(1) *[6 points]* Consider the random-transposition process associated with the ('$S=0$') XXX model on a graph $G=(V,E)$. Thus, the edges carry independent, rate 1 Poisson processes of transpositions ('crosses') and we consider the disjoint cycles of the resulting random permutation at time β.

(a) If $G=K_n$, the complete graph, and $\beta=0.99/n$, show that there is a constant $C>0$ such that

$$\mathbb{P}\big(|\gamma_1|\le C\log n\big)\to 1, \quad \text{as } n\to\infty,$$

where γ_1 is the largest cycle.

(b) If $G=\mathbb{Z}$, the line, and $\beta>0$, show that $\mathbb{P}\big(|\gamma(0)|=\infty\big)=0$ where $\gamma(0)$ is the cycle containing the origin 0.

(2) *[6 points]* Consider the XXZ spin chain in an external magnetic field of strength $h\in\mathbb{R}$, which has Hamiltonian

$$H'(J,\Delta,h)=H_{\mathrm{XXZ}}(J,\Delta)-h\,S^z\in\mathrm{End}(\mathcal{H}), \qquad S^z=\sum_{l\in\mathbb{Z}_L}S^z_l\,.$$

(a) Use symmetry arguments, like in the lecture, to obtain the full spectrum in the zero- and one-particle sectors.

(b) Work out the details for the coordinate Bethe ansatz (CBA) to characterize the spectrum in the two-particle sector, $M=2$, in terms of the quasimomenta p_1 and p_2.
Hint: The case $h=0$ can be found in Appendix B of the lecture notes; be aware of the typo, and make sure to explain everything in your own words.

(3) *[12 points]* Consider the Bethe-ansatz equations (BAE) for the XXZ spin chain,

$$e^{i\,p_m L}=\prod_{\substack{n=1\\ n\neq m}}^{M}S(p_n,p_m)\,, \qquad S(p,p'):=-\frac{1-2\,\Delta\,e^{i\,p'}+e^{i\,(p+p')}}{1-2\,\Delta\,e^{i\,p}+e^{i\,(p+p')}}\,, \qquad 1\le m\le M\,.$$

(a) Assume we are given a parametrization of a, b, c of the six vertex weights that is entire in the spectral parameter u; then so should be the eigenvalues

$$\Lambda_M(a,b,c;\vec{z}\,)=a^L\prod_{m=1}^{M}\frac{b\,(a-b\,z_m)+c^2 z_m}{a\,(a-b\,z_m)}+b^L\prod_{m=1}^{M}\frac{a\,(a-b\,z_m)-c^2}{b\,(a-b\,z_m)}$$

of the transfer matrix, where we abbreviate $z_m:=\mathrm{e}^{i\,p_m}$. Show that the BAE guarantee that $\Lambda_M(a,b,c;\vec{z}\,)$ is non singular in u.
Hint: Write $f(u):=\Lambda_M\big(a(u),b(u),c(u);\vec{z}\,\big)$ in the form $f(u)=g(u)/h(u)$ for g and h polynomial in the vertex weights. Eliminate c in favour of $\Delta:=(a^2+b^2-c^2)/(2\,a\,b)$. For any zero u_* of h, i.e. solving $a(u_*)=b(u_*)\,z_m$ for some $1\le m\le M$, show that the residue $\mathrm{Res}_{u=u_*}f=g(u_*)/h'(u_*)$ vanishes due to the mth Bethe-ansatz equation for $\vec{z}$. (You may assume that u_* is a simple zero. It might be convenient to write $b_*:=b(u_*)$.)

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In the remainder of this exercise we consider the completely isotropic (XXX) case $\Delta = 1$.

(b) Show that in terms of *rapidities* $\lambda_m \coloneqq \frac{1}{2}\cot\frac{p_m}{2}$ the BAE read

$$\left(\frac{\lambda_m + i/2}{\lambda_m - i/2}\right)^L = \prod_{\substack{n=1\\n\neq m}}^{M} \frac{\lambda_m - \lambda_n + i}{\lambda_m - \lambda_n - i}\,, \qquad 1 \leq m \leq M\,.$$

Focus on the two-particle sector, $M = 2$.

(c) Argue that *bound states*, with $\mathrm{Im}(p_1) = -\mathrm{Im}(p_2) \neq 0$, are given by $\lambda_1 = \lambda + i/2 + \cdots$, $\lambda_2 = \lambda - i/2 + \cdots$ for some $\lambda \in \mathbb{R}$, where the dots represent terms that should vanish in the thermodynamic limit $L \to \infty$.

(d) Show that such a bound state contributes $\varepsilon_2(p_1, p_2) = 1/(\lambda^2 + 1) + \cdots$ to the energy.

(4) *[8 points]* Consider the six-vertex model on $\mathbb{Z}_K \times \mathbb{Z}_L$ for K and L even, and suppose that $c \gg a, b$. Use the graphical notation to verify that to ninth order in a, b the partition function is given by

$$\tfrac{1}{2}Z = 1 + N\,a^2\,b^2 + N\,a^2\,b^2\,(a^2+b^2) + \tfrac{1}{2}N(N+1)\,a^4\,b^4 + N\,a^2\,b^2\,(a^4+b^4) + \cdots\,,$$

where $N \coloneqq K\,L$, we have set $c = 1$ for convenience, and the dots represent higher-order terms in a, b.

Hint: For $c \gg a, b$, there are two configurations of maximal weight, each involving a chequerboard-like pattern of the two vertices of weight c (see Figure). You can focus on one such ground-state configuration in order to calculate the leading behaviour of $Z/2$.

(5) *[12 points]* Parametrize the vertex weights as $a(u) = r\,\sinh(u+\gamma)$, $b(u) = r\,\sinh u$, $c(u) = r\,\sinh\gamma$, so that $\tilde{\Delta}(a,b,c) = \cosh\gamma = \Delta$. Consider the Lax operator

$$L_{al}(u) = \begin{pmatrix} a(u) & & & \\ & b(u) & c(u) & \\ & c(u) & b(u) & \\ & & & a(u) \end{pmatrix}_{al} \in \mathrm{End}(V_a \otimes V_l)\,,$$

where zeroes are suppressed, and let $P_{al} \in \mathrm{End}(V_a \otimes V_l)$ be the permutation (transposition) operator that acts by $P_{al}|\alpha,\beta\rangle = |\beta,\alpha\rangle$ on the basis vectors of $V_a \otimes V_l$.

(a) Compute $\mathrm{tr}_a\, P_{al}$ graphically as well as algebraically.

(b) Find all u_* such that $L_{al}(u_*) = c_*\,P_{al}$ and give the proportionality constant c_*.

(c) Calculate $H_0 = \log t(u_*) \in \mathrm{End}(\mathcal{H})$ in terms of the momentum operator $P = -i\log U$, where $U \in \mathrm{End}(\mathcal{H})$ is the shift operator acting as $U|l_1,\cdots,l_M\rangle = |l_1+1,\cdots,l_M+1\rangle$ on the coordinate basis, and relate your computation to the graphical version given in the lecture.

(d) Express $P_{al}\,L'_{al}(u_*)$ in terms of $\vec{S}_a \cdot_\Delta \vec{S}_l \coloneqq S^x_a\,S^x_l + S^y_a\,S^y_l + \Delta\,S^z_a\,S^z_l$.

(e) Compute

$$H_1 = \frac{\mathrm{d}}{\mathrm{d}u}\bigg|_{u=u_*} \log t(u) = t(u_*)^{-1}\,t'(u_*) \in \mathrm{End}(\mathcal{H})$$

in terms of H_{XXZ} and the pseudovacuum energy $E_0 = -J\,L\,\Delta/4$.

(f) Give a graphical version of the computation in (e).

Hint: You could draw $P_{al}\,L'_{al}(u_*)$ from (d) as a crossing, like for the Lax operator, but decorated with a thick dot at the intersection. Do not spend time trying to find a graphical representation of $\log t(u)$.

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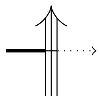
(6) *[6 points]* Consider the Yang–Baxter algebra associated to the R-matrix of the six-vertex model/XXZ spin chain,

$$R_{ab}(u-v) = \begin{pmatrix} a(u-v) & & & \\ & b(u-v) & c(u-v) & \\ & c(u-v) & b(u-v) & \\ & & & a(u-v) \end{pmatrix}_{ab} \in \mathrm{End}(V_a \otimes V_b)\,.$$

(a) Use the graphical notation to obtain the Yang–Baxter-algebra relations

$$\begin{aligned} D(u)\,B(v) &= \frac{a(u-v)}{b(u-v)}\,B(v)\,D(u) - \frac{c(u-v)}{b(u-v)}\,B(u)\,D(v)\,, \\ B(u)\,B(v) &= B(v)\,B(u) \end{aligned}$$

from the RTT-relation. For each diagram in your computation give the algebraic expression; for example

$B(u) = \; u$  $= {}_a\langle +|\,T_a(u)\,|-\rangle_a\,.$

$1 \cdots L$

(b) Use the relations from (a) to show that D acts on $|\Psi_M;\vec{u}\,\rangle = \prod_{m=1}^{M} B(u_m)\,|\Omega\rangle$ as

$$D(u_0)\,|\Psi_M;\vec{u}\,\rangle = N_0(u_0;\vec{u}\,)\,|\Psi_M;\vec{u}\,\rangle + \sum_{m=1}^{M} N_m(u_0;\vec{u}\,)\,|\Psi_M;u_0,\cdots,\widehat{u_m},\cdots,u_M\,\rangle\,,$$

where the hat indicates that the mth spectral parameter is omitted, and find the coefficients $N_0(u_0;\vec{u}\,)$ and $N_m(u_0;\vec{u}\,)$.
Hint: You can look at the computation of $A(u_0)\,|\Psi_M;\vec{u}\,\rangle$ in Section 4.3 of the lecture notes; be aware of typos, and make sure to explain the calculation in your own words.