HOMEWORK SHEET 1 (of 3) | DUE: March 1
Classical and quantum particle systems | Hand in to J. Björnberg

(1) *[6 points]* Let $\mathbb{P}$ be an arbitrary probability measure on $\{0,1\}^n$ and $A \subseteq \{0,1\}^n$. Show that $\mathbb{P}(A)$ can be written as a linear combination of probabilities of increasing events.

(2) *[10 points]* Let $X : \{0,1\}^n \to \mathbb{R}$ and let $\mathbb{E}_p[X]$ denote the expected value of X when the bits are independent and take value 1 with probability p. Prove that the derivative

$$\frac{d}{dp}\mathbb{E}_p[X] = \sum_{i=1}^{n} \mathbb{E}_p[\Delta_i X],$$

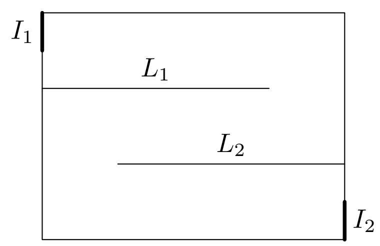
where $[\Delta_i X](\omega) = X(\omega^{[i]\to 1}) - X(\omega^{[i]\to 0})$ and

$$(\omega^{[i]\to x})_j = \begin{cases} \omega_j, & \text{if } j \neq i, \\ x, & \text{if } j = i \end{cases}$$

is obtained by setting the i:th bit of ω to x.
Hint: use the def of derivative and the coupling in Lect 1.

(3) *[6 points]* Consider percolation on the complete graph K_n with $p = n^{-2/5}$.
(a) Using Chernoff's estimates from the lectures (or otherwise) show that, with probability converging to 1, all vertices have degree at most $n^{2/3}$
(b) Show the following: with probability $\to 1$, for any pair A, B of disjoint sets of vertices satisfying $|A|, |B| \geq n^{1/2}$, there are $a \in A$ and $b \in B$ such that the edge ab is open. *Hint:* you may require Stirling's estimate.

(4) *[6 points]* Consider site-percolation on the triangular lattice $\subseteq \mathbb{C}$ (equivalently, black-white percolation on the faces of the hexagonal lattice) with mesh $\delta > 0$ and $p = \frac{1}{2}$, within the rectangle $R \subseteq \mathbb{C}$ with corners $0, 4, 3i, 4 + 3i$. Consider also the intervals $I_1 = [5i/2, 3i]$,



$I_2 = [4, 4 + i/2]$ and $L_1 = [2i, 3 + 2i]$, $L_2 = [1 + i, 4 + i]$ depicted. Show that the following probability is uniformly bounded away from 0 as $\delta \to 0$:

$$\mathbb{P}(I_1 \leftrightarrow I_2,\ I_1 \not\leftrightarrow (L_1 \cup L_2)).$$

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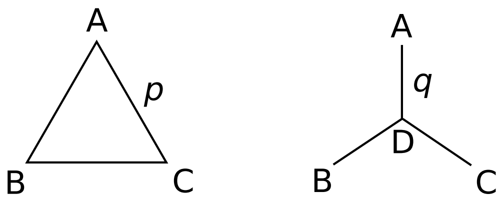
(5) *[6 points]* Still considering site-percolation on the triangular lattice $\subseteq \mathbb{C}$ with $p = \frac{1}{2}$ and now $\delta = 1$: use the estimate in the lectures for the probability $\mathbb{P}(O_n)$ of having a circuit in the annulus $A_n = \Lambda_{3n} \setminus \Lambda_{n-1}$ to show that $\theta(\frac{1}{2}) = 0$ (and thus $p_c \geq \frac{1}{2}$).

(6) *[10 points]* Let G be a connected graph containing 3 vertices A, B, C having an edge between each pair AB, AC and BC. Let G' be obtained from G by (i) deleting the edges AB, AC and BC, and then (ii) inserting a new vertex D connected to each of A, B, C (see Figure). Consider the two *bond* (edge) percolation measures: $\mathbb{P}$ on G where each edge is open with



probability p, and $\mathbb{P}'$ on G' where each edge *except* AD, BD, CD has probability p, and these 3 edges have probability $q = 1 - p$. Assume that the following relation holds:

$$3p - p^3 = 1.$$

(a) Show, for all x, y vertices of G, that $\mathbb{P}(x \leftrightarrow y) = \mathbb{P}'(x \leftrightarrow y)$.

(b) Let $\theta_{\mathbb{T}}(p)$ and $\theta_{\mathbb{H}}(p)$ denote the bond-percolation-probabilities on the triangular and hexagonal lattices $\subseteq \mathbb{C}$, respectively. Deduce that $\theta_{\mathbb{T}}(p) = \theta_{\mathbb{H}}(1 - p)$.

(7) *[6 points]* Write T for the triangle in $\mathbb{C}$ with vertices $0, 1, e^{i\pi/3}$ and T' for the triangle with vertices $1, \tau, \tau^2$ (where $\tau = e^{2\pi i/3}$). Consider the analytic function

$$G_2(z) = H_1(z) + \tau H_\tau(z) + \tau^2 H_{\tau^2}(z), \qquad z \in T$$

discussed in the lectures.

(a) Show that $G_2(z)$ is a convex combination of $1, \tau, \tau^2$, that G_2 maps the boundary of T to the boundary of T', and maps the vertices of T to the vertices of T'.

(b) Assuming the fact that there is a unique such function (a consequence of the Riemann mapping theorem) write an explicit formula for $G_2(z)$ and hence verify that $H_{\tau^2}(z) = \frac{2}{\sqrt{3}}\mathrm{Im}(z)$.