Stochastic Processes and their Applications 32 (1989) 183-212
North-Holland

GENERAL BRANCHING PROCESSES AS MARKOV FIELDS

Peter JAGERS
School of Mathematical and Computing Sciences, Chalmers University of Technology and Gothenburg University, S-412 96 Göteborg, Sweden

Received 5 January 1988
Revised 17 November 1988

The natural Markov structure for population growth is that of genetics: newborns inherit types from their mothers, and given those they are independent of the history of their earlier ancestry. This leads to Markov fields on the space of sets of individuals, partially ordered by descent. The structure of such fields is investigated.

It is proved that this Markov property implies branching, i.e. the conditional independence of disjoint daughter populations. The process also has the strong Markov property at certain optional sets of individuals. An intrinsic martingale (indexed by sets of individuals) is exhibited, that catches the stochastic element of population development. The deterministic part is analyzed by Markov renewal methods.

Finally the strong Markov property found is used to divide the population into conditionally independent subpopulations. On those classical limit theory for sums of independent random variables can be used to catch the asymptotic population development, as real time passes.

branching processes * population growth * Markov fields

1. Introduction

Is there any natural Markovian structure in population growth? Or to be more precise, in tolerably general models for populations of independently reproducing individuals?

Certainly we cannot take the population size itself as Markovian in continuous, physical time. That would have well-known absurd consequences at the individual level: life spans must be exponentially distributed and mothers give birth as an age-homogeneous Poisson process, and possibly also by splitting at death.

But even the much more sophisticated Markovianness in age distributions, assumed in most demography and biological population dynamics, subsumes undesired, or at least, quite special properties of individual life. Indeed, the process which gives at each instant not only the number of individuals born but also their ages, can only be Markovian if individual reproduction point processes have independent increments. (Otherwise additional information about who is whose daughter would be relevant for population forecasts.) But by Kingman's theorem (Kallenberg, 1983, pp. 56-59) the births then still form a Poisson process, or possibly occur at fixed ages.

This work has been supported by the Swedish Natural Science Research Council.

It thus seems that the only acceptably general Markov population processes of independently reproducing individuals are those in discrete time—the generation counting (multi-type) Galton–Watson processes. However, from an empiric point of view these are of little avail except for the study of time-independent phenomena, like ultimate extinction. Still, their existence provides a hint: maybe the difficulty in Markov modelling of population growth consists not so much in finding a proper state space, but rather in realizing that the natural "time" is the genealogical, only partially ordered, family tree, rather than physical time.

Indeed, from genetics we may take the basic dependence structure to be mothers passing on a type (the "genotype") to the newborns at their birth, that type determining a probability law over a space of possible life careers, the latter being thus independent of everything else, once the type is given.

We shall use this idea to formulate general branching processes in abstract type spaces as Markov random fields, indexed by sets of individuals, partially ordered by descent. Existence, uniqueness and a strong Markov branching property will be exhibited. Here the Markov property means that, given the types of a set of individuals, then the population stemming from the set will be independent of the history of its earlier ancestry. The branching property is our name for the conditional independence between daughter populations of different individuals, given their prehistory and provided none of them stem from the others. It turns out to be a beautiful, simple consequence of the theory that the Markov property actually implies branching. In other words, there are no other population processes Markovian over descent trees than branching processes.

We shall also exhibit an intrinsic martingale, indexed by sets of individuals, which catches the stochastic element in population growth. The mean growth is analyzed in Markov renewal terms. In the supercritical case the classical $x \log x$ condition will turn out to guarantee uniform integrability and hence L^1-convergence of this martingale with only partially ordered indices. The martingale convergence will finally be combined with some classical limit theory for sums of independent random variables, to yield limit theorems on population growth in real time and the stabilization of population composition over ages, types, etc., in the supercritical case.

The strict formulation of the model will be in terms of a general Ulam–Harris family space, as in Nerman (1984), or for the one type case in Jagers (1975) and in Jagers and Nerman (1984). The main impetus for this work, however, comes from the tree-space ideas by Neveu (1986) and Chauvin (1986). In particular, the fundamental concept of 'stopping line' (i.e. a set of individuals where no member stems from any other member), first appeared in Chauvin's paper. In 1980 D. Grey (1988) had similar ideas, looking for a framework for various martingale appearing in branching processes. A methodological prerequisite for the work is modern Markov renewal theory where I have chosen to rely upon the formulation by Niemi and Nummelin (1986). The intrinsic, set indexed martingale we use is derived from Nerman's (1981 and 1984) real time martingale and the results on the real-time process in our last section are close to those obtained by Nerman (1984) by

L^2-arguments. The methods however, a combination of weak L^1-convergence with classical limit theory, bear more resemblance to Cohn's (1985) proof for the one-type case. For a law-of-large-numbers approach to this case cf. Nerman (1981).

2. Basic notions

In the classical Ulam–Harris family space each (possible) individual is identified with his descent: an individual is a vector of positive integers, $x=(x_1, x_2, \ldots, x_n)$ being read as the x_nth child of the ... of the x_2th child of the x_1th child of the ancestor, the latter denoted by zero. In other words, the space of *individuals* is

$$I=\bigcup_{n=0}^{\infty} N^n,$$

where $N^0=\{0\}$ and $N=\{1,2,\ldots\}$. On individuals it is good to have notation for some simple operators or functions: For $0\neq x=(x_1,\ldots,x_{n-1},x_n)$,

$$mx=(x_1,\ldots,x_{n-1})$$

is x's *mother*, $mx=0$ if $x\in N$. Also

$$rx=x_n$$

is x's *rank* (in her sibship). If $x, y\in I$ we write xy for the concatenated individual, having first x's and then y's coordinates. In particular $0x=x=x0$ and $mxrx=x$. For any $x\in I$, $g(x)$ is x's dimension or *generation*,

$$g(x)=n \iff x\in N^n.$$

Obviously $m^{n+1}x$ must be x's nth grandmother, provided $g(x)>n$. It will be suitable to stop this regression at the ancestor, so that $m0:=0$.

If for some n $y=m^n x$, we write $x>y$ and say that x *stems from* y. By convention this includes $m^0x=x$. This obviously renders I a partially ordered set, and indeed a semilattice (every non-empty finite subset has a lower bound, the last common ancestor). The partial order will play a crucial rôle in the sequel and much of the theory will be valid for abstract semilattices. Here, let us only mention three further examples:

a. Binary splitting, $\bigcup_{n=0}^{\infty}\{1,2\}^n$, or any subset of I, which is a tree in the sense of Joffe (1978) and Neveu (1986).

b. The continuous semilattice $I\cup N^{\infty}$.

c. The doubly infinite pedigree (Jagers and Nerman (1984)). Write $Z_-=(0,-1,-2,\ldots)$ to be interpreted as a special (e.g. randomly chosen) individual, 'Ego', followed by her mother, grandmother etc. Then

$$Z_-\times I$$

is a doubly infinite space of individuals, centered around Ego. In this space $-j$ has daughters $-j+1$, $(-j, i)$, $i\in N$.

However, we shall stick to the Ulam–Harris space I. In this case the *progeny* of any set $M \subset I$,

$$\operatorname{Pr} M := \{x \in I;\ x \succ M\},$$

can be written $M \times I$. (Some algebraic literature uses $\uparrow M$ for this set.) Here, of course $x \succ M$ means that $x \succ y$ for some $y \in M$. Generally we write $L \succ M$ if all $x \in L$ stem from M.

As pointed out in the introduction *stopping lines* or, for short, *lines* are particularly interesting sets of individuals: $L \subset I$ is a (stopping) line if $x, y \in L, x \neq y \Rightarrow x \nprec y$. Thus, a line here is not at all the same as a genealogical line, but rather something cutting those. Any set M of individuals is initiated by a line, that might be called the *head* of the set, hM:

$$hM := \{x \in M;\ y < x,\ y \neq x \Rightarrow y \notin M\}.$$

Several properties of these sets of individuals are easily realized, like $h \operatorname{Pr} M = hM$, $\operatorname{Pr}(L \cup M) = (\operatorname{Pr} L) \cup (\operatorname{Pr} M)$ and the following.

Proposition 2.1. *If $L, M \subset I$ are stopping lines, then $L = M \Leftrightarrow \operatorname{Pr} L = \operatorname{Pr} M$.* □

We turn now to the *life space* $(\Omega, \mathscr{A})$. An element $\omega \in \Omega$ is a possible life career and any property of individuals like their mass at some age or their life span is viewed as a measurable function on the life space. In particular this applies to reproduction, defined by a sequel of maps $0 \leq \tau(1) \leq \tau(2) \leq \cdots \leq \infty$ from the life space into the extended positive real line, $\tau(k)(\omega)$ being the age of an individual with life career ω when she begets her kth child. If $\tau(k)(\omega) = \infty$, the kth child is never born.

At birth any child gets a type in an abstract type space S with a countably generated sigma-algebra $\mathscr{S}$. These types are given by measurable functions $\rho(k): \Omega \to S$, $k \in N$, $\rho(k)$ being the type of the kth child. The *reproduction process* is thus the point process

$$\xi(A \times B) := \#\{k;\ \rho(k) \in A,\ \tau(k) \in B\}, \qquad A \in \mathscr{S},\ B \in \mathscr{B},$$

$\mathscr{B}$ the Borel algebra on $\mathbb{R}_+$.

From the life and type spaces we construct the *population space*

$$(\Omega, \mathscr{A}) := (S \times \Omega^I, \mathscr{S} \times \mathscr{A}^I),$$

an outcome of which thus consists of a starting type, for the ancestor, and then a life career for each individual in I. For $M \subset I$ write U_M for the projection $S \times \Omega^I \to \Omega^M$ and $U_x = U_{\{x\}}$. Of great importance are the *pre-L-sigma algebras*

$$\mathscr{F}_L := \mathscr{S} \times \sigma(U_x;\ x \succ L) = \mathscr{S} \times \sigma(U_x;\ x \notin \operatorname{Pr} L),$$

for $L \subset I$. (Properly speaking, they are "ex-Pr L" rather than "pre-L".) As above we write $\mathscr{F}_x$ for $\mathscr{F}_{\{x\}}$.

Since $L < M \Rightarrow \Pr L \supset \Pr M \Rightarrow \mathscr{F}_L \subset \mathscr{F}_M$, it holds that:

Proposition 2.2. *$\{\mathscr{F}_L;\ L \subset I\}$ is a filtration under $<$.* □

The following can also be immediately seen:

Proposition 2.3. *$\mathscr{F}_{L \cup M} = \mathscr{F}_L \cap \mathscr{F}_M$ and $\mathscr{F}_L = \mathscr{F}_{hL}$.* □

To lift an entity χ defined on the life space into the population space write

$$\chi_x = \chi \circ U_x$$

for the χ-value pertaining to $x \in I$. In particular $\xi_x = \xi \circ U_x$ is x's reproduction process. However, we let τ_x denote $\tau(rx) \circ U_{mx}$, i.e. x's mother's age at x's birth. Similarly $\rho_x := \rho(rx) \circ U_{mx}$ is x's type. If χ is defined on $S \times \Omega$, it will be fitting to let χ_x denote $\chi(\rho_x, U_x)$.

Finally assume the ancestor born at time zero, i.e. define the *birth times* of $x \in I$ by

$$\sigma_0 = 0, \qquad \sigma_x = \sigma_{mx} + \tau_x, \quad 0 \neq x \in I.$$

Here $\sigma_x = \infty$ has the interpretation that x is never born and

$$\mathscr{R} = \{x \in I;\ \sigma_x < \infty\}$$

might be termed the set of *realized* individuals.

3. The probability measure

Now, as hinted in the introduction, assume for each $s \in S$ a probability measure $P(s, \cdot)$ given on the life space. The functions $s \to P(s, A)$ should be measurable, $A \in \mathscr{A}$. We shall see that such a kernel defines, to each $s \in S$, a unique probability measure $\mathbb{P}_s$ on the population space, which has the property that, given $\rho_0 = s$ and the type ρ_x of x, x's life follows the law $P(\rho_x, \cdot)$ independently of the process for the rest.

Indeed, the space of individuals can be enumerated in such a way that a mother always precedes her daughters. For any $x \in I$ we may therefore define a transition probability for the life law of x, given ρ_0 and the lives of individuals preceding x in the enumeration, simply to be $P(\rho_x, \cdot)$. By Ionesco Tulcea's theorem (Neveu, 1965, p. 162) this defines a unique probability measure $\mathbb{P}_s$ on $(S \times \Omega^I, \mathscr{S} \times \mathscr{A}^I)$ such that $\rho_0 = s$ and the following holds for finite dimensional sets: For any finite set, M, of individuals write

$$\mathrm{An}\, M := \{x \in I \setminus M;\ \exists y \in M, x < y\}$$

for the set of proper ancestors of M, and

$$\omega_M = \{\omega_x;\ x \in M\}$$

for the restriction of an outcome to life careers of individuals in M, $\omega_M = U_M(\omega)$, $\omega = (\rho_0, \omega_x; x \in I)$. Define

$$\mathbb{P}_s U_M^{-1}(\mathrm{d}\omega) = \int_{\Omega^{\mathrm{An}\, M}} \cdots \int \prod_{x \in (\mathrm{An}\, M) \cup M} P(\rho_x(\omega), \mathrm{d}\omega_x).$$

(Recall that under $\mathbb{P}_s$,

$$\rho_x(\omega) = \begin{cases} \rho(rx)(\omega_{mx}), & x \neq 0, \\ s, & x = 0.) \end{cases}$$

It also follows that for any $A \in \mathcal{S} \times \mathcal{A}^I$, $s \to \mathbb{P}_s(A)$ is measurable.

The obvious fact that the formation of the finite product measure under the integral is associative leads to Neveu's and Chauvin's (1986) succinct formulation of the Markov and branching properties. To state it write $S_x := (\rho_x, U_{\mathrm{Pr}(x)})$ for the type of x and the projection on the progeny of $x \in I$. In another interpretation, this is the translation that renders x the ancestor (cf. Jagers, 1975).

Theorem 3.1. *Let $L \subset I$ be a stopping line and φ_x, $x \in L$, non-negative measurable functions on the population space. Then, for any $s \in S$,*

$$\mathbb{E}_s\left[\prod_{x \in L} \varphi_x \circ S_x \,\middle|\, \mathcal{F}_L\right] = \prod_{x \in L} \mathbb{E}_{\rho_x}[\varphi_x],$$

$\mathbb{E}_s$ denoting expectation with respect to $\mathbb{P}_s$, $s \in S$.

Proof. Assume L to be finite and consider for any $x \in L$ finite $J_x \subset I$, writing $xJ_x := \{xy : y \in J_x\}$. Assume that $\mathrm{An}\, J_x = \emptyset$, i.e. that J_x with any individual contains all her ancestors. Then with $\rho_0 = s$,

$$\mathbb{P}_s U^{-1}(\mathrm{An}\, L) \cup \bigcup_{x \in L} xJ_x(\mathrm{d}\omega) = \prod_{y \in (\mathrm{An}\, L) \cup \bigcup_{x \in L} xJ_x} P(\rho_y(\omega), \mathrm{d}\omega_y)$$

$$= \prod_{y \in \mathrm{An}\, L} P(\rho_y(\omega), \mathrm{d}\omega_y) \prod_{x \in L} \prod_{z \in J_x} P(\rho_{xz}(\omega), \mathrm{d}\omega_{xz})$$

$$= \mathbb{P}_s \bigcup_{\mathrm{An}\, L}^{-1} (\mathrm{d}\omega) \prod_{x \in L} \mathbb{P}_{\rho_x} \bigcup_{J_x}^{-1} (\mathrm{d}\omega).$$

The general result follows from considerations of a sequence of finite $L_n \uparrow L$, $0 \leq \varphi_x \leq 1$, monotone convergence and backwards martingale convergence applied to the conditional expectations given $\mathcal{F}_{L_n}$ when $\mathcal{F}_{L_n} \downarrow \mathcal{F}_L$. □

Here we used an argument that will appear several times, that $L_n \subset L_{n+1} \Rightarrow L_{n+1} < L_n$ and hence $\mathcal{F}_{L_{n+1}} \subset \mathcal{F}_{L_n}$.

The Neveu-Chauvin form directly exhibits the announced beautiful fact that the Markov property implies branching, i.e. that conditional independence of disjoint daughter processes is actually a consequence of the Markovianess on the population three:

Theorem 3.2. *Consider a probability measure* $\tilde{\mathbb{P}}_s$, $s \in S$, *on the population space* $(S \times \Omega^I, \mathcal{S} \times \mathcal{A}^I)$ *such that for any* $x \in I$ $S_x | \mathcal{F}_x$ *has the distribution* $\tilde{\mathbb{P}}_{\rho_x}$. *Then, for any line* $L \subset I$ *the* S_x, $x \in L$, *are conditionally independent given* $\mathcal{F}_L$.

Proof. It suffices to consider finite lines, so assume $L = \{x_1, x_2, \ldots, x_n\}$. Let $\varphi_{x_i} : S \times \Omega^I \to [0, 1]$ be arbitrary measurable functions, $i = 1, 2, \ldots n$. Then, $\tilde{\mathbb{E}}_s$ denoting expectation with respect to $\tilde{\mathbb{P}}_s$, $s \in S$,

$$\tilde{\mathbb{E}}_s\left[\prod_{i=1}^n \varphi_{x_i} \circ S_{x_i} | \mathcal{F}_L\right]$$

$$= \tilde{\mathbb{E}}_s\left[\tilde{\mathbb{E}}_s\left[\prod_{i=1}^n \varphi_{x_i} \circ S_{x_i} | \mathcal{F}_{x_1}\right] \Bigg| \mathcal{F}_L\right] = \tilde{\mathbb{E}}_s\left[\prod_{i=2}^n \varphi_{x_i} \circ S_{x_i} \tilde{\mathbb{E}}_s[\varphi_{x_1} \circ S_{x_1} | \mathcal{F}_{x_1}] | \mathcal{F}_L\right]$$

$$= \tilde{\mathbb{E}}_s\left[\prod_{i=2}^n \varphi_{x_i} \circ S_{x_i} \tilde{\mathbb{E}}_{\rho_{x_1}}[\varphi_{x_1}] | \mathcal{F}_L\right] = \tilde{\mathbb{E}}_{\rho_{x_1}}[\varphi_{x_1}] \tilde{\mathbb{E}}_s\left[\prod_{i=2}^n \varphi_{x_i} \circ S_{x_i} | \mathcal{F}_L\right].$$

We used first that $\mathcal{F}_L \subset \mathcal{F}_{x_1}$, and then the Markov assumption. Repeating the argument we conclude that

$$\tilde{\mathbb{E}}_s\left[\prod_{i=1}^n \varphi_{x_i} \circ S_{x_i} | \mathcal{F}_L\right] = \prod_{i=1}^n \tilde{\mathbb{E}}_{\rho_{x_i}}[\varphi_{x_i}] = \prod_{i=1}^n \tilde{\mathbb{E}}[\varphi_{x_i} \circ S_{x_i} | \mathcal{F}_L]. \qquad \square$$

4. Optionality and strong Markov branching

The purpose of this section is to extend Theorem 3.1 to random stopping lines, which are "predetermined" or optional in the appropriate sense. Thus we define a random set of individuals

$$\mathcal{J} : S \times \Omega^I \to 2^I$$

to be *optional* if $\{\mathcal{J} < L\} \in \mathcal{F}_L$ for all $L \subset I$. If $\mathcal{J}$ is both optional and a stopping line, we shall call it an optional line. The definition is obviously patterned after that of optional times, for general such work in partially ordered index sets cf. Helms (1958), Hürzeler (1986), Kurtz (1980) and Neveu (1975). Note, however that by the tree structure any random set always containing the common ancester 0 satisfies

$$\{\mathcal{J} < L\} \supset \{0 < L\} = S \times \Omega^I \in \mathcal{F}_L,$$

for any $L \subset I$, and so must be optional.

Here are some examples of random sets of individuals:

a. $\mathcal{Y}_t = \{x \in I; \sigma_x \leq t\}$ is the set of *realized* individuals *by* t. Since $0 \in \mathcal{Y}_t$, the set is optional.

b. Obviously the set $\mathscr{R}=\{x \in I; \sigma_x < \infty\}$ of ever realized individuals is also optional.

c. Assume that $\lambda : \Omega \to \mathbb{R}_+$ is a *life span*. Write $\lambda_x = \lambda \circ U_x$, as agreed, for x's life length. Those *alive* at t,

$$\mathscr{Z}_t = \{x \in I; \sigma_x \leq t < \sigma_x + \lambda_x\}$$

is neither optional, nor a line.

d. But the *coming generation* at t,

$$\mathscr{I}_t = \{x \in I; \sigma_{mx} \leq t < \sigma_x < \infty\}$$

is both, since it is obviously a stopping line and, for any $L \subset I$,

$$\{\mathscr{I}_t < L\} = \{\forall x \in L; \exists k \in \mathbb{Z}_+; t < \sigma_{m^k x} < \infty\} \in \mathscr{F}_L.$$

e. Define the *nth individual* X_n appearing, $n = 0, 1, \ldots$, by ordering individuals according to their birth times, and by some genealogical rule guaranteeing that mothers precede daughters, if several individuals are born simultaneously. Since $\{X_n\}$ is a singleton it is trivially a stopping line and since

$$\{X_n < L\} = \{\forall x \in hL \; \exists k; m^k x = X_n\} \in \mathscr{F}_L,$$

it is also optional.

f. The *realized nth generation* $N^n \cap \mathscr{R} = \{x \in N^n; \sigma_x < \infty\}$ is an optional line. Following the text-book pattern we define for any optional set $\mathscr{I}$ the *pre-$\mathscr{I}$-algebra* $\mathscr{F}_{\mathscr{I}}$ by

$$A \in \mathscr{F}_{\mathscr{I}} \Leftrightarrow \forall L \subset I: A \cap \{\mathscr{I} < L\} \in \mathscr{F}_L.$$

It is easy to check that $\mathscr{F}_{\mathscr{I}}$ is really a σ-algebra and that several other simple facts hold:

Proposition 4.1. *If $\{\mathscr{I} < L\} \in \mathscr{F}_L$ for all finite L, then $\mathscr{I}$ is optional. If $\mathscr{I}$ is optional then $A \in \mathscr{F}_{\mathscr{I}} \Leftrightarrow A \cap \{\mathscr{I} < L\} \in \mathscr{F}_L$ for all finite L.*

Proof. Let finite $L_n \uparrow L$. Since $L_{n+1} \supset L_n$, $L_{n+1} < L_n$, $\mathscr{F}_{L_{n+1}} \subset \mathscr{F}_{L_n}$ and $\mathscr{F}_L = \bigcap_n \mathscr{F}_{L_n}$. Thus $\{\mathscr{I} < L_n\} \in \mathscr{F}_{L_n}$ for all n implies that

$$\{\mathscr{I} < L\} = \bigcap_n \{\mathscr{I} < L_n\} \in \mathscr{F}_L.$$

The second assertion follows in the same manner. □

Proposition 4.2. *If $\mathscr{I} = M$ (a constant set $\subset I$), then $\mathscr{I}$ is optional and $\mathscr{F}_{\mathscr{I}} = \mathscr{F}_M$.* □

Proposition 4.3. *For any optional $\mathcal{I}$ and any $L \subset I$, $\{\mathcal{I} < L\} \in \mathcal{F}_{\mathcal{I}}$.* $\square$

Proposition 4.4. *If $\mathcal{I}$, $\mathcal{J}$ are optional and $\mathcal{I} < \mathcal{J}$, then $\mathcal{F}_{\mathcal{I}} \subset \mathcal{F}_{\mathcal{J}}$.* $\square$

Lemma 4.5. *If $\mathcal{I}$ is an optional line, then $\mathcal{I} \in \mathcal{F}_{\mathcal{I}}$ in the sense that, for all $x \in I$, $\{x \in \mathcal{I}\} \in \mathcal{F}_{\mathcal{I}}$. Also $\{x \in \mathcal{I}\} \in \mathcal{F}_x$.*

Proof. Take any $x \in I$. Clearly $\{x \in \mathcal{I}\} = \{\mathcal{I} < x\} \setminus \{\mathcal{I} < mx\} \in \mathcal{F}_{\mathcal{I}}$ by Proposition 4.3. By optionality the same set belongs to $\mathcal{F}_x$, since $\mathcal{F}_{mx} \subset \mathcal{F}_x$. $\square$

Note that Lemma 4.5 is not valid for general optional sets. E.g. $\{x \in \mathcal{Y}_t\} = \{\sigma_x \leqslant t\}$ and

$$\{x \in \mathcal{Y}_t\} \cap \{\mathcal{Y}_t < mx\} = \{\sigma_x \leqslant t\} \notin \mathcal{F}_{mx}.$$

Proposition 4.6. *For any optional line $\mathcal{I}$ and any set L of individuals, $\{L \subset \mathcal{I}\} \in \mathcal{F}_{\mathcal{I}} \cap \mathcal{F}_L$ and $\{\mathcal{I} \subset L\} \in \mathcal{F}_{\mathcal{I}}$.*

Proof. Since L is countable,

$$\{L \subset \mathcal{I}\} = \bigcap_{x \in L} \{x \in \mathcal{I}\} \in \mathcal{F}_{\mathcal{I}}.$$

Since $\mathcal{I}$ is a stopping line,

$$\{L \subset \mathcal{I}\} = \bigcap_{x \in L} (\{\mathcal{I} < L\} \cap \{\mathcal{I} < mx\}').$$

But as each $\{\mathcal{I} < mx\}' \in \mathcal{F}_{\mathcal{I}}$ by Lemma 4.5, the whole intersection must be in $\mathcal{F}_L$. Similarly

$$\{\mathcal{I} \subset L\} = \{L' \subset \mathcal{I}\} = \bigcap_{x \in L'} \{x \in \mathcal{I}\}' \in \mathcal{F}_{\mathcal{I}}. \qquad \square$$

Corollary 4.7. *For $\mathcal{I}$, L as above $\{\mathcal{I} = L\} \in \mathcal{F}_{\mathcal{I}}$.* $\square$

Proposition 4.8. *If $\mathcal{I}$ is an optional line and $A \in \mathcal{F}_{\mathcal{I}}$ or $A \in \mathcal{F}_L$, $L \subset I$, then $A \cap \{\mathcal{I} = L\} \in \mathcal{F}_{\mathcal{I}} \cap \mathcal{F}_L$.*

Proof. In the first case, $A \in \mathcal{F}_{\mathcal{I}}$ the corollary directly yields $A \cap \{\mathcal{I} = L\} \in \mathcal{F}_{\mathcal{I}}$ first and then

$$A \cap \{\mathcal{I} = L\} = A \cap \{\mathcal{I} = L\} \cap \{\mathcal{I} < L\} \in \mathcal{F}_L.$$

The case $A \in \mathcal{F}_L$ follows similarly. $\square$

Lemma 4.9. *If $\{\mathcal{I}_n\}$ is a sequence of optional lines stemming from one another, $\mathcal{I}_{n+1} < \mathcal{I}_n$, then $\mathcal{I} = h \bigcup_n \mathrm{Pr}\, \mathcal{I}_n$ is an optional line and $\mathcal{F}_{\mathcal{I}} = \bigcap_n \mathcal{F}_{\mathcal{I}_n}$.*

Proof. For any finite $L \subset I$,

$$\{\mathcal{I} < L\} = \{L \subset \bigcup_n \mathrm{Pr}\, \mathcal{I}_n\} = \bigcup_n \{L \subset \mathrm{Pr}\, \mathcal{I}_n\} = \bigcup_n \{\mathcal{I}_n < L\} \in \mathcal{F}_L,$$

showing that $\mathcal{J}$ is optional by Proposition 4.1. It is a stopping line by definition. Since $\mathcal{J} < \mathcal{J}_n$ for all n, $\mathcal{F}_{\mathcal{J}} \subset \bigcap_n \mathcal{F}_{\mathcal{J}_n}$. But for finite L and $A \in \bigcap_n \mathcal{F}_{\mathcal{J}_n}$,

$$A \cap \{\mathcal{J} < L\} = \bigcup_n A \cap \{\mathcal{J}_n < L\} \in \mathcal{F}_L$$

proving that $A \in \mathcal{F}_{\mathcal{J}}$ and hence $\mathcal{F}_{\mathcal{J}} \supset \bigcap_n \mathcal{F}_{\mathcal{J}_n}$. □

Proposition 4.10. *The intersection of an optional line with a fixed set of individuals remains an optional line.*

Proof. We must only check optionality. Take finite $K, L \subset I$ and write $\mathcal{J}$ for the optional line. Then

$$\{\mathcal{J} \cap K < L\} = \bigcup_{\substack{M \subset K \\ M < L}} \{M \subset \mathcal{J}\} \in \mathcal{F}_L$$

since each $\{M \subset \mathcal{J}\} \in \mathcal{F}_M \subset \mathcal{F}_L$ by Proposition 4.6. But for general K there are finite sets $K_n \uparrow K$, $\mathcal{J} \cap K_{n+1} < \mathcal{J} \cap K_n$ and $\mathcal{J} \cap K = h \bigcup_n \operatorname{Pr} \mathcal{J} \cap K_n$. □

Proposition 4.11. *If $\mathcal{J}$ is optional, so is $\mathcal{J} \cap \mathcal{R}$.*

Proof. $\{\mathcal{J} \cap \mathcal{R} < L\} = \bigcap_{x \in hL} \bigcup_{k=0}^{g(x)} \{m^k x \in \mathcal{J}\} \cap \{\sigma_{m^k x} < \infty\} \in \mathcal{F}_L$. □

Note. Obviously Propositions 4.10 and 4.11 are special cases of an assertion about $\mathcal{J} \cap \mathscr{J}$, where $\mathscr{J}$ should have some property like $\{x \in \mathscr{J}\} \in \mathcal{F}_L$ for $x \in L \cup \operatorname{An} L$, $L \subset I$. In the sequel we often write $\mathbb{P}$ or $\mathbb{E}$ for $\mathbb{P}_s, \mathbb{E}_s$, $s \in S$.

Lemma 4.12. *Let $\varphi : S \times \Omega^I \to \mathbb{R}_+$ be measurable, $L \subset I$, and $\mathcal{J}$ an optional line. Then, on $\{\mathcal{J} = L\}$,*

$$\mathbb{E}[\varphi \mid \mathcal{F}_{\mathcal{J}}] = \mathbb{E}[\varphi \mid \mathcal{F}_L].$$

Proof. The set $\mathcal{H}$ of φ such that $\mathbb{E}[\varphi \mid \mathcal{F}_L] 1_{\{\mathcal{J}=L\}} \in \mathcal{F}_{\mathcal{J}}$ is obviously closed under differences and increasing limits. By Corollary 4.7, $1 \in \mathcal{H}$. By Theorem 3.1 it holds for $A \in \mathcal{F}_L$ and measurable $\varphi_x \geq 0$, $x \in L$, that

$$\mathbb{E}[1_A \prod_{x \in L} \varphi_x \circ S_x \mid \mathcal{F}_L] 1_{\{\mathcal{J}=L\}} = 1_{A \cap \{\mathcal{J}=L\}} \prod_{x \in L} \mathbb{E}_{\rho_x}[\varphi_x] = 1_{A \cap \{\mathcal{J}=L\}} \prod_{x \in \mathcal{J}} \mathbb{E}_{\rho_x}[\varphi_x].$$

If L is not a stopping line equality still holds, since both sides vanish.

By Proposition 4.8, $A \cap \{\mathcal{J} = L\} \in \mathcal{F}_L$. As the measurability of $s \to \mathbb{E}_s[\varphi_x]$ is well established, Proposition 4.5 yields that the right-hand side of the equality is measurable with respect to $\mathcal{F}_{\mathcal{J}}$ and hence that all functions, $A \in \mathcal{F}_L$,

$$1_A \prod_{x \in L} \varphi_x \circ S_x \in \mathcal{H}.$$

By the closure argument known as Dynkin's lemma it follows that all $\mathbb{E}[\varphi|\mathcal{F}_L]1_{\{\mathcal{J}=L\}}$ must be measurable with respect to $\mathcal{F}_{\mathcal{J}}$ (φ assumed integrable).

Now let $A\in\mathcal{F}_{\mathcal{J}}$. Then, $A\cap\{\mathcal{J}=L\}\in\mathcal{F}_L$ and thus

$$\mathbb{E}[\mathbb{E}[\varphi|\mathcal{F}_L]1_{\{\mathcal{J}=L\}};A]=\mathbb{E}[\mathbb{E}[\varphi|\mathcal{F}_L];A\cap\{\mathcal{J}=L\}]=\mathbb{E}[\varphi;A\cap\{\mathcal{J}=L\}]$$
$$=\mathbb{E}[\varphi 1_{\{\mathcal{J}=L\}};A],$$

proving the lemma, as $\{\mathcal{J}=L\}\in\mathcal{F}_{\mathcal{J}}$. □

Corollary 4.13. *Let φ be as above and $\mathcal{J}$ an optional line that can equal only countably many sets, $\mathcal{L}$ being the class of these. Then*

$$\mathbb{E}[\varphi|\mathcal{F}_{\mathcal{J}}]=\sum_{L\in\mathcal{L}}\mathbb{E}[\varphi|\mathcal{F}_L]1_{\{\mathcal{J}=L\}}.$$

In particular this holds with $\mathcal{L}$ the class of finite subsets of I, if $\mathcal{J}$ is finite. □

Let us now consider such a finite optional line $\mathcal{J}$ and a φ of the form

$$\varphi=\prod_{x\in\mathcal{J}}\varphi_x\circ S_x.$$

Then, by the corollary and Theorem 3.1,

$$\mathbb{E}[\varphi|\mathcal{F}_{\mathcal{J}}]=\sum_L\prod_{x\in L}\mathbb{E}_{\rho_x}[\varphi_x]1_{\{\mathcal{J}=L\}}=\prod_{x\in\mathcal{J}}\mathbb{E}_{\rho_x}[\varphi_x].$$

Now, do not any longer ask that $\mathcal{J}$ is necessarily finite, and assume $0\leq\varphi_x\leq 1$ to avoid some nuisance. Then for finite sets $K_n\uparrow I\mathcal{J}_n=\mathcal{J}\cap K_n$ will be finite optional lines such that $\mathcal{J}_{n+1}<\mathcal{J}_n$ and $h(\bigcup_n \Pr\mathcal{J}_n)=h(\bigcup_n\mathcal{J}\cap K_n)=h(\mathcal{J})=\mathcal{J}$. Hence, by Lemma 4.9, $\mathcal{F}_{\mathcal{J}}=\bigcap_n\mathcal{F}_{\mathcal{J}_n}$. By backwards martingale convergence (cf. Neveu, 1975, p. 118 e.g.) it follows that, as $n\to\infty$,

$$\prod_{x\in\mathcal{J}}\mathbb{E}_{\rho_x}[\varphi_x]\downarrow\prod_{x\in\mathcal{J}_n}\mathbb{E}_{\rho_x}[\varphi_x]=\mathbb{E}[\varphi]\mathcal{F}_{\mathcal{J}_n}]\xrightarrow[\text{a.s.}]{L^1}\mathbb{E}[\varphi|\mathcal{F}_{\mathcal{J}}].$$

Thus, the strong Markov branching property is proved:

Theorem 4.14. *Let $\mathcal{J}$ be an optional stopping line and $\varphi_x:S\times\Omega^I\to[0,1]$, $x\in I$ measurable functions. Then, $s\in S$,*

$$\mathbb{E}_s\left[\prod_{x\in\mathcal{J}}\varphi_x\circ S_x\middle|\mathcal{F}_{\mathcal{J}}\right]=\prod_{x\in\mathcal{J}}\mathbb{E}_{\rho_x}[\varphi_x].\qquad\square$$

5. The intrinsic martingale

We shall now single out for study the wide class of populations that turn out to grow in a Malthusian manner, i.e. exponentially, thereby also stabilizing their composition, as we shall see. The crucial rôle in this analysis belongs to the *reproduction kernel* μ,

$$\mu(s, A\times B)=\int_{\Omega} \xi(A\times B)(\omega)P(s, \mathrm{d}\omega)$$
$$=\mathbb{E}_s[\xi_0(A\times B)], \quad s\in S,\ A\in\mathcal{S},\ B\in\mathcal{B},$$

and its Markov renewal properties. Not to get off the main track we give notation and conditions for this theory rather quickly, following mainly Niemi and Nummelin, 1986. This means that we shall work with spread-out-ness and in the coupling tradition rather than following the analytic, Feller-style approach. Markov renewal theory for general state spaces in the latter style has been developed by Shurenkov (1984).

For any $\lambda\in\mathbb{R}$ we define

$$\mu_\lambda(r, \mathrm{d}s\times\mathrm{d}u)=\mathrm{e}^{-\lambda u}\mu(r, \mathrm{d}s\times\mathrm{d}u).$$

The composition operation '$*$' denotes Markov transition on S and convolution on $\mathbb{R}_+$, so that

$$\mu^{*2}(s, A\times B)=\mu*\mu(s, A\times B)=\int_{S\times\mathbb{R}_+}\mu(r, A\times(B-u))\mu(s, \mathrm{d}r\times\mathrm{d}u).$$

Further any kernel to the $*$-power 0 is $1_{A\times B}(s, 0)$ giving all mass to $(s, 0)$. The renewal measures are

$$\nu_\lambda:=\sum_{n=0}^{\infty}\mu_\lambda^{*n}$$

and it is assumed throughout that the *Malthusian parameter* α,

$$\alpha:=\inf\{\lambda;\ \nu_\lambda(s, S\times\mathbb{R}_+)<\infty \text{ for some } s\in S\},$$

is finite. The kernel $\mu(\cdot, \cdot\times\mathbb{R}_+)$ should also be π-irreducible for some sigma-finite measure π, π denoting the maximal such measure (cf. Nummelin, 1984, p. 13), and it should be what Niemi and Nummelin call α-recurrent, i.e. $\nu_\alpha(s, A\times\mathbb{R}_+)=\infty$ for all $s\in S$ and $A\in\mathcal{S}$ with $\pi(A)>0$. (For some discussion about when this might be the case cf. Jagers, 1984.)

If all this holds there is (Nummelin, 1984, p. 70) a strictly positive and π-almost everywhere finite eigenfunction h,

$$\int_S h(s)\mu_\alpha(r, \mathrm{d}s\times\mathbb{R}_+)=h(r), \quad r\in S,$$

and π can (and will) be also taken as invariant for μ_α,

$$\int_S \mu_\alpha(s, A\times\mathbb{R}_+)\pi(\mathrm{d}s)=\pi(A).$$

Further we require even strong α-recurrence, namely that

$$0<\beta=\int_{S\times S\times\mathbb{R}_+} t\,\mathrm{e}^{-\alpha t}h(s)\mu(r,\mathrm{d}s\times\mathrm{d}t)\pi(\mathrm{d}r)<\infty,$$

meaning in branching process parlance that the mean age at child bearing is finite and positive. Then (cf. Nummelin, 1978, p. 124) $h\in L^1[\pi]$ so that we can (and shall) norm to

$$\int_S h(s)\pi(\mathrm{d}s)=1.$$

Clearly then $hd\pi$ is the stationary probability measure of the Markov transition kernel $Q(s, A\times\mathbb{R}_+)$, $s\in S$, $A\in\mathcal{S}$, defined by

$$Q(r,\mathrm{d}s\times\mathrm{d}t)=h(s)\,\mathrm{e}^{-\alpha t}\mu(r,\mathrm{d}s\times\mathrm{d}t)/h(r).$$

All these conditions will be summarized by saying that the population considered is *Malthusian*. If the Malthusian parameter $\alpha>0$, the population is *supercritical*, being *critical* or *subcritical* according as $\alpha=0$ or $\alpha<0$.

As mentioned we shall further ask that the kernel is *spread out*, i.e. that for each s, $\mu^{*n}(s,\cdot)$ is for some n non-singular with respect to the product of π with Lebesgue measure.

From now on we consider only Malthusian populations (though for some results this is unnecessarily restrictive). The *intrinsic* process or (super)martingale $\{w_M;\ M\subset I\}$ is defined to be

$$w_M=\sum_{x\in M}\mathrm{e}^{-\alpha\sigma_x}h(\rho_x).$$

Note that $w_M=w_{M\cap\mathcal{R}}$. For lines L, $\{w_L\}$ is clearly adapted to the filtration $\{\mathcal{F}_L\}$. To formulate the martingale property of $\{w_M\}$ we need a definition: A stopping line M *covers* L if $M>L$ and any individual stemming from L either stems from M or has progeny in M. If M covers the ancestor it may simply be called *covering*. The class of covering stopping lines will be denoted by $\mathcal{C}$. We also write $g(M)=\sup_{x\in M}g(x)$.

The origin of this intrinsic process is the real time martingale

$$\sum_{\sigma_{mx}\leqslant t<\sigma_x}\mathrm{e}^{-\alpha\sigma_x}h(\rho_x),\quad t\in\mathbb{R}_+,$$

introduced by Nerman (1984). In our terminology Nerman's martingale is $\{w_{\mathcal{I}_t};\ t\in\mathbb{R}_+\}$. Indeed $\mathcal{I}_t<\mathcal{I}_{t'}$, if $t\leqslant t'$, and they are optional lines.

Theorem 5.1. *If* $L<M$ *are stopping lines, then*

$$\mathbb{E}[w_M\,|\,\mathcal{F}_L]\leqslant w_L.$$

If M *with* $g(M)<\infty$ *covers* L, *equality holds.*

Proof. We consider first the case of inequality for finite M and use induction over the generations following L: If $M \subset L$, then obviously $L < M$ and

$$\mathbb{E}[w_M \mid \mathcal{F}_L] \leq \mathbb{E}[w_L \mid \mathcal{F}_L] = w_L.$$

Make the induction hypothesis that if

$$M \subset L \times \bigcup_{k=0}^{n-1} N^k,$$

then the inequality asked for holds, and consider an M included in

$$L \times \bigcup_{k=0}^{n} N^k,$$

so that

$$\mathbb{E}[w_M \mid \mathcal{F}_L] = \mathbb{E}[w_{M \cap (L \times \bigcup_{k=0}^{n-1} N^k)} \mid \mathcal{F}_L] + \mathbb{E}[w_{M \cap (L \times N^n)} \mid \mathcal{F}_L].$$

Write

$$A := \{mx;\ x \in M \cap (L \times N^n)\}.$$

Since $L < L \times N^{n-1}$, $\mathcal{F}_L \subset \mathcal{F}_{L \times N^{n-1}}$, and

$$\begin{aligned}
\mathbb{E}[w_{M \cap (L \times N^n)} \mid \mathcal{F}_L] &= \mathbb{E}[\mathbb{E}[w_{M \cap (L \times N^n)} \mid \mathcal{F}_{L \times N^{n-1}}] \mid \mathcal{F}_L] \\
&\leq \mathbb{E}\left[\sum_{x \in A} \sum_{i \in N} \mathbb{E}[\mathrm{e}^{-\alpha\sigma_{xi}} h(\rho_{xi}) \mid \mathcal{F}_{L \times N^{n-1}}] \,\middle|\, \mathcal{F}_L\right] \\
&= \mathbb{E}\left[\sum_{x \in A} \mathrm{e}^{-\alpha\sigma_x} \sum_{i \in N} \mathbb{E}[\mathrm{e}^{-\alpha\sigma_i \circ S_x} h(\rho_i \circ S_x) \mid \mathcal{F}_{L \times N^{n-1}}] \,\middle|\, \mathcal{F}_L\right] \\
&= \mathbb{E}\left[\sum_{x \in A} \mathrm{e}^{-\alpha\sigma_x} \sum_{i \in N} \mathbb{E}_{\rho_x}[\mathrm{e}^{-\alpha\sigma_i} h(\rho_i)] \,\middle|\, \mathcal{F}_L\right] \\
&= \mathbb{E}\left[\sum_{x \in A} \mathrm{e}^{-\alpha\sigma_x} \mathbb{E}_{\rho_x}\left[\int_{S \times \mathbb{R}_+} \mathrm{e}^{-\alpha t} h(s) \xi_0(\mathrm{d}s \times \mathrm{d}t)\right] \,\middle|\, \mathcal{F}_L\right] \\
&= \mathbb{E}\left[\sum_{x \in A} \mathrm{e}^{-\alpha\sigma_x} \int_{S \times \mathbb{R}_+} \mathrm{e}^{-\alpha t} h(s) \mu(\rho_x, \mathrm{d}s \times \mathrm{d}t) \,\middle|\, \mathcal{F}_L\right] \\
&= \mathbb{E}\left[\sum_{x \in A} \mathrm{e}^{-\alpha\sigma_x} h(\rho_x) \,\middle|\, \mathcal{F}_L\right] = \mathbb{E}[w_A \mid \mathcal{F}_L].
\end{aligned}$$

But $(M \cap (L \times \bigcup_{k=0}^{n-1} N^k)) \cup A$ must be a stopping line included in $L \times \bigcup_{k=0}^{n-1} N^k$. So the induction hypothesis yields that

$$\begin{aligned}
\mathbb{E}[w_M \mid \mathcal{F}_L] &\leq \mathbb{E}[w_{M \cap (L \times \bigcup_{k=0}^{n-1} N^k)} + w_A \mid \mathcal{F}_L] \\
&= \mathbb{E}[w_{(M \cap (L \times \bigcup_{k=0}^{n-1} N^k)) \cup A} \mid \mathcal{F}_L] \leq w_L.
\end{aligned}$$

A general M can always be approximated from within by finite $M_n \uparrow M$. Monotone convergence concludes the proof of the inequality. The covering case is harder because no finite covering lines exist, besides $\{0\}$. However, the proof will follow from the subsequent lemmas. □

Lemma 5.2. *The class of covering stopping lines is a lattice in the sense that if $L, M \in \mathscr{C}$, there is a "first" element $L \vee M$ in $\mathscr{C}$ such that $L < L \vee M$ and $M < L \vee M$, and $L < K, M < K \in \mathscr{C} \Rightarrow L \vee M < K$. Similarly, there is a "last" element $L \wedge M$ in $\mathscr{C}$, such that L and M both stem from $L \wedge M$.*

The proof follows from consideration of sets

$$L_1 = \{x \in L; \exists y \in M, x < y\}, \qquad L_2 = \{x \in L; M < x\}$$

and the corresponding partition M_1, M_2 of M. The asked for sets are $L_2 \cup M_2$ and $L_1 \cup M_1$. □

Lemma 5.3. *If $L \in \mathscr{C}$ and $g(L) = \sup_{x \in L} g(x) < \infty$, then $\mathbb{E}_s[w_L] = h(s)$, $s \in S$.*

Proof. For any $k \in N$ repeated use of the eigenfunction property of h yields

$$\begin{aligned}\sum_{x \in N^k} \mathbb{E}_s[\mathrm{e}^{-\alpha\sigma_x} h(\rho_x)] &= \sum_{x \in N^{k-1}} \sum_{i \in N} \mathbb{E}_s[\mathrm{e}^{-\alpha\sigma_x} \mathbb{E}[\mathrm{e}^{-\alpha\sigma_i \circ S_x} h(\rho_i \circ S_x) \mid \mathscr{F}_x]] \\ &= \sum_{x \in N^{k-1}} \mathbb{E}_s\left[\mathrm{e}^{-\alpha\sigma_x} \sum_{i \in N} \mathbb{E}_{\rho_x}[\mathrm{e}^{-\alpha\tau(i)} h(\rho_i)]\right] \\ &= \sum_{x \in N^{k-1}} \mathbb{E}_s[\mathrm{e}^{-\alpha\sigma_x} h(\rho_x)] = \cdots = h(s).\end{aligned}$$

For any $x \in I$ and $n \geq g(x)$ therefore

$$\begin{aligned}\mathbb{E}_s[\mathrm{e}^{-\alpha\sigma_x} h(\rho_x)] &= \mathbb{E}_s\left[\mathrm{e}^{-\alpha\sigma_x} \sum_{y \in N^{n-g(x)}} \mathbb{E}_{\rho_x}[\mathrm{e}^{-\alpha\sigma_y} h(\rho_y)]\right] \\ &= \mathbb{E}_s\left[\mathrm{e}^{-\alpha\sigma_x} \sum_{y \in N^{n-g(x)}} \mathbb{E}[\mathrm{e}^{-\alpha\sigma_y \circ S_x} h(\rho_y \circ S_x) \mid \mathscr{F}_x]\right] \\ &= \sum_{y \in N^{n-g(x)}} \mathbb{E}_s[\mathrm{e}^{-\alpha\sigma_{xy}} h(\rho_{xy})].\end{aligned}$$

Now let L be a covering stopping line and write $I_n = \bigcup_{k=0}^{n-1} N^k$ for short. By the proved part of Theorem 5.1,

$$\begin{aligned}h(s) = \mathbb{E}_s[w_{N^0}] \geq \mathbb{E}_s[w_{L \cap I_n}] &= \sum_{x \in L \cap I_n} \mathbb{E}_s[\mathrm{e}^{-\alpha\sigma_x} h(\rho_x)] \\ &= \sum_{x \in L \cap I_n} \sum_{y \in N^{n-g(x)}} \mathbb{E}_s[\mathrm{e}^{-\alpha\sigma_{xy}} h(\rho_{xy})] \\ &= \sum_{\substack{x \in N^n \\ x > L}} \mathbb{E}_s[\mathrm{e}^{-\alpha\sigma_x} h(\rho_x)].\end{aligned}$$

Since $w_{L \cap I_n} \uparrow w_L$, monotone convergence yields

$$\begin{aligned}h(s) \geq \mathbb{E}_s[w_L] &= \lim_{n \to \infty} \sum_{\substack{x \in N^n \\ x > L}} \mathbb{E}_s[\mathrm{e}^{-\alpha\sigma_x} h(\rho_x)] \\ &= \lim_{n \to \infty} \sum_{x \in N^n} \mathbb{E}_s[\mathrm{e}^{-\alpha\sigma_x} h(\rho_x)] = h(s),\end{aligned}$$

if only $g(L) < \infty$, so that finally all $x \in N^n$ stem from L. □

Now we can finish the proof of Theorem 5.1: Assume that M with $g(M)<\infty$ covers L. Then, for each $x\in L$, $\{y;\, xy\in M\}$ is covering with finite maximal generation. Lemma 5.3 yields

$$\mathbb{E}[w_M\,|\,\mathcal{F}_L]=\sum_{x\in L}\mathrm{e}^{-\alpha\sigma_x}\sum_{y;\,xy\in M}\mathbb{E}[\mathrm{e}^{-\alpha\sigma_y\circ S_x}h(\rho_y\circ S_x)\,|\,\mathcal{F}_L]$$

$$=\sum_{x\in L}\mathrm{e}^{-\alpha\sigma_x}\mathbb{E}_{\rho_x}[w_{\{y:xy\in M\}}]=\sum_{x\in L}\mathrm{e}^{-\alpha\sigma_x}h(\rho_x)=w_L. \qquad \square$$

In terms of $\mathcal{C}_0$, the class of covering stopping lines with finite maximal generation, we can state the obvious

Corollary 5.4. *$\{w_L;\, L\in\mathcal{C}_0\}$ is a martingale with respect to $\{\mathcal{F}_L;\, L\in\mathcal{C}_0\}$.* $\square$

6. Uniform integrability and convergence of the intrinsic martingale

Since $\{w_L;\, L\in\mathcal{C}_0\}$ is thus a non-negative martingale, any sequence $L_n<L_{n+1}\in\mathcal{C}_0$ yields a.s. convergence of w_{L_n} to some limit. Convergence of the whole martingale itself is slightly harder to establish (cf. Neveu, 1972, p. 95ff). Indeed, L^1-convergence is what we should hope for. It turns out that uniform integrability is guaranteed by the classical $x\log x$ criterion, which now takes the following form:

Define

$$\bar{\xi}:=\int_{S\times\mathbb{R}_+}\mathrm{e}^{-\alpha t}h(s)\xi(\mathrm{d}s\times\mathrm{d}t)$$

and write E_π for expectation with respect to $\int_S P(s,\mathrm{d}\omega)\pi(\mathrm{d}s)$. The $x\log x$ condition is

$$E_\pi[\bar{\xi}\log^+\bar{\xi}]<\infty.$$

Theorem 6.1. *Consider a Malthusian branching population, satisfying $x\log x$. For almost all $s\in S[\pi]$, $\{w_L;\, L\in\mathcal{C}_0\}$ is uniformly $\mathbb{P}_s$-integrable.*

Proof. Take any $L\in\mathcal{C}$ and some $K\subset I$ with $g(K)<\infty$, and any set $A\in\mathcal{F}_K$. Write $L_n=\{x\in L;\, g(x)\leq n\}$. By monotone convergence

$$\mathbb{E}[w_{L_n};A]\uparrow\mathbb{E}[w_L;A]$$

and by Theorem 5.1,

$$\mathbb{E}[w_{L_n};A]\leq\mathbb{E}[w_{L\wedge N^n};A]\leq\mathbb{E}[w_{N^n};A]\leq\mathbb{E}[\sup_n w_{N^n};A].$$

Since the sigma-algebras $\mathcal{F}_K$ with $g(K)<\infty$ generate $\mathcal{S}\times\mathcal{A}^I$, it follows that $w_L\leq\sup_n w_{N^n}$ almost surely $[\mathbb{P}_s]$, $s\in S$. Therefore the claimed uniform integrability would follow from $\sup_n w_{N^n} \in L^1[\mathbb{P}_s]$, π-almost all $s \in S$. This will be proved by one of Asmussen's methods (Asmussen and Hering, 1983, p. 23ff).

Write

$$\eta_x = \bar{\xi}_x - h(\rho_x), \quad x \in I,$$

$$\eta = \bar{\xi} - h(\rho_0),$$

$$\delta(s, t) = e^{\alpha t} E_s[\eta; |\eta| > e^{\alpha t}], \quad s \in S,$$

E_s being integration on Ω with respect to $P(s, \cdot)$ for $\rho_0 = s$. Decompose

$$w_{N^{n+1}} = \sum_{x \in N^n} e^{-\alpha\sigma_x} \sum_{i \in N} e^{-\alpha\sigma_i \circ S_x} h(\rho_i \circ S_x)$$

$$= \sum_{x \in N^n} e^{-\alpha\sigma_x} \bar{\xi}_x = \sum_{x \in N^n} e^{-\alpha\sigma_x} \eta_x + w_{N^n},$$

and

$$w_{N^{n+1}} - w_{N^n} = \sum_{x \in N^n} (e^{-\alpha\sigma_x} \eta_x 1_{\{|\eta_x| \leq e^{\alpha\sigma_x}\}} + \delta(\rho_x, \sigma_x))$$

$$+ \sum_{x \in N^n} (e^{-\alpha\sigma_x} \eta_x 1_{\{|\eta_x| > e^{\alpha\sigma_x}\}} - \delta(\rho_x, \sigma_x))$$

$$= a_{n+1} + b_{n+1},$$

say. Since

$$E_s[\eta; |\eta| \leq e^{\alpha t}] = -e^{\alpha t} \delta(s, t),$$

$$\mathbb{E}[a_{n+1}] = \mathbb{E}[a_{n+1} | \mathcal{F}_{N^n}] = \mathbb{E}[b_{n+1}] = \mathbb{E}[b_{n+1} | \mathcal{F}_{N^n}] = 0.$$

Further

$$\mathbb{V}\mathrm{ar}[a_{n+1}] = \mathbb{E}[\mathbb{V}\mathrm{ar}[a_{n+1} | \mathcal{F}_{N^n}]] + \mathbb{V}\mathrm{ar}[\mathbb{E}[a_{n+1} | \mathcal{F}_{N^n}]] = \mathbb{E}[\mathbb{V}\mathrm{ar}[a_{n+1} | \mathcal{F}_{N^n}]]$$

$$\leq \mathbb{E}\left[\sum_{x \in N^n} e^{-2\alpha\sigma_x} E_{\rho_x}[\eta^2; |\eta| \leq e^{\alpha t}]|_{t=\sigma_x}\right].$$

In the notation

$$Y(A \times B) = \#\{x \in I; \rho_x \in A, \sigma_x \in B\},$$

so that

$$\mathbb{E}_s[Y(A \times B)] = \sum_{n=0}^{\infty} \mu^{*n}(s, A \times B) = \nu(s, A \times B),$$

thus

$$\sum_{n=0}^{k} \mathbb{V}\mathrm{ar}_r[a_{n+1}] \leq \sum_{n=0}^{\infty} \mathbb{V}\mathrm{ar}_r[a_{n+1}] \leq \int_{S \times \mathbb{R}_+} e^{-2\alpha t} E_s[\eta^2; |\eta| \leq e^{\alpha t}] \nu(r, ds \times dt).$$

If we could replace $e^{-\alpha t}\nu(r, ds \times dt)$ by its Markov renewal theory limit, as $t \to \infty$, we would obtain, but for constants,

$$\int_{S\times\mathbb{R}} e^{-\alpha t} E_s[\eta^2; |\eta| \leq e^{\alpha t}]\pi(ds)\, dt = \int_S \pi(ds) E_s[\eta^2 \int_{\alpha^{-1}\log|\eta|}^{\infty} e^{-\alpha t}\, dt]$$

$$= E_\pi[|\eta|] \leq E_\pi[\bar{\xi}] + \int_S h(s)\pi(ds) < \infty,$$

By assumption. But there is (cf. Jacod, 1971, p. 98 e.g.) a unique measure G on $S \times \mathbb{R}_+$,

$$G(A \times B) = \int_{S\times B} Q(s, A \times (t, \infty)) h(s) \pi(ds)\, dt/\beta$$

$$= \int_{S\times B} \left\{ \int_A h(r) \mu_\alpha(s, dr \times (t, \infty)) \right\} \pi(ds)\, dt/\beta,$$

such that

$$G * \sum_{n=0}^{\infty} Q^{*n} = (h\pi) \otimes \lambda,$$

here λ stands for Lebesgue measure,

$$h\pi(A) = \int_A h\, d\pi,$$

and Q was defined in Section 5. Since

$$e^{-\alpha t} h(s) \nu(r, ds \times dt)/h(r) = \sum_{n=0}^{\infty} Q^{*n}(r, ds \times dt),$$

$$\int_{S\times\mathbb{R}_+} \left\{ \int_{S\times\mathbb{R}_+} e^{-2\alpha t} E_s[\eta^2; |\eta| \leq e^{\alpha t}] \nu(r, ds \times dt - u) \right\} (1/h(r)) G(dr \times du)$$

$$= \int_{S\times\mathbb{R}_+} \left\{ \int_{S\times\mathbb{R}_+} e^{-\alpha(t+u)} E_s[\eta^2; |\eta| \geq e^{\alpha t}] \right.$$

$$\left. \times (1/h(s)) \sum_{n=0}^{\infty} Q^{*n}(r, ds \times dt - u) \right\} G(dr \times du)$$

$$\leq \int_{S\times\mathbb{R}_+} \left\{ \int_{S\times\mathbb{R}_+} e^{-\alpha t} E_s[\eta^2; |\eta| \leq e^{\alpha t}] \right.$$

$$\left. \times (1/h(s)) \sum_{n=0}^{\infty} Q^{*n}(r, ds \times dt - u) \right\} G(dr \times du)$$

$$= \int_{S\times\mathbb{R}_+} e^{-\alpha t} E_s[\eta^2; |\eta| \leq e^{\alpha t}] \pi(ds)\, dt < \infty,$$

as we have already seen. Hence a.e. $[G]$,

$$\infty > \int_{S\times\mathbb{R}_+} e^{-2\alpha t} E_s[\eta^2; |\eta| \leqslant e^{\alpha t}]\nu(r, ds\times dt - u)$$

$$= \int_{S\times\mathbb{R}_+} e^{-2\alpha(t+u)} E_s[\eta^2; |\eta| \leqslant e^{\alpha(t+u)}]\nu(r, ds\times dt)$$

$$\geqslant e^{-2\alpha u} \int_{S\times\mathbb{R}_+} e^{-2\alpha t} E_s[\eta^2; |\eta| \leqslant e^{\alpha t}]\nu(r, ds\times dt).$$

It follows that a.e. $[\pi]$,

$$\sum_{n=0}^{\infty} \mathbb{V}\mathrm{ar}_r[a_{n+1}] \leqslant \int_{S\times\mathbb{R}_+} e^{2\alpha t} E_s[\eta^2; |\eta| \leqslant e^{\alpha t}]\nu(r, ds\times dt) < \infty$$

and by a basic martingale result (Neveu, 1975, p. 68 e.g.),

$$\sup_n \sum_{n=0}^{n} a_{n+1}$$

is in $L^2[\mathbb{P}_r]$, $r \in S[\pi]$.

We proceed to $\{b_{n+1}\}$ in the decomposition:

$$0 \leqslant \sum_n |b_{n+1}| \leqslant \sum_{x\in I} e^{-\alpha\sigma_x} |\eta_x| 1_{\{|\eta_x| > e^{\alpha\sigma_x}\}} |\delta(\rho_x, \sigma_x)|,$$

to be called ζ. Since

$$|\delta(\rho_x, \sigma_x)| \leqslant e^{-\alpha\sigma_x} E_{\rho_x}[|\eta|; |\eta| > e^{\alpha t}]|_{t=\sigma_x} = \mathbb{E}[e^{-\alpha\sigma_x} |\eta_x| 1_{\{|\eta_x| > e^{\alpha\sigma_x}\}} | \mathcal{F}_x],$$

it follows that

$$\mathbb{E}_r[\zeta] \leqslant 2\mathbb{E}_r\left[\sum_{x\in I} e^{-\alpha\sigma_x} |\eta_x| 1_{\{|\eta_x| > e^{\alpha\sigma_x}\}}\right]$$

$$= 2\int_{S\times\mathbb{R}_+} e^{-\alpha t} E_s[|\eta|; |\eta| > e^{\alpha t}]\nu(r, ds\times dt).$$

As before, if integration with respect to $e^{-\alpha t}\nu(r, ds\times dt)$ could be replaced by integration with respect to the limit measure we would have an expression proportional to

$$\int_{S\times\mathbb{R}_+} E_s[|\eta|; |\eta| > e^{\alpha t}]\pi(ds)\, dt = \int_S E_s[|\eta| \log|\eta|; |\eta| > 1]\pi(ds)/\alpha$$

$$\leqslant 2\int_S E_s[|\xi| \log|\xi|]\pi(ds),$$

since $|\eta| \leqslant \bar{\xi} + h(s)$, $E_s[\bar{\xi}] = h(s)$, and by Jensen's inequality. Thus, through convolution with G we may again conclude that a.e. $[G]$,

$$\int_{S\times\mathbb{R}_+} e^{-\alpha(t+u)} E_s[|\eta|; |\eta| > e^{\alpha(t+u)}]\nu(r, ds\times dt) < \infty.$$

But

$$E_s[|\eta|; |\eta| > e^{\alpha(t+u)}] = E_s[|\eta|; |\eta| > e^{\alpha t}] - E_s[|\eta|; e^{\alpha t} < |\eta| \leq e^{\alpha(t+u)}]$$
$$\geq E_s[|\eta|; |\eta| > e^{\alpha t}] - E_s[\eta^2; |\eta| \leq e^{\alpha(t+u)}]\, e^{-\alpha t}$$
$$= E_s[|\eta|; |\eta| > e^{\alpha t}] - E_s[\eta^2; |\eta| > e^{\alpha(t+u)}]\, e^{-\alpha(t+u)}\, e^{\alpha u}.$$

We have already seen that the second term has an integral that is a.e. $[G]$ finite. It follows that again a.e. $[\pi]$,

$$\mathbb{E}_r\left[\sum_n |b_{n+1}|\right] \leq 2\int_{S\times\mathbb{R}_+} e^{\alpha t} E_s[|\eta|; |\eta| > e^{\alpha t}]\nu(r, ds \times dt) < \infty.$$

Finally,

$$0 \leq \sup_n w_{N^n} \leq h(\rho_0) + \sup_n \left|\sum_{k=1}^{n} a_k\right| + \sum_{n=1}^{\infty} |b_n|$$

and thus must be $\mathbb{P}_s$ integrable for almost all $s \in S$ $[\pi]$. □

From one-type branching processes, it is clear that $x \log x$ is the best possible condition. A blemish on our theorem is however that its conclusion is only valid for s in an unspecified subset of S, albeit of full π-measure. In default of anything better this can be mitigated by an "$x(\log x)^{1+\varepsilon}$" condition plus spread-out-ness:

Theorem 6.2. *If the population is Malthusian and spread out and satisfies*

$$E_\pi[\bar{\xi}(\log^+ \bar{\xi})^{1+\varepsilon}] < \infty$$

for some $\varepsilon > 0$, *then* $\{w_L;\, L \in \mathscr{C}_0\}$ *is uniformly integrable with respect to any* $\mathbb{P}_s$, $h(s) < \infty$.

Sketch of proof. The procedure is as in the preceding proof up to the two integrals

$$\int_{S\times\mathbb{R}_+} e^{-2\alpha t} E_s[\eta^2; |\eta| \leq e^{\alpha t}]\nu(r, ds \times dt)$$

and

$$\int_{S\times\mathbb{R}_+} e^{-\alpha t} E_s[|\eta|]; |\eta| > e^{\alpha t}]\nu(r, ds \times dt).$$

Since $x/(\log x)^{1+\varepsilon}$ increases for $x \geq e$,

$$e^{-\alpha t} E_s[\eta^2; |\eta| \leq e^{\alpha t}] \leq e^{2-\alpha t} + E_s[|\eta|(\log^+ |\eta|)^{1+\varepsilon}]/(\alpha t)^{1+\varepsilon}.$$

Also

$$E_s[|\eta|; |\eta| > e^{\alpha t}] \leq E_s[|\eta|(\log^+ |\eta|)^{1+\varepsilon}]/(\alpha t)^{1+\varepsilon}.$$

The convergences

$$\nu_\alpha(r, A \times (B+n)) \to h(r)\pi(A)\lambda(B)/\beta,$$

valid for any $A \in \mathscr{S}$, $B \in \mathscr{B}$ if $h(r) < \infty$ (Niemi and Nummelin, 1986), conclude the proof. □

Under the conditions of these theorems L^1-convergence for martingales with a directed index set (Helms, 1958, cf. Neveu, 1975, p. 96) yields that for almost all $s \in S$, or all s with $h(s) < \infty$, there is a w_s such that

$$w_L = \mathbb{E}_s[w_s \mid \mathcal{F}_L],$$

and as L filters to the right ($<$) in $\mathscr{C}_0$

$$w_L \to w_s$$

in $L^1(\mathbb{P}_s)$. In particular this holds for any sequence $\{w_{L_n}\}$, with $L_n < L_{n+1}$ and $\inf_{L_n} g(x) \to \infty$, which also must converge a.s. $\mathbb{P}_s$, as $n \to \infty$. $L^1(\mathbb{P}_s)$-convergence as L filters to the right ($<$) of course means that to any $\varepsilon > 0$ there is an L_ε such that $L_\varepsilon < L \Rightarrow \mathbb{E}_s|w_L - w_s| < \varepsilon$.

It may seem discomforting that there is a w_s for each $\mathbb{P}_s$. However w_s is a function $S \times \Omega^I \to \mathbb{R}_+$ and if we can show that s only enters this way w_s can be identified with a w on $S \times \Omega^I, \mathbb{P}_s$ forcing the first coordinate of the argument to be s. And, actually,

$$\liminf_{n \to \infty} w_{N^n} := w$$

exists everywhere, $\{w_{N^n}\}$ is a non-negative martingale under any $\mathbb{P}_s$, and therefore $\mathbb{P}_s(w \neq w_s) = 0$. These last conclusions deserve to be summarized:

Theorem 6.3. *Under the conditions of Theorem* 6.1 (*or* 6.2) *there is a random variable* $w \geq 0$ *such that for* π*-almost all* $s \in S$ (*or all* s *such that* $h(s) < \infty$)

$$w_L = \mathbb{E}_s[w \mid \mathcal{F}_L]$$

and $w_L \xrightarrow{L^1(\mathbb{P}_s)} w$, *as* $L \in \mathscr{C}_0$ *filters* ($<$). *If* $L_n < L_{n+1} \in \mathscr{C}_0$ *and to any* $x \in I$ *there is an* L_n *such that* x *has progeny in* L_n, $w_{L_n} \to w$, *as* $n \to \infty$, *also a.s.* $\mathbb{P}_s$. □

As hinted in its presentation, the intrinsic martingale has particular interest when evaluated at certain random sets of individuals, like the coming generation at t, $\mathcal{I}_t$, or the (realized) nth generation, $\mathcal{X}_n = N^n \cap \mathcal{R}$.

Lemma 6.4. *Let* $\mathcal{I}$ *and* L *be an optional and a fix stopping line, respectively. Then, on* $\{\mathcal{I} < L \cap \mathcal{R}\}, \mathbb{E}_s[w_L \mid \mathcal{F}_{\mathcal{I}}] \leq w_{\mathcal{I}}$. *If* L *covers* $\mathcal{I}$ *and* $L \cap \mathcal{R}$ *is finite, then equality holds.*

Note. $N^n \cap \mathcal{R}$, e.g., is finite if only $P_s(\xi(S \times \mathbb{R}_+) < \infty) = 1$, $s \in S$. Weak conditions on the reproductions kernel yield the finiteness of

$$\nu(s, S \times [0, t]) = \mathbb{E}_s[\# \mathcal{Y}_t].$$

The finiteness of any $\mathcal{Y}_t$ combined with that of $\xi(S \times R_+)$ forces $\mathcal{I}_t$ to be finite, as well.

Proof. First assume $\mathscr{I}$ finite and use that the set fI of finite subsets of I has only countably many members. Corollary 4.13 and Theorem 5.1 yield

$$\mathbb{E}[w_L \mid \mathscr{F}_{\mathscr{I}}]1_{\{\mathscr{I}<L\cap\mathscr{R}\}} = \sum_{\substack{M\in fI\\ M<L\cap\mathscr{R}}} \mathbb{E}[w_L \mid \mathscr{F}_M]1_{\{\mathscr{I}=M\}} \overset{\leqslant}{(=)} \sum_{\substack{M\in fI\\ M<L\cap\mathscr{R}}} w_M 1_{\{\mathscr{I}=M\}}$$

$$= w_{\mathscr{I}} 1_{\{\mathscr{I}<L\cap\mathscr{R}\}},$$

equality holding in the covering case.

For $\mathscr{I}$ general, let $K_n \uparrow I$ be finite sets and write $\mathscr{I}\cap K_n = \mathscr{I}_n$. Then $\mathscr{I}<\mathscr{I}_{n+1}<\mathscr{I}_n$ and (Lemma 4.9)

$$\mathscr{F}_{\mathscr{I}} = \bigcap_n \mathscr{F}_{\mathscr{I}_n}.$$

By backwards martingale convergence (Neveu, 1975, p. 118),

$$\mathbb{E}[w_L \mid \mathscr{F}_{\mathscr{I}_n}] \to \mathbb{E}[w_L \mid \mathscr{F}_{\mathscr{I}}].$$

But since $L\cap\mathscr{R}$ is finite, $\{\mathscr{I}_n < L\cap\mathscr{R}\}\uparrow\{\mathscr{I}<L\cap\mathscr{R}\}$. Hence

$$\mathbb{E}[w_L \mid \mathscr{F}_{\mathscr{I}}]1_{\{\mathscr{I}<L\cap\mathscr{R}\}} \leftarrow \mathbb{E}[w_L \mid \mathscr{F}_{\mathscr{I}_n}]1_{\{\mathscr{I}_n<L\cap\mathscr{R}\}} \overset{\leqslant}{(=)} w_{\mathscr{I}} 1_{\{\mathscr{I}_n<L\cap\mathscr{R}\}},$$

$$\to w_{\mathscr{I}} 1_{\{\mathscr{I}<L\cap\mathscr{R}\}},$$

with equality if L covers $\mathscr{I}$. □

We shall now widen the definition of covering slightly, saying that $\mathscr{I}$ covers $\mathscr{J}$ if $\mathscr{I}$ covers $\mathscr{J}\cap\mathscr{R}$ in the old sense. Thus only realized individuals enter the definition.

Theorem 6.5. *Assume the conditions of Theorem* 6.1 *and that, for all* $s\in S$, $\mathbb{P}_s(\xi(S\times\mathbb{R}_+)<\infty)=1$. *If* $\mathscr{I}$ *is a covering optional stopping line with* $g(\mathscr{I})<\infty$, *then for* π*-almost all* $s\in S$,

$$\mathbb{E}_s[w \mid \mathscr{F}_{\mathscr{I}}] = w_{\mathscr{I}}\,[\mathbb{P}_s].$$

Proof. If $g(\mathscr{I})\leqslant n$, then certainly $\mathscr{I}<N^n\cap\mathscr{R}$. Hence by Lemma 6.4 in the case of equality

$$\mathbb{E}[w_{N^n} \mid \mathscr{F}_{\mathscr{I}}]1_{\{g(\mathscr{I})\leqslant n\}} = w_{\mathscr{I}} 1_{\{g(\mathscr{I})\leqslant n\}}.$$

As $n\to\infty$, $1_{\{g(\mathscr{I})\leqslant n\}}\uparrow 1$ and $w_{N^n}\to w$ a.s. and in L^1, yielding

$$\mathbb{E}[w \mid \mathscr{F}_{\mathscr{I}}] = w_{\mathscr{I}}.$$

(We allow ourselves to omit the a.e. π and a.s. $\mathbb{P}_s$ qualifications.) □

Corollary 6.6. *Under the assumptions of the preceding theorem* $w_{\mathscr{I}}\to w$ *(almost all* $L^1(\mathbb{P}_s)$, $s\in S$*) as* $\mathscr{I}$ *filters* ($<$) *in the class of covering optional stopping lines with* $g(\mathscr{I})<\infty$. *If* $\{\mathscr{I}_n\}$ *is a sequence of such elements* $\mathscr{I}_n<\mathscr{I}_{n+1}$ *tending to infinity in the sense that for any* $x\in\mathscr{R}$ *there is an* n *such that* x *has progeny in* $\mathscr{I}_n$, *then* $w_{\mathscr{I}_n}\to w$ *a.s.* $[\mathbb{P}_s]$ *and in* $L^1(\mathbb{P}_s)$, $s\in S[\pi]$. □

Lemma 6.4 makes it possible to exhibit the submartingale structure also at optional stopping lines:

Theorem 6.7. *If $\mathcal{I} < \mathcal{J}$ are covering optional stopping lines with $g(\mathcal{J}) < \infty$, then*

$$\mathbb{E}[w_{\mathcal{J}} \mid \mathcal{F}_{\mathcal{I}}] \leq w_{\mathcal{I}}.$$

(Under the conditions of Theorem 6.5 equality holds.)

Proof. Arguing as above we write

$$\begin{aligned} w_{\mathcal{I}} 1_{\{g(\mathcal{I}) \leq n\}} &= \mathbb{E}[w_{N^n} \mid \mathcal{F}_{\mathcal{I}}] 1_{\{g(\mathcal{I}) \leq n\}} = \mathbb{E}[\mathbb{E}[w_{N^n} \mid \mathcal{F}_{\mathcal{J}}] \mid \mathcal{F}_{\mathcal{I}}] 1_{\{g(\mathcal{I}) \leq n\}} \\ &= \mathbb{E}[w_{\mathcal{J}} 1_{\{g(\mathcal{J}) \leq n\}} \mid \mathcal{F}_{\mathcal{I}}] 1_{\{g(\mathcal{I}) \leq n\}} + \mathbb{E}[\mathbb{E}[w_{N^n} \mid \mathcal{F}_{\mathcal{J}}] 1_{\{g(\mathcal{J}) > n\}} \mid \mathcal{F}_{\mathcal{I}}]. \end{aligned}$$

Let $n \to \infty$. □

General optional sampling theorems for martingales with partially ordered indices have been given by Kurtz (1980) and Hürzeler (1986).

7. Asymptotics in real time

The—non-Markovian—development in real time of some aspect of a branching population (like the total number of births, $y_t = \#\mathcal{Y}_t$, the number of individuals "alive", or the number below some age at time t may be followed through so called random characteristics (Jagers, 1975, Jagers and Nerman, 1984). These are additive functionals formed from a measurable function

$$\chi : S \times \Omega^I \times \mathbb{R} \to \mathbb{R}_+,$$

the *characteristic*. This one is assumed to vanish if its last argument (interpreted as age) is negative (or $-\infty$), and also to have realizations which are right continuous and have left limits in this argument (i.e. are D-valued). The *χ-counted population* at time t, z_t^χ, is then defined by

$$z_t^\chi := \sum_{x \in I} \chi(S_x, t - \sigma_x) = \sum_{x \in I} \chi_x(t - \sigma_x).$$

Thus χ_x is the caracteristic pertaining to the individual $x \in I$. But note that χ_x need not be determined by x's life only. Characteristics which are so, i.e. can be written $\chi(\rho_x U_x, t - \sigma_x)$, are called *individual*. For examples of processes thus arising, cf. Jagers (1975).

If $\mathcal{I}$ is an optional stopping line and x an individual, we shall write $x < \mathcal{I}$ if x precedes $\mathcal{I}$ strictly in the sense that it does not itself belong to $\mathcal{I}$ but has descendants in it. For any covering $\mathcal{I}$ a *fundamental equation* holds,

$$z_t^\chi = \sum_{x < \mathcal{I}} \chi(S_x, t - \sigma_x) + \sum_{x \in \mathcal{I}} z_{t-\sigma_x}^\chi \circ S_x.$$

We assume throughout this section that the population is Malthusian. Then we can norm the fundamental equation by $e^{\alpha t}h(s)$. With notation

$$\zeta_t = e^{-\alpha t} z_t^\chi / h(\rho_0), \qquad \varphi_{\mathcal{J}}(t) = \sum_{x < \mathcal{J}} \chi(S_x, t - \sigma_x)/h(\rho_0),$$

we then obtain that

$$\zeta_t = \varphi_{\mathcal{J}}(t) + \sum_{x \in \mathcal{J}} \zeta_{t-\sigma_x} \circ S_x \, e^{-\alpha\sigma_x} h(\rho_x)/h(\rho_0).$$

In terms of

$$w_M(A \times B) = \sum_{\substack{x \in M \\ \rho_x \in A, \sigma_x \in B}} e^{-\alpha\sigma_x} h(\rho_x)/h(\rho_0).$$

this leads on to

$$\mathbb{E}[\zeta_t \mid \mathcal{F}_{\mathcal{J}}] = e^{-\alpha t} E[\varphi_{\mathcal{J}}(t) \mid \mathcal{F}_{\mathcal{J}}] + \int_{S \times \mathbb{R}_+} \mathbb{E}_s[\zeta_{t-u}] w_{\mathcal{J}}(ds \times du)$$

and the Markov renewal equation holding for means,

$$\mathbb{E}_r[\zeta_t] = e^{-\alpha t}\mathbb{E}_r[\varphi_{\mathcal{J}}(t)] + \int_{S \times \mathbb{R}_+} \mathbb{E}_s[\zeta_{t-u}]\mathbb{E}_r[w_{\mathcal{J}}(ds \times du)].$$

A classical form of this is obtained by the choice $\mathbb{Z}_1 = N \cap \mathcal{R}$ for $\mathcal{J}$. Then

$$\begin{aligned} \varphi_{\mathcal{J}} &= \chi/h(\rho_0), \; w_{\mathcal{J}}(ds \times du) \\ &= e^{-\alpha u} h(s)\xi(ds \times du)/h(\rho_0), \mathbb{E}_r[w_{\mathcal{J}}(ds \times du)] \\ &= e^{-\alpha u} h(s)\mu(r, ds \times du)/h(r). \end{aligned}$$

Analysis of this, or direct arguments as in Nerman (1984), yields the asymptotic mean behaviour, as time passes:

Theorem 7.1. *Consider a spread-out supercritical Malthusian branching population counted by a characteristic χ such that $\sup_t e^{-\alpha t}\mathbb{E}_s[\chi(t)]$ is integrable with respect to π, $e^{-\alpha t}\mathbb{E}_s[\chi(t)]$ is so with respect to $\pi \otimes \lambda$ (λ being Lebesgue measure on $\mathbb{R}_+$), and $\lim_{t\to\infty} e^{-\alpha t}\mathbb{E}_s[\chi(t)] = 0$. Then, for π-almost all $s \in S$, as $t \to \infty$,*

$$e^{-\alpha t}\mathbb{E}_s[z_t^\chi] \to h(s) \int_{S \times \mathbb{R}_+} e^{-\alpha u} \varepsilon_r[\chi(u)]\pi(dr)\, du/\beta := h(s)\mathbb{E}_\pi[\hat{\chi}(\alpha)]/\alpha\beta,$$

in an obvious notation for Laplace transform. (The mean age at childbearing, β, was introduced in Section 5.)

Note. If $\sup_{s,t} e^{-\alpha t}\mathbb{E}_s[\chi(t)]/h(s) < \infty$, then the first integrability condition holds, as $\int h \, d\pi = 1$. If $\sup_{s,t} \mathbb{E}_s[\chi(t)]/h(s) < \infty$ also the second must be satisfied. As pointed out, the theorem can be reformulated, essentially by replacing spread-out-ness by direct Riemann integrability of expected characteristics (cf. Shurenkov, 1984).

Note also that the convergence $e^{-\alpha t}\mathbb{E}_s[y_t] \to h(s)\pi(S)/\alpha\beta(y_t = z_t^{1_{\mathbb{R}_+}})$ does not follow from this, unless $\inf h > 0$.

Proof. In terms of

$$Y(A \times B) = \#\{x \in I;\ \rho_x \in A,\ \sigma_x \in B\}$$

we can write

$$\mathbb{E}_r[z_t^\chi] = \mathbb{E}_r\left[\sum_{x \in I} \mathbb{E}[\chi_x(t-\sigma_x) \mid \mathscr{F}_x]\right] = \mathbb{E}_r\left[\sum_{x \in I} \mathbb{E}_{\rho_x}[\chi(t-\sigma_x)]\right]$$

$$= \mathbb{E}_r\left[\int_{S \times \mathbb{R}_+} \mathbb{E}_s[\chi(t-u)]\, Y(\mathrm{d}s \times \mathrm{d}u)\right]$$

$$= \int_{S \times \mathbb{R}_+} \mathbb{E}_s[\chi(t-u)]\, \nu(r, \mathrm{d}s \times \mathrm{d}u).$$

Hence

$$\mathrm{e}^{-\alpha t}\mathbb{E}_r[z_t^\chi] = \int_{S \times \mathbb{R}_+} \mathrm{e}^{-\alpha(t-u)}\mathbb{E}_s[\chi(t-u)]\, \nu(r, \mathrm{d}s \times \mathrm{d}u)$$

and the assertion follows from Niemi and Nummelin (1986) and Nummelin (1978, p. 133). □

From this first asymptotic result we proceed to weak L^1-convergence, then convergence in distribution, in probability and, finally, in L^1. (Weak L^1-convergence may not be a terribly popular concept in probability theory—it means that expectations over any set converge, cf. Neveu, 1965, p. 118.)

Theorem 7.2. *Add $x \log x$, finiteness of $\xi(S \times \mathbb{R}_+)$, and $\inf h > 0$ to the assumptions of Theorem 7.1 and assume that, for each fixed t, y_t is uniformly integrable over its starting type $\rho_0 = s \in S$. Then, as $t \to \infty$,*

$$\mathrm{e}^{-\alpha t} z_t^\chi \to h(s) w \mathbb{E}_\pi[\hat{\chi}(\alpha)]/\alpha\beta$$

weakly in $L^1(\mathbb{P}_s)$ for almost all $s \in S$ $[\pi]$.

Note. Under our general assumptions the uniform integrability condition that, for fixed t,

$$\lim_{u \to \infty} \mathbb{E}_s[y_t;\ y_t > u] = 0$$

uniformly in s, is trivially satisfied in the single- (or finite-) type case. For short we shall refer to this condition by calling the branching population itself uniformly integrable.

Proof. Assume that $\chi(t)$ vanishes for $t > n$ and choose for $t \geq t_0 > n$, $\mathscr{I}$ as

$$\mathscr{I}_{t_0} = \{x;\ \sigma_{mx} \leq t_0 < \sigma_x < \infty\}.$$

Then, in the fundamental equation above, $\varphi_{\mathscr{I}}(t)=0$ provided $t>2t_0$ and, since $w_{\mathscr{I}}(\mathrm{d}s\times\mathrm{d}u)$ only gives mass to a finite number of points,

$$\mathbb{E}[\zeta_t\,|\,\mathscr{F}_{\mathscr{I}}]=\int \mathbb{E}_s[\zeta_{t-u}]w_{\mathscr{I}}(\mathrm{d}s\times\mathrm{d}u)\to\mathbb{E}_\pi[\hat{\chi}(\alpha)]w_{\mathscr{I}}/\alpha\beta,$$

as $t\to\infty$ (Theorem 7.1). For any $A\in\mathscr{F}_{\mathscr{I}_{t_0}}$ therefore

$$\mathbb{E}_s[\zeta_t;A]\to\mathbb{E}_\pi[\hat{\chi}(\alpha)]\mathbb{E}_s[w_{\mathscr{I}_{t_0}};A]/\alpha\beta=\mathbb{E}_\pi[\hat{\chi}(\alpha)]\mathbb{E}_s[w;A]/\alpha\beta.$$

But $\{\mathscr{F}_{\mathscr{I}_{t_0}};\,t_0>n\}$ generates all of $\mathscr{S}\times\mathscr{A}^I$.

Now assume for the time that χ is bounded by n and write $\kappa=\inf h>0$. Then, since $x>\mathscr{I}_{t-n}\Rightarrow t-\sigma_x\leq n$, we have ($\mathscr{I}=\mathscr{I}_{t-n}$)

$$\zeta_t\leq\sum_{x\in\mathscr{I}}\mathrm{e}^{-\alpha\sigma_x}h(\rho_x)ny_n\circ S_x/\kappa h(\rho_0).$$

We shall show that ζ_t is uniformly integrable. Therefore let $\varepsilon>0$ be given. By the uniform integrability of the $y_n\circ S_x$, $x\in I$, there is a $\delta>0$ such that $\mathbb{P}(A\,|\,\mathscr{F}_{\mathscr{I}})<\delta\Rightarrow\mathbb{E}[y_n\circ S_x;A\,|\,\mathscr{F}_{\mathscr{I}}]<\varepsilon\kappa/n^2$. (Recall that the $y_n\circ S_x$ given $\mathscr{I}$ are distributed like a bunch of $y_n(\rho_0)$ for different starting types ρ_0.)

Now consider an A with $\mathbb{P}_s(A)<\delta_0$, the latter to be chosen shortly. If

$$B=\{\mathbb{P}_s(A\,|\,\mathscr{F}_{\mathscr{I}})<\delta\},$$

then

$$\delta_0>\mathbb{P}_s(A)\geq\mathbb{P}_s(A\cap B)\geq\delta\mathbb{P}_s(B).$$

Hence, with prime for complement,

$$\begin{aligned}\kappa h(s)\mathbb{E}_s[\zeta_t;A]&\leq\mathbb{E}_s\left[\mathbb{E}\left[\sum_{x\in\mathscr{I}}\mathrm{e}^{-\alpha\sigma_x}h(\rho_x)ny_n\circ S_x;A\,\middle|\,\mathscr{F}_{\mathscr{I}}\right];B'\right]\\&\quad+\mathbb{E}_s\left[\mathbb{E}\left[\sum_{x\in\mathscr{I}}\mathrm{e}^{-\alpha\sigma_x}h(\rho_x)ny_n\circ S_x;A\,\middle|\,\mathscr{F}_{\mathscr{I}}\right];B\right]\\&\leq\mathbb{E}_s[\varepsilon\kappa w_{\mathscr{I}}]+n\mathbb{E}_s\left[\sum_{x\in\mathscr{I}}\mathrm{e}^{-\alpha\sigma_x}h(\rho_x)\mathbb{E}[y_n\circ S_x\,|\,\mathscr{F}_{\mathscr{I}}];B\right]\\&\leq\varepsilon\kappa h(s)+n\sup_r\mathbb{E}_r[y_n]\mathbb{E}_s[w_{\mathscr{I}};B].\end{aligned}$$

This can be made arbitrarily small by choice of δ_0 small enough (by Theorem 6.5), proving the uniform integrability $[\mathbb{P}_s]$ of ζ_t, and thus of $\mathrm{e}^{-\alpha t}z_t^\chi$, for $t\geq t_0$.

By the Dunford–Pettis theorem (Neveu, 1965, p. 118, or Dunford and Schwarz, p. 289) these $\mathrm{e}^{-\alpha t}z_t^\chi$ are weakly L^1-compact, and the weak L^1-convergence follows, for $\chi:s$ of the kind assumed.

The general case follows by truncation: Let $\varphi:S\times\Omega^I\to[0,1]$ be measurable and write for a given χ,

$$\chi^{(n)}(s,\omega,t)=1_{[0,n]}(t)\chi(s,t,\omega)\wedge n,$$

$\wedge$ for minimum. By the special case considered

$$\mathbb{E}_s[\mathrm{e}^{-\alpha t} z_t^{\chi^{(n)}} \varphi] \to h(s)\mathbb{E}_s[w\varphi]\mathbb{E}_\pi[\hat{\chi}^{(n)}(\alpha)]/\alpha\beta.$$

Writing $c\mathbb{E}_\pi[\hat{\chi}^{(n)}(\alpha)]$ for this limit, we see that

$$|\mathbb{E}_s[\mathrm{e}^{-\alpha t} z_t^{\chi} \varphi] - c\mathbb{E}_\pi[\hat{\chi}(\alpha)]| \leq \mathbb{E}_s[\mathrm{e}^{-\alpha t}(z_t^{\chi} - z_t^{\chi^{(n)}})\varphi] + |\mathbb{E}_s[\mathrm{e}^{-\alpha t} z_t^{\chi^{(n)}} \varphi] - c\mathbb{E}_\pi[\hat{\chi}^{(n)}(\alpha)]| + c\{\mathbb{E}_\pi[\hat{\chi}(\alpha)] - \mathbb{E}_\pi[\hat{\chi}^{(n)}(\alpha)]\}.$$

The first term is majorized by

$$\mathbb{E}_s[\mathrm{e}^{-\alpha t} z_t^{\chi}] - \mathbb{E}_s[\mathrm{e}^{-\alpha t} z_t^{\chi^{(n)}}] \to h(s)\mathbb{E}_\pi[\hat{\chi}(\alpha) - \hat{\chi}^{(n)}(\alpha)]/\alpha\beta$$

by Theorem 7.1. By choice of n large and monotone convergence this can be made arbitrarily small. The same applies to the last term, whereas the middle one is the case treated. □

We consider now again a χ of the special kind considered in the proof and write

$$m_s(t) = \mathbb{E}_s[\zeta_t], \qquad \eta_t = \zeta_t - m_s(t).$$

For any $t > c > n$ consider

$$\zeta_t - \mathbb{E}[\zeta_t \mid \mathscr{F}_{\mathscr{I}_{t-c}}] = \sum_{x \in \mathscr{I}_{t-c}} \{\zeta_{t-\sigma_x} \circ S_x - m_{\rho_x}(t - \sigma_x)\} \mathrm{e}^{-\alpha\sigma_x} h(\rho_x)/h(\rho_0)$$
$$= \sum_{x \in \mathscr{I}_{t-c}} \eta_{t-\sigma_x} \circ S_x \, \mathrm{e}^{-\alpha\sigma_x} h(\rho_x)/h(\rho_0).$$

This is a sum of random variables, which are independent and have expectation zero under the conditional law $\mathbb{P}(\cdot \mid \mathscr{F}_{\mathscr{I}_{t-c}})$. For any $v > 0$, $r \in S$,

$$\sum_{x \in \mathscr{I}_{t-c}} \mathbb{P}_r(|\eta_{t-\sigma_x} \circ S_x| \mathrm{e}^{-\alpha\sigma_x} h(\rho_x)/h(\rho_0) > v \mid \mathscr{F}_{\mathscr{I}_{t-c}})$$
$$= \int_{S \times \mathbb{R}_+} \mathbb{P}_s(|\eta_{t-u}| > v\, \mathrm{e}^{\alpha u} h(r)/h(s))\, \mathrm{e}^{\alpha u}/h(s) w_{\mathscr{I}_{t-c}}(\mathrm{d}s \times \mathrm{d}u)$$
$$\leq \int_{S \times \mathbb{R}_+} \mathbb{E}_s[|\eta_{t-u}|; |\eta_{t-u}| > v'\, \mathrm{e}^{\alpha u}] w_{\mathscr{I}_{t-c}}(\mathrm{d}u \times \mathrm{d}s)/v',$$

where $v' = vh(r)$. But since $\chi \leq n$, $x \in \mathscr{I}_{t-c} \Rightarrow t - \sigma_x \leq c$, and $\inf h(s) = \kappa > 0$, the η_{t-u} are uniformly integrable. Thus for $\varepsilon > 0$ given choose t large enough, so that $u = \sigma_x \geq t - c$, $x \in \mathscr{I}_{t-c}$ and, for all $r \in S$,

$$\mathbb{E}_r[|\eta_{t-u}|; |\eta_{t-u}| > v'\, \mathrm{e}^{\alpha u}] < \varepsilon v'.$$

Then

$$\sum_{x \in \mathscr{I}_{t-c}} \mathbb{P}_r(|\eta_{t-s_x} \circ S_x| \mathrm{e}^{-\alpha\sigma_x} h(\rho_x)/h(\rho_0) > v \mid \mathscr{F}_{\mathscr{I}_{t-c}}) < \varepsilon w_{\mathscr{I}_{t-c}}.$$

By the a.s. convergence of any subsequence of $\{w_{\mathscr{I}_t};\ t \geq 0\}$, any subsequence of $\{\zeta_t\}$ under $\mathbb{P}(\mid \mathscr{F}_{\mathscr{I}_{t-c}})$ is not only a sum of uniformly asymptotically negligible summands but must even have canonical limit measures without mass outside $[-v, v]$. And since v was arbitrary, the only possible limits are infinitely divisible with canonical measures, which can only have mass in the origin (cf. Feller, pp. 583–588 or Loève, in particular p. 317).

To check this mass consider, for any $v>0$,

$$\sum_{x\in\mathscr{I}_{t-c}} \mathbb{E}[\eta_{t-c}^2\, e^{-2\alpha\sigma_x} h^2(\rho_x)/h^2(\rho_0);\, |\eta_{t-\sigma_x}| e^{-\alpha\sigma_x} h(\rho_x)/h(\rho_0)\leqslant v\,|\,\mathscr{F}_{\mathscr{I}_{t-c}}]$$

$$\leqslant v \sum_{x\in\mathscr{I}_{t-c}} \mathbb{E}[|\eta_{t-\sigma_x}|\,|\,\mathscr{F}_{\mathscr{I}_{t-c}}]\, e^{-\alpha\sigma_x} h(\rho_x)/h(\rho_0)$$

$$\leqslant v \int_{S\times\mathbb{R}_+} \mathbb{E}_s[|\eta_{t-u}|]\, w_{\mathscr{I}_{t-c}}(ds\times du)/h(\rho_0)$$

$$\leqslant vn \int_S \mathbb{E}_s[y_c]\, w_{\mathscr{I}_{t-c}}(ds\times\mathbb{R})/h(\rho_0)\leqslant vn \sup_{s\in S}\mathbb{E}_s[y_c]\, w_{\mathscr{I}_{t-c}}/h(\rho_0).$$

Since $\mathbb{E}_s[y_c]$ is bounded by the uniform integrability assumption, it follows that no weak limit of $\eta_t|\mathscr{F}_{t-c}$, $t\to\infty$, can have any mass canonical measure mass at the origin either. Thus, for almost all $s\in S[\pi]$, $u\in\mathbb{R}$,

$$\mathbb{P}_s[\zeta_t-\mathbb{E}[\zeta_t\,|\,\mathscr{F}_{t-c}]\leqslant u)=\mathbb{E}_s[\mathbb{P}_s(\eta_t\leqslant u\,|\,\mathscr{F}_{t-c})]\to 1_{\mathbb{R}_+}(u).$$

It follows that, for π-almost all $s\in S$ and any $c\geqslant n$,

$$\zeta_t-\mathbb{E}[\zeta_t\,|\,\mathscr{F}_{t-c}]\to 0$$

in $\mathbb{P}_s$-probability.

However, if we write

$$\gamma := \mathbb{E}[\hat{\chi}(\alpha)]/\alpha\beta$$

for $\lim_{t\to\infty} m_s(t)$ by Theorem 7.1, then

$$|\mathbb{E}[\zeta_t\,|\,\mathscr{F}_{\mathscr{I}_{t-c}}]-\gamma w|\leqslant \int_{S\times\mathbb{R}_+} |m_s(t-u)-\gamma|\, w_{\mathscr{I}_{t-c}}(ds\times du)+\gamma|w_{\mathscr{I}_{t-c}}-w|.$$

In terms of

$$R=\sum_{n=0}^{\infty} Q^{*n},$$

$$R(s, A\times B)=\mathbb{E}_s[w_{\mathscr{Y}_t}(A\times B)]/h(s),$$

the expectation on $w_{\mathscr{I}_t}$ satisfies

$$\mathbb{E}_s[w_{\mathscr{I}_t}(A\times B)]/h(s)=\int_{S\times[0,t]} Q(r, A\times B\cap(t-u,\infty))R(s, dr\times du).$$

By strong recurrence (and the other conditions for the Markov renewal theory already assumed) this converges as $t\to\infty$ to

$$\int Q(s, A\times B\cap(t,\infty))h(s)\pi(ds)\, dt/\beta$$

(Nummelin, 1978, p. 133) and if we write τ for $t-c$,

$$\int_{S\times\mathbb{R}_+} |m_s(\tau+c-u)-\gamma|\mathbb{E}_r[w_{\mathscr{S}_r}(\mathrm{d}s\times\mathrm{d}u]$$

can be made arbitrarily small for large τ by choice of c large. By Corollary 6.6 the same is true for π-almost all $L^1(\mathbb{P}_s)$-norms of $\mathbb{E}[\zeta_t\,|\,\mathscr{F}_{t-c}]-\gamma w$. The convergence

$$\zeta_t\to\gamma w$$

in probability follows for almost all $\mathbb{P}_s$. Since it holds also L^1-weakly by Theorem 7.2 it holds even strongly in $L^1(\mathbb{P}_s)$, a.e. π (cf. Zaanen, 1967, p. 385).

The approximation argument used in the proof of Theorem 7.2 can now be used to extend this L^1-convergence to $\mathrm{e}^{-\alpha t}z_t^\chi$ for characteristics not necessarily bounded or vanishing outside some bounded interval. In conclusion:

Theorem 7.3. *Consider a branching population and a characteristic such that the conditions of Theorem 7.2 are satisfied. Then, for π-almost all $s\in S$,*

$\mathrm{e}^{-\alpha t}z_t^\chi\to$ [redacted] $)\mathbb{E}_\pi[\hat\chi(\alpha)]w/\alpha\beta$

strongly in $L^1(\mathbb{P}_s)$, as $t\to\infty$. □

References

S. Asmussen and H. Hering, Branching Processes (Birkhäuser, Boston etc., 1983).

B. Chauvin, Arbres et processus de Bellman-Harris, Ann. Inst. H. Poincaré 22 (1986) 209–232.

B. Chauvin, Sur la proprieté de branchement, Ann. Inst. H. Poincaré 22 (1986) 233–266.

H. Cohn, A martingale approach to supercritical (CMJ) branching processes, Ann. Prob. 13 (1985) 1179–1191.

N. Dunford and J.T. Schwartz, Linear Operators, I (Interscience, New York, 1958).

W. Feller, An Introduction to Probability Theory and its Applications II (2nd ed., J. Wiley and Sons, New York etc., 1971).

D. Grey, Personal communication (1988).

L.L. Helms, Mean convergence of martingales, Trans. Am. Math. Soc. 87 (1958) 439–446.

H.E. Hürzeler, The optional sampling theorem for processes indexed by a partially ordered set, Ann. Probab. 13 (1986) 1224–1235.

J. Jacod, Theorème de renouvellement et classifications pour les chaînes semi-Markoviennes, Ann. Inst. H. Poincaré, sect. B7 (1971) 83–129.

P. Jagers, Branching Processes with Biological Applications (J. Wiley and Sons, London, New York etc., 1975).

P. Jagers, On the Malthusianness of general branching processes in abstract type spaces, in: A. Gut und L. Holst, eds., Probability and Mathematical Statistics. Essays in Honour of Carl-Gustaf Esseen (Dept. of Mathematics, Uppsala, 1983).

P. Jagers and O. Nerman, The growth and composition of branching populations, Adv. Appl. Prob. 16 (1984) 221–259.

A. Joffe, Remarks on the structure of trees with application to supercritical Galton–Watson processes, in: A. Joffe and P. Ney, ed., Advances in Probability and Related Topics. (M. Dekker, New York, 1978).

O. Kallenberg, Random Measures (3rd. ed., Akademie-Verlag, Berlin, Academic Press, London, New York etc., 1983).

T.G. Kurtz, The optional sampling theorem for martingales indexed by a directed set, Ann. Probab. 8 (1980) 675–668.

M. Loève, Probability Theory (Van Nostrand, New York, 1955).

O. Nerman, On the convergence of supercritical general (C-M-J) branching processes, Z. Wahrscheinlichkeitstheorie verw. Geb. 57 (1981) 365–395.

O. Nerman, The Growth and Composition of Supercritical Branching Populations on General Type Spaces (Dep. Mathematics Chalmers U. Tech. and Gothenburg U. 1984-18).

J. Neveu, Mathematical Foundations of the Calculus of Probability (Holden-Day, San Francisco, 1965), translation of: Bases Mathématiques du Calcul des Probabilités (Masson, Paris, 1964).

J. Neveu, Discrete-Parameter Martingales (North-Holland, Amsterdam, 1975). [Translation of: Martingales à Temps Discret (Masson, Paris, 1972).]

J. Neveu, Arbres et processus de Galton–Watson, Ann. Inst. H. Poincaré 22 (1986) 199–207.

S. Niemi and E. Nummelin, On non-singular renewal kernels with an application to a semigroup of transition kernels, Stoch. Proc. Appl. 22 (1986) 177–202.

E. Nummelin, Uniform and ratio limit theorems for Markov renewal and semi-regenerative processes on a general state space, Ann. Inst. H. Poincaré 14 (1978) 119–143.

E. Nummelin, General Irreducible Markov Chains and Non-Negative Operators (Cambridge Univ. Press, Cambridge, 1984).

V.M. Shurenkov, On Markov renewal theory, Teor. Ver. Primenen. 24 (1984) 248–278.

A.C. Zaanen, Integration (North-Holland, Amsterdam, 1967)