lecture notes). So color T with 2 colors. Now in G there is just one more edge. The worst that can happen is that this last edge joins two vertices which have been given the same color. In that case, give either of these vertices a third color.

This describes a means of coloring G with at most 3 colors, hence $\chi(G) \leq 3$, v.s.v.

(ii) If x is an odd integer then $x \equiv \pm 1 \pmod 4$, so $x^2 \equiv 1 \pmod 4$. Thus a sum of two odd integer squares will always be congruent to 2 (mod 4). But no integer square is congruent to 2 (mod 4) : we've already shown that an odd square is 1 (mod 4), and an even square is evenly divisible by 4. So we're done !