The "No Justice in the Universe" phenomenon: Why honesty of effort may not be rewarded in matchplay tournaments

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(Joint work with Anders Martinsson and Edvin Wedin)

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But "justice" should mean that "the best team won".

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Condition 2: $\pi_1 \geq \pi_2 \geq \cdots \geq \pi_n$.

It's obvious (?) that both the League and the Cup satisfy Condition 2, so we were originally more interested in Condition 1.

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$$\pi_1(\text{League}) \to 0, \quad \pi_1(\text{Cup}) \to 1, \quad \text{provided } 1/n \ll \varepsilon \ll 1/\ln n.$$

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  Solution: Knockout, but each contest is "best of $\omega(n)$ matches".

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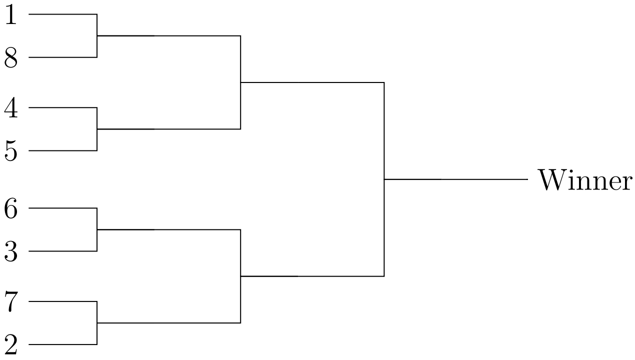

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Thus $\pi_2 > \pi_1$ for any $a < b$.

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We could find no evidence that this phenomenon is commonly understood. On the other hand, there are well-documented instances where upsets encouraged a team to (apparently) throw a game to avoid an ostensibly stronger opponent in Phase 2.

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The algorithm outputs a **winner** once all matches have been played.

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This definition is meant to capture the intuition that the rules "are the same for everyone". Lack of symmetry is a common, and obvious source of unfairness in many real-life tournaments.

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- We say that **T** is **honest** if, for any possible such state of **T** and any matrix P, we have $\pi_i^+(P) \geq \pi_i^-(P)$.

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ANSWER:

No, for every $n \geq 3$.

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Already for $N = 2$ the tournament is unfair: $\left(\frac{3}{8}, \frac{5}{12}, \frac{5}{24}\right)$.

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Already for $N = 2$ the tournament is unfair: $\left(\frac{7}{12}, \frac{1}{6}, \frac{1}{4}\right)$.

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- $\mathcal{S}$ is a convex polygon with five vertices:

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The previous examples $\mathbf{T}_{1,N}$ and $\mathbf{T}_{2,N}$ allow us to approach V_1 and V_2 as $N \to \infty$. It is easy to construct families of (fair) tournaments approaching the other three vertices.

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Remark: The convex polytope $\mathcal{A}_n^*$ has $\frac{3^{n-1}+1}{2}$ corners.

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The vector $v(G) = (v_1, \ldots, v_n)$ is defined as

$$v_i = \frac{\text{indeg}_G(i)}{2n} = \frac{1}{n} + \frac{\text{indeg}_G(i) - \text{outdeg}_G(i)}{2n}.$$

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(a) He makes an arbitrary list $(t_1, t_2, \ldots, t_{n-2})$ of the other $n-2$ remainers and computes the elements q_{ij} of an $(n-2) \times (n-2)$ matrix such that q_{ij} is the fraction of the matches between t_i and t_j which were won by t_i.

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(b) He tries to find a subset $\{u_1, \dots, u_{n-2}\} \subset [n]$ such that, for all $1 \leq i < j \leq n-2$,

$$|q_{ij} - p_{u_i, u_j}| < \varepsilon.$$

Note that he can find at most one such $(n-2) \times (n-2)$ submatrix of P. If he does so, we say that he **succeeds** in Step 3.

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Step 5: Assign to the player eliminated in Step 2 a token of weight $1 - s$, where s is the sum of the weights of the tokens distributed in Step 4. The winner of the tournament is now chosen at random, weighted in accordance with the distribution of tokens.