Homework 2 — solutions

①

(a) $\mathbb{P}_{p,q}(\omega) \propto \left(\frac{p}{1-p}\right)^{|\omega|} q^{k(\omega)}$

trees have $k(\omega) = |V| - |\omega|$

So $\mathbb{P}_{p,q}(\omega) \propto \left(\frac{p/q}{1-p}\right)^{|\omega|}$

This is $= \left(\frac{\pi}{1-\pi}\right)^{|\omega|}$ if $\frac{p/q}{1-p} = \frac{\pi}{1-\pi}$

So $\pi = \frac{p}{p+q(1-p)}$.

(b) $\langle \sigma_x \sigma_y \rangle = \mathbb{P}_{p,2}(x \leftrightarrow y) = \mathbb{P}_{\pi,1}(x \leftrightarrow y)$
with $p = 1-e^{-2\beta}$

For trees with d offspring,
percolation-threshold is $p_c = \frac{1}{d}$.

$\mathbb{Z}$ = tree with $d=1$.

So $p_c = 1$: no phase-transition.

(2) $$\mathbb{P}_{p,q}(\omega) = \frac{\left(\frac{p}{1-p}\right)^{|\omega|} q^{k(\omega)}}{\sum_{\omega'} \left(\frac{p}{1-p}\right)^{|\omega'|} q^{k(\omega')}}$$

(a) Only the ω with minimal $|\omega|$ $(=0)$ gets >0 limit, so: empty graph gets prob. 1

(b) All support on minimal $k(\omega)$ $(=1)$ so get all connected subgraphs prob. $\propto \left(\frac{p}{1-p}\right)^{|\omega|}$.

(c) Write numerator as $\underbrace{\left(\frac{1}{1-p}\right)^{|\omega|}}_{\text{stays }>0,\ \to 1} p^{|\omega|+k(\omega)}$.

Minimal $|\omega|+k(\omega)$ survives, this is for any graph with no cycles (forest). All forests get equal weight (uniform).

(d) Write numerator as $p^{|\omega|+k(\omega)} \left(q/p\right)^{k(\omega)} \left(\frac{1}{1-p}\right)^{|\omega|}$.

Need to minimize both $|\omega|+k(\omega)$ (no cycles) and $k(\omega)$ $(\Rightarrow k(\omega)=1)$. So get all connected trees, with equal weight (uniform spanning tree).

3

(a) $\frac{\partial}{\partial h_x} \log Z = \frac{1}{Z} \frac{\partial Z}{\partial h_x} = \langle \sigma_x \rangle$

(b) $\frac{\partial}{\partial \beta} \log Z = \sum_{xy} \langle \sigma_x \sigma_y \rangle$

$\frac{\partial^2}{\partial h_x \partial h_y} \log Z = \langle \sigma_x \sigma_y \rangle - \langle \sigma_x \rangle \langle \sigma_y \rangle.$

(c) By the FK-representation, can write $\langle \sigma_x \sigma_y \rangle$ and $\langle \sigma_x \rangle$ as probabilities:

$\langle \sigma_x \sigma_y \rangle = \mathbb{P}(x \leftrightarrow y)$

$\langle \sigma_x \rangle = \mathbb{P}(x \leftrightarrow \partial)$ [see lectures]

Probabilities are ≥ 0

So $\log Z$ (hence also Z) incr. in β, h_x.

(4) Orient edges as in hint.

(a) If $e = \nearrow$ then $W_\gamma(e \to b) = \pi/2 + 2\pi k$

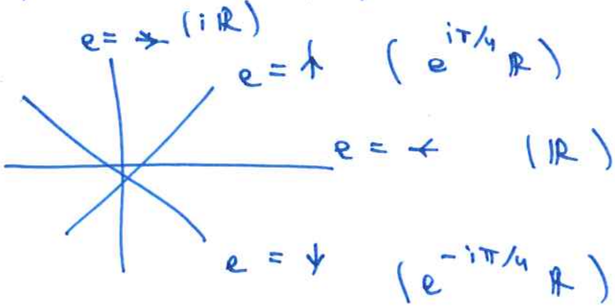
(some $k \in \mathbb{Z}$)

So $e^{\frac{i}{2}W} = \pm e^{i\pi/4}$

$\uparrow$ random (dep. on ω)

Hence $F(e = \nearrow) \in e^{i\pi/4}\mathbb{R}$.

Similarly:



(b) We saw in the lectures:

$[*]$ $F(N) - F(S) = i\,(F(E) - F(W))$.

Can either mimic the proof of $[*]$ and put in $q = 2$, or note that (a) and $[*]$ give (by complex conj.)

$$\overline{F(N)} - \overline{F(S)} = -i\left(\overline{F(E)} - \overline{F(W)}\right)$$

$$\Leftrightarrow \quad (-i)F(N) - i\,F(S) = -i\left(F(E) + F(W)\right)$$

as required.

5

(a) Check that $A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ works.

(b) Let $\mathcal{A} = \prod_{x \in V} A_x$ (factors commute)

Then $\mathcal{A} H \mathcal{A}^{-1} = H$ so

$$\langle S^{(1)}_x S^{(1)}_y \rangle = \frac{1}{2}\, \mathrm{tr}\left(\mathcal{A} S^{(3)}_x S^{(3)}_y \mathcal{A}^{-1} \mathcal{A} e^{-\beta H} \mathcal{A}^{-1} \right)$$

(cyclicity of tr)

$$= \langle S^{(3)}_x S^{(3)}_y \rangle .$$

Using $B = \frac{1}{\sqrt{2}}\begin{pmatrix} -i & -i \\ 1 & -1 \end{pmatrix}$ from the lectures $(1 \to 3 \to 2 \to 1)$ and $\mathcal{B} = \prod_x B_x$ also gives $\langle S^{(1)}_x S^{(1)}_y \rangle = \langle S^{(2)}_x S^{(2)}_y \rangle$.

(c) Writing $H^{\pm} = -\sum_{xy} \mathbf{S}_x \cdot \mathbf{S}_y \pm h \sum_x S^{(2)}_x$

we have $\mathcal{A} H^{\pm} \mathcal{A}^{-1} = H^{\mp}$.

Hence

$$\langle S^{(1)}_x S^{(1)}_y \rangle_{\pm} = \langle S^{(3)}_x S^{(3)}_y \rangle_{\mp}$$

$$\langle S^{(2)}_x S^{(2)}_y \rangle_{+} = \langle S^{(2)}_x S^{(2)}_y \rangle_{-} .$$

6

(a) Write $\Sigma^{(j)} = S_x^{(j)} + S_y^{(j)}$

and $C_{xy} = \sum_{j=1}^{3} (\Sigma^{(j)})^2$

$$= \frac{3}{2} + 2\, \mathbf{S}_x \cdot \mathbf{S}_y$$

So $S_x^1 S_y^1 + S_x^2 S_y^2 + \Delta S_x^3 S_y^3$

$$= \mathbf{S}_x \cdot \mathbf{S}_y + (\Delta - 1) S_x^3 S_y^3$$

$$= \frac{1}{2} C_{xy} - \frac{3}{4} + (\Delta - 1) S_x^3 S_y^3 .$$

Now $C_{xy} = \begin{cases} 2 & \text{on span}\{|++\rangle, |+-\rangle + |-+\rangle, |--\rangle\} \\ 0 & \text{on span}\{|+-\rangle - |-+\rangle\}. \end{cases}$

Consequently the spectrum is:

$$\begin{cases} \frac{\Delta}{4} & \text{twice} \quad (\text{on } |++\rangle \text{ and } |--\rangle) \\ \frac{1}{2} - \frac{\Delta}{4} & \text{once} \quad (\text{on } |+-\rangle + |-+\rangle) \\ -\frac{1}{2} - \frac{\Delta}{4} & \text{once} \quad (\text{on } |+-\rangle - |-+\rangle) \end{cases}$$

Most degenerate: $\Delta = \pm 1$.

(b) $\Delta = +1$: singlet spanned by $|+-\rangle - |-+\rangle$
with eigenvalue $-\frac{3}{4}$
triplet: eigenval $+\frac{1}{4}$.

Hence $-\mathbf{S}_x \cdot \mathbf{S}_y + \frac{1}{4}$ $\begin{cases} \text{kills triplet} \\ \text{preserves singlet} \end{cases}$

hence $-\mathbf{S}_x \cdot \mathbf{S}_y + \frac{1}{4} = P.$

For $\Delta = -1$, $P = + \mathbf{S}_x \cdot \mathbf{S}_y + \frac{1}{4}$ similarly.

⑦

(a) We have

$$C_\Sigma = \frac{3}{4}n + 2\sum_{xy} \mathbf{S}_x \cdot \mathbf{S}_y$$

So (using also 5(b))

$$\langle C_\Sigma \rangle = \frac{3}{4}n + 2\binom{n}{2}\langle \mathbf{S}_x \cdot \mathbf{S}_y \rangle$$

$$= \frac{3}{4}n + 3\cdot 2\binom{n}{2}\langle S_x^{(3)} S_y^{(3)} \rangle .$$

(b) For $n=3$, C_Σ has two eigenspaces of dim. 4, namely one copy of $\mathcal{U}^{(3/2)}$ and two copies of $\mathcal{U}^{(1/2)}$.

The eigenvalues are $\frac{15}{4}$ and $\frac{3}{4}$.

Using $-H = \frac{1}{2}C_\Sigma - \frac{9}{4}$ (from (a))

we get

$$\mathrm{tr}(e^{-\beta H}) = e^{-\frac{9}{4}\beta}\left(4e^{\frac{15\beta}{8}} + 4e^{\frac{3\beta}{8}}\right)$$

$$\mathrm{tr}(C_\Sigma e^{-\beta H}) = e^{-\frac{9}{4}\beta}\left(15e^{15\beta/8} + 3e^{3\beta/8}\right)$$

and hence

$$\langle S_x^{(3)} S_y^{(3)} \rangle = -\frac{1}{8} + \frac{5e^{15\beta/8} + e^{3\beta/8}}{24\left(e^{15\beta/8} + e^{3\beta/8}\right)} .$$