Solutions #1

(1)

$$\mathbb{1}_A = \sum_{\omega \in A} \mathbb{1}_\omega = \sum_{\omega \in A} \left(\mathbb{1}_{\geq \omega} - \mathbb{1}_{> \omega} \right)$$

So $$\mathbb{P}(A) = \sum_{\omega \in A} \Big(\mathbb{P}(\geq \omega) - \mathbb{P}(> \omega) \Big)$$

(2)

$$\mathbb{E}_p(X) = \mathbb{E}\Big(X\big((\mathbb{1}\{U_i \leq p\})_{i=1}^n \big) \Big)$$

$h > 0$

$$\mathbb{E}_{p+h}(X) - \mathbb{E}_p(X)$$
$$= \mathbb{E}\Big[X\big((\mathbb{1}\{U_i \leq p+h\})_{i=1}^n \big) - X\big((\mathbb{1}\{U_i \leq p\})_{i=1}^n \big) \Big]$$

Consider the ~~event:~~
~~$A = \{$ exac~~ (random)
~~number $N = \# i : p < U_i$~~

Set $A = \{ i : p < U_i \leq p+h \}$

(i) $$\mathbb{P}(|A| \geq 2) = \sum_{k=2}^n \mathbb{P}(|A| = k)$$
$$= \sum_{k=2}^n \binom{n}{k} h^k (1-h)^{n-k}$$
$$= O(h^2)$$

(ii) If $A = \phi$ then integrand $= 0$.

So $$\mathbb{E}_{p+h}(X) - \mathbb{E}_p(X)$$
$$= O(h^2) + \mathbb{E}\left[\{\text{diff}\} \cdot \mathbb{1}_{|A|=1} \right]$$
$$= O(h^2) + \sum_{i=1}^n \mathbb{E}\left[\{\text{diff}\} \, \mathbb{1}_{A = \{i\}} \right]$$

~~If $A = \{i\}$ then~~

$$= \cancel{O(h^2) + \sum_{i=1}^{n} \mathbb{E}\left[\{\text{diff}\} \mid A = \{i\} \right] \cdot h(1-h)^{n-1}}$$

$$= O(h^2) + \sum_{i=1}^{n} \sum_{\omega \in \{0,1\}^{[n]\setminus\{i\}}} [\Delta_i X](\omega) \cdot h \cdot p^{|\omega|}(1-p)^{n-1-|\omega|}$$

$$= O(h^2) + h \sum_{i=1}^{n} \mathbb{E}_p[\Delta_i X]$$

note that

$$\mathbb{E}_p[\Delta_i X] = \sum_{\omega \in \{0,1\}^{[n]\setminus\{i\}}} [\Delta_i X](\omega)\, p^{|\omega|}(1-p)^{n-1-|\omega|}$$

since the value of ω_i does not matter.

Similar for $p-h$

So

$$\frac{d}{dp} \mathbb{E}_p(X) = \lim_{h \to 0} \frac{\mathbb{E}_{p+h}(X) - \mathbb{E}_p(X)}{h}$$

$$= \sum_{i=1}^{n} \mathbb{E}_p[\Delta_i X].$$

Alternatively: write $\mathbb{E}(X) = \sum_{\omega} \varphi(\omega_1) \cdots \varphi(\omega_n) X(\omega)$

with $\varphi(0) = 1-p$, $\varphi(1) = p$,

and use $\frac{d\varphi}{dp}(\omega_i) = \delta_{\omega_i,1} - \delta_{\omega_i,0}$.

(3)

$3/5 = 0.6$

$2/3 = 0.666\ldots$

(a) Degrees $\sim \text{Bin}(n-1, p)$

$\mu = (n-1)p \approx n^{3/5}$

X_v = deg. of v.

$$\mathbb{P}(X_v \geq \mu + t) \leq \exp\left(-\frac{t^2}{2(\mu + t/3)}\right)$$

Put $t = \frac{1}{2} n^{2/3}$

$$\mathbb{P}(X_v \geq n^{2/3}) = \mathbb{P}(X_v - \mu \geq n^{2/3} - \mu)$$

$$n^{2/3} - \mu = n^{2/3} - n^{3/5} \geq \frac{1}{2} n^{2/3}$$

if n large

$$\leq \mathbb{P}(X_v - \mu \geq t (= \tfrac{1}{2} n^{2/3}))$$

$$\leq \exp\left(-\frac{\frac{1}{4} n^{4/3}}{2(n^{3/5} + \frac{1}{6} n^{2/3})}\right)$$

$$\leq \exp(-c \cdot n^{2/3})$$

So $\mathbb{P}(\exists v: X_v > n^{2/3})$

$$\leq n \cdot \exp(-c n^{2/3}) \to 0.$$

(b) Fix A, B with $|A|, |B| \geq n^{1/2}$.

$$\mathbb{P}(A \not\leftrightarrow B) \leq (1-p)^{(n^{1/2})^2} = \left(1 - \frac{1}{n^{2/5}}\right)^n$$

$$\leq e^{-n^{3/5}}$$

The # pairs A, B like that

is $\leq \binom{n}{n^{1/2}}^2$

$$\cancel{\text{Stirling}: \quad n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n}$$

$$\cancel{\binom{n}{n^{1/2}} = \frac{n!}{n^{1/2}!}}$$

$$\binom{n}{m} = \frac{n!}{m!\,(n-m)!} \approx \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{\sqrt{2\pi m}\left(\frac{m}{e}\right)^m \sqrt{2\pi(n-m)}\left(\frac{n-m}{e}\right)^{n-m}}$$

$$\cancel{= \sqrt{\frac{2\pi n}{2\pi m \cdot 2\pi(n-m)}} \cdot \frac{n^n}{m^m (n-m)^{n-m}}}\,.$$

$m = \sqrt{n}$:

$$\cancel{\sqrt{\frac{n^{1/2}}{2\pi(n-n^{1/2})}} \cdot \frac{n^n}{n^{\frac{1}{2}n^{1/2}}(n-\sqrt{n})^{n-\sqrt{n}}}}$$

Stirling:

$$\log(n!) \approx n\log n - n + C\log n$$

$$\binom{n}{m} = \exp\Big(\log(n!) - \log(m!) - \log((n-m)!)\Big)$$

$$= \exp\Big(n\log n - \cancel{n} - m\log m + \cancel{m} - (n-m)\log(n-m) + \cancel{n-m} + c\log n\Big)$$

$m = \sqrt{n}$

$$\leq \exp\Big(n\log n - \tfrac{1}{2}n^{1/2}\log n - (n-n^{1/2})\log(n-n^{1/2}) + C\log n\Big)$$

$$= \exp\Big(n \cdot \big(\log n - \log n(1-n^{-1/2}) \big) - \tfrac{1}{2} n^{1/2} \log n + C \log n$$
$$+ n^{1/2} \log (n - n^{1/2})$$

$$= \exp\Big(n \cdot \log\Big(\frac{1}{1-n^{-1/2}}\Big) + \cancel{\tfrac{1}{4}}\, n^{1/2} \log\Big(\frac{n-n^{1/2}}{n^{1/2}}\Big) + C \log n \Big)$$

So $\quad \mathbb{P}(\exists A, B : A \leftrightarrow B)$

$$\leq e^{-n^{3/5}} \cdot \exp\Big(n^{1/2} \log\big(n^{1/2} - 1\big) + n \log\Big(\frac{1}{1-n^{-1/2}}\Big) + C \log n \Big)$$

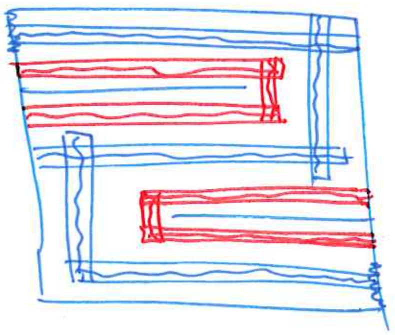
$$\log \frac{1}{1-x} = -\log(1-x)$$
$$= x + \frac{x^2}{2} + \cdots$$
$$n \log \frac{1}{1-n^{-1/2}} = n \cdot \Big(n^{-1/2} + \frac{n^{-1}}{2} + \cdots \Big)$$
$$\leq c n^{1/2}$$

So get

$$\exp\Big(-n^{0.6} + c \cdot n^{0.5} + n^{1/2} \log n + (\text{lower}) \Big)$$

$$\longrightarrow 0$$

(4)



white paths in red. rects
~~red~~ black paths in blue.

$\{red\} \cap \{blue\} = \phi$

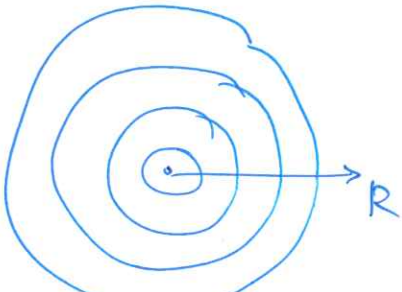
$\Rightarrow$ indep.

Use RSW.

(5) $\mathbb{P}(\mathcal{O}_n) \geq \alpha > 0 \quad \forall n, \text{~~big~~}$

Holds also for <u>white</u> circuit.

Surrond 0 with N annuli:



Can fit $N \geq c_1 \log R$ within Λ_R.

$\mathbb{P}(0 \leftrightarrow \partial\Lambda_R) \leq \mathbb{P}($none of the N annuli has a white circuit$)$

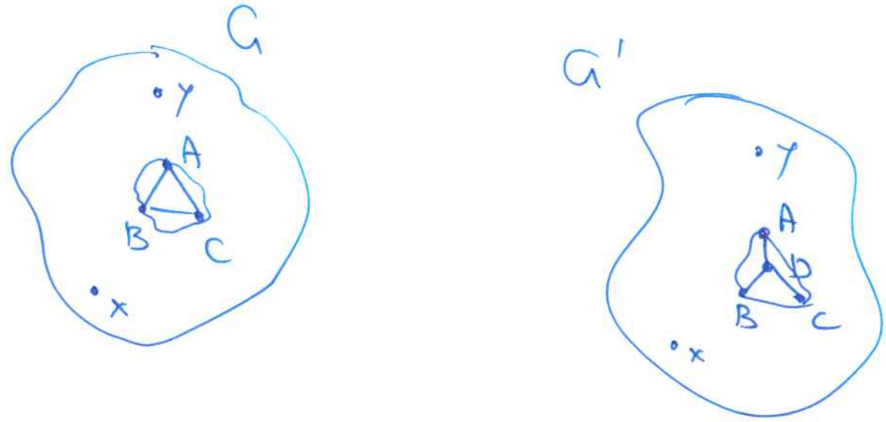
$\leq (1-\alpha)^N$ (indep. since disj.)

$= c_2 R^{-c_3}$, some $c_2, c_3 > 0$

Let $R \to \infty$:

$\mathbb{P}(0 \leftrightarrow \infty) = 0$.

(6)



(a) Consider all possible local connections among A, B, C using edges AB, AC, BC and AD, ~~A~~ BD, CD resp:

~~A~~ <u>connected</u>

(i) none: $(1-p)^3$ vs $\cancel{q}(1-q)^3 + 3q(1-q)^2$

(ii) AB, no other: $p(1-p)^2$ vs $q^2(1-q)$
(etc)

(iii) all: $p^3 + 3(1-p)p^2$ vs q^3

Clearly (ii) are the same, and (i) same $\Leftrightarrow$ (iii) same.

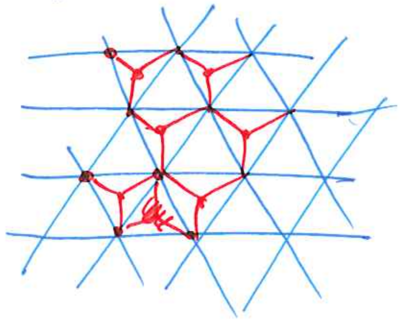
Check (i):

$$(1-p)^3 - (1-q)^3 - 3q(1-q)^2$$
$$= (1-p)^3 - p^3 - 3p^2(1-p)$$
$$= 1 - 3p + \cancel{3p^2} - p^3 - p^3 - \cancel{3p^2} + 3p^3$$
$$= 1 - 3p + p^3 = 0$$

Hence all local connections among A, B, C ~~are~~ have same prob. under $\mathbb{P}$, $\mathbb{P}'$.

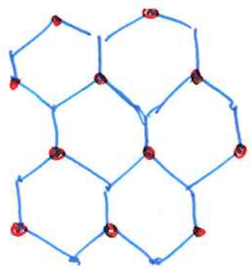
So $\mathbb{P}(x \leftrightarrow y) = \mathbb{P}'(x \leftrightarrow y)$.

(b) $\mathbb{T}$



$\rightsquigarrow$ transform all ∇'s (or all Δ's)

$\mathbb{H}$



$$\mathbb{P}(0 \leftrightarrow \text{dist} \geq R) = \mathbb{P}'(0 \leftrightarrow \text{dist} \geq R)$$

$$\downarrow (R \to \infty) \qquad \qquad \downarrow (R \to \infty)$$

$$\theta_{\mathbb{T}}(p) \qquad \qquad \theta_{\mathbb{H}}(1-p)$$

(7)

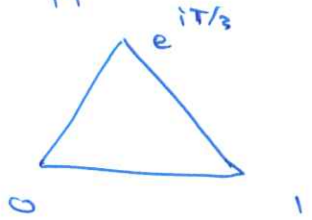
(a) Know: $G_1 = H_1 + H_\tau + H_{\tau^2} \equiv 1$

and $H_{\tau^j} \geq 0$ (probs.)

So G_2 = conv. comb.

$H_{\tau^j}(z) \to 1$ for $z \to \tau^j$

0 for $z \to$ opp side



So eg on $[1, e^{i\pi/3}]$ get

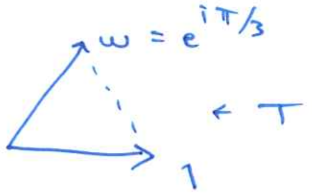
conv. comb. $\tau H_\tau(z) + \tau^2 H_{\tau^2}(z)$

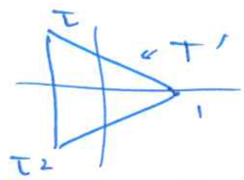
etc

(b) Write $G_2(z) = f(z)$.
Can check that $\boxed{f(z) = 1 - z(1-\tau)}$

works. One way to arrive at

this:





Write $z = t + sw$ $\quad s, t \in \mathbb{R}$

~~Want~~ Try
$$\begin{cases} f(t) = (1-t) + t\tau & \text{for } t \in [0,1] \\ f(sw) = (1-s) + s\tau^2 & \text{for } s \in [0,1]. \end{cases}$$

and $f(t+sw) = f(t) + f(sw) - 1$.

The remaining side $[1, w]$ is given by $s+t=1$.

Write s, t using $z, \bar{z}$ and simplify.

Note that

$$\tau G_2(z) = \tau H_1(z) + \tau^2 H_\tau(z) + H_{\tau^2}(z)$$

So that

$$\mathrm{Re}(\tau G_2(z)) = -\frac{1}{2}\left(H_1(z) + H_\tau(z)\right) + H_{\tau^2}(z)$$

$$= -\frac{1}{2}\left(1 - H_{\tau^2}(z)\right) + H_{\tau^2}(z).$$

Hence

$$H_{\tau^2}(z) = \frac{1}{3}\left(1 + 2\,\mathrm{Re}(\tau G_2(z))\right)$$

But
$$\mathrm{Re}(\tau f(z)) = \frac{\tau f(z) + \overline{\tau f(z)}}{2}$$

and

$$\tau f(z) + \overline{\tau f(z)} = \tau(1 - z(1-\tau)) + \tau^2(1 - \bar{z}(1-\tau^2))$$

$$= \underbrace{(\tau + \tau^2)}_{-1} + \underbrace{(\tau - \tau^2)}_{i\sqrt{3}}\underbrace{(\bar{z} - z)}_{-2i\,\mathrm{Im}(z)}$$

So
$$H_{\tau^2}(z) = \frac{1}{3}\left(-2i^2\sqrt{3}\cdot \mathrm{Im}(z)\right)$$

$$= \frac{2}{\sqrt{3}}\,\mathrm{Im}(z).$$