**TMA 055 : Diskret matematik**

**Tentamen 231006**

**Lösningar**

**F.1** The homogeneous equation is

$$u_n - 8u_{n-1} + 15u_{n-2} = 0.$$

The characteristic equation for this is

$$x^2 - 8x + 15 = 0,$$

which factorises as

$$(x-3)(x-5) = 0,$$

and hence has the roots $x = 3$ and $x = 5$. Hence the general solution to the homogeneous equation is

$$u_n^h = C_1 \cdot 3^n + C_2 \cdot 5^n.$$

Since $3^n$ is already a solution to the homogeneous equation, our guess for a particular solution should have the form

$$u_n^p = A \cdot n \cdot 3^n + B.$$

Substituting into the recurrence relation, the requirement on $A$ is that

$$A \cdot [n3^n - 8(n-1)3^{n-1} + 15(n-2)3^{n-2}] = 3^n,$$

whereas the requirement on $B$ is that

$$B - 8B + 15B = 1.$$

Solving, we obtain $A = -3/2$ and $B = 1/8$. Hence the general solution to our recurrence relation is

$$u_n = \left(C_1 - \frac{3n}{2}\right) \cdot 3^n + C_2 \cdot 5^n + \frac{1}{8}.$$

It remains to insert the initial conditions :

$$n = 0 \Rightarrow u_0 = 1 = C_1 + C_2 + \frac{1}{8},$$
$$n = 1 \Rightarrow u_1 = 1 = 3\left(C_1 - \frac{3}{2}\right) + 5C_2 + \frac{1}{8}.$$

Solving, we obtain $C_1 = -1/2$, $C_2 = 11/8$. Hence the final answer is

$$u_n = \frac{11}{8} \cdot 5^n - \frac{1}{2}(1 + 3n) \cdot 3^n + \frac{1}{8}.$$

**F.2** $90 = 2 \cdot 3^2 \cdot 5$ so $\phi(90) = 90(1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{5}) = 24$. Hence, Euler's Theorem states that, if $n$ is an integer relatively prime to 90, then

$$n^{24} \equiv 1 \pmod{90}.$$

Note that both 11 and 19 are relatively prime to 90. Hence (all congruences are modulo 90)

$$11^{73} = (11^{24})^3 \cdot 11^1 \equiv 1^3 \cdot 11 \equiv 11,$$

and

$$19^{50} = (19^{24})^2 \cdot 19^2 \equiv 1^2 \cdot 361 \equiv 361 \equiv 1.$$

Thus,

$$(11^{73} + 19^{50} + 5)^{95} \equiv (11 + 1 + 5)^{95} = 17^{95}.$$

But 17 is also relatively prime to 90, so we can apply Euler's theorem again and deduce that

$$17^{95} = (17^{24})^4 \cdot 17^{-1} \equiv 1^4 \cdot 17^{-1} \equiv 17^{-1}.$$

So it remains to compute the inverse of 17 modulo 90. To do this, we run Euclid's algorithm forwards and backwards. Forwards, we get

$$\begin{aligned} 90 &= 5 \cdot 17 + 5, \\ 17 &= 3 \cdot 5 + 2, \\ 5 &= 2 \cdot 2 + 1, \\ 2 &= 2 \cdot 1 + 0. \end{aligned}$$

Backwards, we then get

$$\begin{aligned} 1 &= 5 - 2 \cdot 2 \\ &= 5 - 2 \cdot (17 - 3 \cdot 5) \\ &= 7 \cdot 5 - 2 \cdot 17 \\ &= 7 \cdot (90 - 5 \cdot 17) - 2 \cdot 17 \\ &= 7 \cdot 90 - 37 \cdot 17. \end{aligned}$$

Thus we deduce that $17^{-1} \equiv -37 \equiv 53 \pmod{90}$.

So the answer is 53.

**F.3 (i)** Clearly, $\chi(G_0) \geq 3$ since $G_0$ contains many triangles. In fact, $\chi(G_0) \geq 4$ because the vertex $f$ is at the centre of a wheel formed by $a, b, c, d, e$. This 5-cycle requires three colours, and then a fourth is needed for $f$. On the other hand, the graph is plane, hence $\chi(G_0) \leq 4$, by the Four-Colour Theorem. It follows that $\chi(G_0) = 4$.

If we apply the greedy algorithm with the nodes ordered $s, a, b, c, d, e, f, g, h, t$, then we get a 4-coloring, namely (the colors are $1, 2, 3, 4$)

| | | | |
|---|---|---|---|
| s | 1 | e | 3 |
| a | 2 | f | 4 |
| b | 3 | g | 1 |
| c | 1 | h | 3 |
| d | 2 | t | 4 |

**(ii)** Starting with the null flow $f \equiv 0$, one can find the following sequence of $f$-augmenting paths from $s$ to $t$ (these are not the only choices, of course) :

$$\begin{aligned} s - e - g - t, &\quad \epsilon = 4, \\ s - a - f - d - t, &\quad \epsilon = 4, \\ s - b - c - h - t, &\quad \epsilon = 3, \\ s - e - d - g - t, &\quad \epsilon = 2, \\ s - a - b - f - d - t, &\quad \epsilon = 2, \\ s - e - d - h - t, &\quad \epsilon = 1. \end{aligned}$$

This yields the following maximal flow with $|f| = 16$

| Edge | Flow | Edge | Flow | Edge | Flow | Edge | Flow |
|---|---|---|---|---|---|---|---|
| $(s,e)$ | 7 | $(e,g)$ | 4 | $(b,f)$ | 2 | $(d,h)$ | 1 |
| $(s,a)$ | 6 | $(e,d)$ | 3 | $(c,f)$ | 0 | $(c,h)$ | 3 |
| $(s,b)$ | 3 | $(f,e)$ | 0 | $(b,c)$ | 3 | $(g,t)$ | 6 |
| $(e,a)$ | 0 | $(a,f)$ | 4 | $(d,c)$ | 0 | $(d,t)$ | 6 |
| $(e,b)$ | 2 | $(f,d)$ | 6 | $(d,g)$ | 2 | $(h,t)$ | 4 |

The corresponding cut of minimal capacity is

$$S = \{s, a, b, c, e, f\}, \qquad T = \{d, g, h, t\}.$$

We verify that

$$c(S,T) = c(e,g) + c(e,d) + c(f,d) + c(c,h) = 4 + 3 + 6 + 3 = 16 = |f|, \text{ v.s.v.}$$

**F.4 (i)** 42 indistinguishable balls (the individual grades, since we are not interested in who got what grade here) are to be placed in 3 distinguishable bins (the grades 3,4 and 5). By Prop. 9 in the lecture notes, the number of ways this can be done is $\begin{pmatrix} 42+3-1 \\ 3-1 \end{pmatrix} = \begin{pmatrix} 44 \\ 2 \end{pmatrix} = 946.$

**(ii)** This time it matters who gets what grade. So we can think of the problem as the number of ways of permuting 42 balls in three distinguishable colours (the grades), where balls of the same colour are indistinguishable. By Prop. 8 in the lecture notes, the number of ways to do this is given by the multinomial coefficient

$$C(42:14,14,14) = \frac{42!}{(14!)^3} = 2120572665910728000.$$

**(iii)** Similar to part **(i)** but we shall use the inclusion-exclusion principle to sieve out the disallowed distributions. Let $x_i$ denote the number of people who obtain grade $i$, where $i = 3, 4, 5$. Thus the problem can be reformulated as the number of integer solutions to

$$x_3 + x_4 + x_5 = 42, \tag{1}$$

such that $0 \le x_i \le 18$. Let $X$ denote the set of all possible solutions to (1) with each $x_i \ge 0$. In part **(i)** we have already computed

$$|X| = \begin{pmatrix} 44 \\ 2 \end{pmatrix} = 946. \tag{2}$$

Let $A_3, A_4$ resp. $A_5$ denote the subsets of $X$ consisting of those solutions to (1) in which $x_3, x_4$ resp. $x_5$ is at least 19. Thus the number we are interested in is $|X \backslash (A_3 \cup A_4 \cup A_5)|$. By the I-E principle (eq.(15) in the lecture notes), this number is given by

$$|X| - |A_3| - |A_4| - |A_5| + |A_3 \cap A_4| + |A_3 \cap A_5| + |A_4 \cap A_5| - |A_3 \cap A_4 \cap A_5|.$$

By obvious symmetry, this simplifies to

$$|X| - 3|A_3| + 3|A_3 \cap A_4| - |A_3 \cap A_4 \cap A_5|. \tag{3}$$

Consider $|A_3|$. At least 19 people must get the grade 3. This leaves 23 people whose grades may be chosen freely. By the same argument as in part **(i)**, the number of ways to choose the grades is then

$$|A_3| = \binom{23+3-1}{3-1} = \binom{25}{2} = 300. \tag{4}$$

Next, consider $A_3 \cap A_4$. At least 19 people must get each of the grades 3 and 4. Thus 38 grades have already been handed out, and only 4 grades may be freely chosen. Hence,

$$|A_3 \cap A_4| = \binom{4+3-1}{3-1} = \binom{6}{2} = 15. \tag{5}$$

Finally, $|A_3 \cap A_4 \cap A_5| = 0$ since we simply don't have enough grades to go round. So substituting (2), (4) and (5) into (3), the final answer is

$$946 - 3 \cdot 300 + 3 \cdot 15 = 91.$$

**F.5** See Theorem 14 in the lecture notes.

**F.6 (i)** (a) cannot work since the sum of the terms of this sequence is odd, whereas the sum of the degrees of the vertices in any graph is even : see eq.(40), p.48 in the lecture notes.

(b) cannot work since such a graph would have ten nodes, of which three would have degree 9. These three nodes would thus be joined to all other nodes, so in particular every other node would have degree at least 3. So we couldn't have a node of degree 2.

**(ii)** There are 3 isomorphism classes of trees on 5 nodes, which I'll call

A,B and C.

A is a chain.
B has a root of degree 3, and one of its children has another child.
C has a root of degree 4.

Given a labelleing of A, there is only one way to relabel without changing the labels of the edges. Namely, the label of the middle vertex along the chain cannot be changed, but the labels of the second and fourth vertices can be interchanged, as long as the labels of the two leaves are simoultaneously interchanged. Thus tree A gives rise to $5!/2 = 60$ labelled trees.

Given a labelling of B, there is also only one way to relabel without changing the labels of the edges. Namely, all we can do is interchange the labels of the two children of the root which are leaves. Thus tree B also gives rise to $5!/2 = 60$ labelled trees.

Given a labelling of C, we can permute the labels of the four leaves freely, without changing the labellings of the edges. Thus tree C gives rise to $5!/4! = 5$ labelled trees.

Adding up, we find that the total number of labelled trees on 5 vertices is $60 + 60 + 5 = 125$, v.s.v.