TMA 055 : Diskret matematik

Tentamen 210807

Lösningar

F.1 The homogeneous equation is

$$u_n - u_{n-1} - 12u_{n-2} = 0.$$

The characteristic equation for this is

$$x^2 - x - 12 = 0,$$

which factorises as

$$(x+3)(x-4) = 0,$$

and hence has the roots $x = 4$ and $x = -3$. Hence the general solution to the homogeneous equation is

$$u_n^h = C_1 \cdot 4^n + C_2 \cdot (-3)^n.$$

Since 4^n is already a solution to the homogeneous equation, our guess for a particular solution should have the form

$$u_n^p = A \cdot n \cdot 4^n + Bn + C.$$

Substituting into the recurrence relation, the requirement on A is that

$$A \cdot [n4^n - (n-1)4^{n-1} - 12(n-2)4^{n-2}] = 4^n,$$

whereas the requirement on B and C is that

$$[Bn + C] - [B(n-1) + C] - 12[B(n-2) + C] = 3n + 1.$$

Solving, we obtain $A = 4/7$, $B = -1/4$ and $C = -29/48$. Hence the general solution to our recurrence relation is

$$u_n = \left(C_1 + \frac{4n}{7}\right) \cdot 4^n + C_2 \cdot (-3)^n - \left(\frac{n}{4} + \frac{29}{48}\right).$$

It remains to insert the initial conditions :

$$n = 0 \Rightarrow u_0 = 1 = C_1 + C_2 - \frac{29}{48},$$
$$n = 1 \Rightarrow u_1 = 1 = 4\left(C_1 + \frac{4}{7}\right) - 3C_2 - \frac{41}{48}.$$

Solving, we obtain $C_1 = 92/147$, $C_2 = 767/784$. Hence the final answer is

$$u_n = \left(\frac{4n}{7} + \frac{92}{147}\right) \cdot 4^n + \frac{767}{784} \cdot (-3)^n - \left(\frac{n}{4} + \frac{29}{48}\right).$$

F.2 $112 = 2^4 \cdot 7$ so $\phi(112) = 112(1-\frac{1}{2})(1-\frac{1}{7}) = 48$. Hence, Euler's Theorem states that, if n is an integer relatively prime to 112, then

$$n^{48} \equiv 1 \pmod{112}.$$

Note that both 11 and 5 are relatively prime to 112. Hence (all congruences are modulo 112)

$$11^{97} = (11^{48})^2 \cdot 11^1 \equiv 1^2 \cdot 11 \equiv 11,$$

and

$$5^{143} = (5^{48})^3 \cdot 5^{-1} \equiv 1^3 \cdot 5^{-1} \equiv 5^{-1}.$$

So here we need to compute the inverse of 5 modulo 112. One can use Euclid's algorithm, or perhaps just observe directly that

$$5 \cdot 45 = 225 = 2 \cdot 112 + 1,$$

which implies that $5^{-1} \equiv 45$. Thus,

$$(11^{97} + 5^{143} + 53)^{50} \equiv (11 + 45 + 53)^{50} = 109^{50} \equiv (-3)^{50} = 3^{50}.$$

But 3 is also relatively prime to 112, so we can apply Euler's theorem again and deduce that

$$3^{50} = 3^{48} \cdot 3^2 \equiv 1 \cdot 3^2 \equiv 9.$$

So the answer is 9.

F.3 (i) Clearly, $\chi(G_0) \geq 3$ since G_0 contains many triangles. In fact, $\chi(G_0) = 4$. If we apply the greedy algorithm with the nodes ordered

$s, a, b, c, d, e, f, g, h, i, j, k, t$, then we get a 4-coloring, namely (the colors are $1, 2, 3, 4$)

s	1	g	2
a	2	h	3
b	3	i	1
c	2	j	1
d	1	k	2
e	2	t	4
f	3		

To see that one cannot do better than this, let's try to color the graph with three colors, starting at vertex t and working backwards. Since $\{t, k, h\}$ form a triangle, without loss of generality they must get colors 1,2,3 respectively. Next, by considering the triangle $\{t, g, j\}$ we see that g must get color 2 and j must get color 3. Also i must get color 1. Next, d must get color 1 and f get color 2. But then e requires a 4th color.

(ii) Starting with the null flow $f \equiv 0$, one can find the following sequence of f-augmenting paths from s to t (these are not the only choices, of course) :

$$\begin{aligned}
s - a - f - j - t, &\quad \epsilon = 4,\\
s - c - i - k - t, &\quad \epsilon = 6,\\
s - b - d - g - t, &\quad \epsilon = 6,\\
s - a - f - e - h - t, &\quad \epsilon = 1,\\
s - b - e - h - t, &\quad \epsilon = 4.
\end{aligned}$$

This yields the following maximal flow with $|f| = 21$

Edge	Flow	Edge	Flow	Edge	Flow	Edge	Flow
(s, a)	5	(d, f)	0	(d, h)	1	(h, k)	0
(s, b)	10	(b, e)	4	(e, h)	5	(i, k)	6
(s, c)	6	(e, d)	0	(h, g)	0	(j, t)	4
(b, a)	0	(c, i)	6	(h, i)	0	(g, t)	6
(c, b)	0	(f, e)	1	(f, j)	2	(h, t)	5
(a, f)	5	(d, g)	6	(g, j)	0	(k, t)	6
(b, d)	6						

The corresponding cut of minimal capacity is

$$S = \{s, a, b, c\}, \qquad T = \{d, e, f, g, h, i, j, k, t\}.$$

We verify that

$c(S,T) = c(a,f) + c(b,d) + c(b,e) + c(c,i) = 5 + 6 + 4 + 6 = 21 = |f|$, v.s.v.

F.4 (a) (i) $P(29,25) = \frac{29!}{4!}$
(ii) $P(29,3) \cdot P(29,6) \cdot P(29,7) \cdot P(29,9) = \frac{(29!)^4}{26!23!22!19!}$.

(b) Letting X_i denote the number of throws of the dice resulting in an i, we are simply looking for the number of solutions in non-negative integers to the equation

$$X_1 + X_2 + X_3 + X_4 + X_5 + X_6 = 10. \tag{1}$$

This number is just $\binom{10+6-1}{6-1} = \binom{15}{5}$.

F.5 See Theorem 14 in the lecture notes.

F.6 (i) Since the graph is complete, each of the $n!$ possible orderings of its vertices corresponds to a Hamilton cycle. But there are $2n$ different orderings corresponding to a single cycle, since we have n choices for the starting point and two choices for the 'orientation', i.e.: backwards or forwards. Thus the number of Hamilton cycles in K_n is $n!/2n = (n-1)!/2$.

(ii) Each region, including the exterior, is bounded by at least three edges. Thus if we count up the edges bounding each region, one region at a time, then the total number of edges we count must be at least $3R$. On the other hand, every edge in the graph is counted at most twice in this counting procedure, since each edge bounds either one or two regions (the former in the case when the edge 'sticks out' into the exterior). Thus the total number of edges counted is at most $2E$. The result follows.

(iii) The number on the right is congruent to 3 (mod 4). Note that this is easy to see, since any decimal number leaves the same remainder upon division by 4 as the number formed by its last two digits : in this case 59.

Now the point is that the sum of two integer squares can never be congruent to 3 (mod 4), since every integer square is congruent to either 0 or 1 (mod 4), so that the sum of two of them can only be 0,1 or 2 (mod 4).